Mixed-Price Auctions for Divisible Goods

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Abstract

In a mixed-price auction, bidders' payments are convex combinations of price discrimination and the market-clearing price. In a symmetric divisible-good model, I prove that all pure-strategy equilibria in mixed-price auctions are symmetric, and give a closed-form expression for equilibrium bids. I show that the set of feasible equilibrium bids shrinks as the auction becomes discriminatory, as aggregate supply becomes deterministic, and as the market becomes large. When bidders have linear marginal values the unique equilibrium of the discriminatory auction raises more revenue than any equilibrium of the uniform-price auction, but an additional bidder may be more valuable than proper selection of auction format. On the whole, sellers implementing uniform-price auctions may reap substantial gains by introducing mild price discrimination.

1 Introduction

Multi-unit auctions are commonly used to allocate homogeneous goods, including government debt, emissions permits, and electricity generation. The two most commonly implemented multi-unit auction formats are uniform-price and discriminatory (Brenner et al., 2009), and trillions of dollars of securities and commodities pass through these auction formats each year.[1] In spite of the vast sums allocated by these auction formats, which of the two yields better outcomes remains both theoretically and empirically ambiguous: the uniform-price auction is commonly believed to be simpler for participants (Friedman, 1991; Chari and Weber, 1992), but allows for a wide range of equilibrium behavior (McAdams, 2007; Burkett and Woodward, 2020b), while the discriminatory auction is more complex but also less susceptible to bidder coordination (Pycia and Woodward, 2021). Because these auction formats allocate substantial financial value, specific guidance for mitigating the drawbacks of these formats may have a significant impact on auction outcomes.

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[1]In both of these auction formats bidders submit demand curves to the seller, and these demand curves are used to determine a market-clearing price and associated quantities. In a uniform-price auction each bidder pays the (constant) market-clearing price for each unit she receives, while in a discriminatory auction each bidder pays her bid for each unit she receives. In 2018, the U.S. Treasury auctioned over \$10 trillion in securities using a uniform-price auction; worldwide, the discriminatory auction is more popular (Brenner et al., 2009; Del Río, 2017).

This paper studies interpolations of the uniform-price and discriminatory auctions known as *mixed-price* auctions. Under a mixed pricing rule, bidders pay a convex combination of their uniform-price and discriminatory payments.[2] Mixed pricing corresponds to a uniform-price auction with a tax on apparent surplus, and has recently been proposed as a method for improving revenue in treasury auctions (Armantier and Sbaï, 2009) as well as for balancing the benefits of technological improvements in electricity markets (Ruddell et al., 2016).[3] My results show that incremental price discrimination may improve action performance by shrinking the set of equilibria, reducing the range of equilibrium outcomes, and increasing expected revenue.[4] Further, when aggregate supply is relatively certain even a small amount of price discrimination can eliminate many seller-suboptimal equilibria of the uniform-price auction. On the other hand, the incremental revenue raised by switching to a discriminatory format can be bounded by the revenue raised by bringing another bidder to the market. Thus in an auction in which bidding is relatively competitive, equilibrium revenue will not depend significantly on auction format, but in an auction with a relatively small number of bidders mild price discrimination can significantly improve outcomes. Practically, sellers implementing uniform-price auctions have little to lose, and potentially much to gain, by introducing mild price discrimination.

My model of divisible-good auctions is similar to those in Klemperer and Meyer (1989), Ruddell et al. (2016), and Pycia and Woodward (2021). Bidders are symmetric and symmetrically-informed, and have strictly decreasing marginal value for quantity; symmetric information is an approximation of markets in which bidders interact frequently and there is little idiosyncratic uncertainty (Holmberg, 2009). Aggregate supply is random, and bidders submit weakly decreasing demand curves before supply is realized. After aggregate supply is realized, the auctioneer uses submitted bids to determine the market-clearing price and associated quantities.[5] Bidders pay to the seller proportion α of the area under their demand curve (the discriminatory payment), and proportion $1-\alpha$ of their quantity at the market-clearing price (the uniform-price payment). Equivalently, they pay the market-clearing price for each unit received, plus proportion α of the area between their demand curve and the market-clearing price. For this reason, I refer to α as the *extent of price discrimination*.

Previous theoretical work has considered the mixed-price auction model, but my approach is distinguished in two ways. Early analyses (Viswanathan and Wang, 2002; Wang and Zender, 2002) employed the mixed-price auction model is as a tool to unify the analysis of discriminatory and uniform-price auctions, but equilibrium outcomes under partial discrimination were largely ignored.

[2]There are other auction formats "between" uniform-price and discriminatory. For example, the "Spanish auction" format price-discriminates against low winning bids and charges a constant price for high winning bids (Armantier and Sbaï, 2009).

[3]In addition to potentially improving the seller's revenue, benefit taxation may also be used to discourage collusive behavior in both one-shot and repeated multi-unit auctions (Marszalec et al., 2020).

[4]Pycia and Woodward (2021) show that when the auctioneer can design the parameters of the auction, the discriminatory auction outperforms the uniform-price auction; they do not consider mixed-price auctions. I hold fixed the parameters of the auction, and allow the auctioneer to select only the auction format and not its parameters.

[5]Quantity allocations are identical in the discriminatory and uniform-price auction formats, so partial discrimination does not affect bidders' realized quantities (conditional on submitted bids).

More recent work (Armantier and Sbaï, 2009; Ruddell et al., 2016) formally considers the possibility of partial discrimination, but does not consider the theoretical properties of mixed-price equilibria except when bidders are price-takers.[6] By contrast, I explicitly consider the theoretical properties of mixed-price auctions which are neither purely discriminatory nor purely uniform-price, and derive comparative statics on this partial discrimination. Additionally, I assume that bidders' marginal values are decreasing in the quantity received. Larger quantities of treasury securities, for example, serve as less effective hedges against market uncertainty, and may be marginally less valuable.

My analysis proceeds by establishing a closed-form expression for equilibrium bids in any mixed-price auction. The first-order optimality conditions for equilibrium bids are derived by applying the calculus of variations, and are a complicated and potentially intractable multidimensional integro-differential system. I simplify this system by proving that equilibrium strategies must be symmetric.[7] The simplified system is a standard first-order differential equation which may be solved with standard techniques. As the solution to a differential equation, equilibrium bids are uniquely determined by their initial condition.

Unlike in standard differential analysis, whether a specific initial condition is valid—i.e., whether the implied bids are an equilibrium—depends on the full bid function derived from the initial condition. Variational analysis of bidding strategies gives conditions under which bids are locally optimal holding fixed the initial condition; that is, conditional on a particular bid at a particular quantity. However, bid deviations might not only alter marginal payments and probabilities of winning, but may also expand the relevant domain of the bid function. In a symmetric equilibrium, for example, bidders will never receive more than their share of the aggregate quantity sold; but by submitting a bid which is everywhere above the minimum possible equilibrium price, a bidder can obtain more than her share of the aggregate quantity. Because the range of obtainable quantities changes when the bidder changes her bid, standard variational analysis cannot be used to ensure that this sort of deviation is not profitable. In other words, the standard first-order approach must be augmented by analyses to ensure the validity of the selected initial conditions, which in turn depends on the solutions obtained from the variational approach.

Considering the validity of initial conditions, and not just the pointwise optimality of submitted bids, allows me to study the effect of partial discrimination on equilibrium uniqueness.[8] By ana-

[6] Armantier and Sbaï (2009) conduct a counterfactual analysis of French treasury auctions assuming that submitted demand curves must be a low-order polynomial. The mixed-price auction I study is equivalent to their "α-discriminatory" auction, but I allow for bidders to submit non-polynomial demand curves. Ruddell et al. (2016) consider the possibility that the seller taxes apparent bidder surplus and characterize equilibrium existence. My equilibrium analysis is similar but my comparative statics are novel.

[7] The argument for equilibrium symmetry is similar to that for discriminatory auctions (Pycia and Woodward, 2021): asymmetric bid profiles induce profitable deviations through discontinuous elasticities for small quantities. Equilibrium symmetry in the uniform-price auction must be analyzed separately, and follows an approach similar to that in Klemperer and Meyer (1989).

[8] Although uniqueness is a binary concept, for simplicity of exposition I discuss decreased equilibrium multiplicity as increased uniqueness. Thus uniform-price auction equilibria are nonunique (Klemperer and Meyer, 1989; Back and Zender, 1993), the discriminatory auction equilibrium is unique (Wang and Zender, 2002; Pycia and Woodward, 2021), and for any strictly mixed-price auction, equilibrium is more unique than in the uniform-price auction, but still (generally) nonunique.

lyzing perturbations of candidate equilibrium bids, I show that the set of feasible initial conditions shrinks as the auction format becomes discriminatory. The set of feasible initial conditions also shrinks as aggregate supply becomes deterministic, except in the (unmixed) uniform-price auction. An immediate implication is that when supply is highly concentrated, all partially-discriminatory auctions are approximately revenue-equivalent and raise more revenue than most equilibria of the unmixed uniform-price auction. Since in practice supply is frequently concentrated, sellers who are otherwise interested in implementing uniform-price auctions may improve auction outcomes by implementing a small tax on apparent benefits (that is, mild price discrimination).

My results on equilibrium uniqueness regard only the set of valid initial conditions for equilibrium bids. Because equilibrium bids depend on the extent of price discrimination and the elasticity of residual supply, equilibrium bid sets are not generally comparable across different mixed-price auction formats.[9] However, by applying my uniqueness results to the closed-form expression for equilibrium bids I show that the set of feasible market-clearing prices shrinks as the auction becomes discriminatory, and grows as the pool of bidders increases: for any equilibrium in a less-discriminatory auction, there is a corresponding equilibrium in a more-discriminatory auction with a broader range of market-clearing prices.[10] Thus incremental discrimination may reduce the volatility of equilibrium outcomes, and increasing the number of bidders may increase it. In the limit, with an infinite number of bidders, equilibrium is unique in all mixed-price auctions; since bids are more elastic the more discriminatory is the auction, price volatility increases as the extent of price discrimination decreases.[11]

While equilibrium bids are expressable in closed form, they do not in general have a simple algebraic formulation. To obtain further comparative statics I constrain attention to the *polynomial-Lomax* model, in which marginal values are polynomials and the distribution of supply is negative Lomax.[12] The order of the marginal value polynomials is unrestricted, and therefore quite general. The negative Lomax distribution allows for arbitrary concentration of supply at low or high levels, and therefore is a natural fit for markets with relatively little uncertainty regarding aggregate supply, such as markets for electricity (Holmberg, 2009) or treasury securities (Pycia and Woodward, 2021). When marginal values and bids are both linear, equilibrium expected revenue is strictly increasing in the extent of price discrimination. This is not because the seller is price-discriminating against fixed

[9] Equilibrium bidding incentives in mixed-price auctions are a convex combination of discriminatory and uniform-price bidding incentives. However, because equilibrium bids are the solution to a differential equation relating elasticity to marginal utility, equilibrium bids are not a convex combination of discriminatory and uniform-price equilibrium bids.

[10] In my analysis, I consider the effect of increasing the number of bidders while holding per-capita supply constant. My conclusions remain valid when the number of bidders increases while holding aggregate supply constant. See, e.g., Jackson and Kremer (2006) for related arguments.

[11] This uniqueness is familiar from other large-market contexts; see, e.g., Swinkels (1999) and Swinkels (2001). Ruddell et al. (2016) show that all mixed-price auctions are revenue-equivalent when bidders are price-takers. Pycia and Woodward (2021) show that seller-optimal equilibria of the discriminatory and uniform-price auctions are revenue-equivalent, conditional on the seller being able to set the distribution of quantity. By contrast, my seller is passive and cannot affect the distribution of quantity.

[12] Equivalently, supply follows a negative Pareto type II distribution with cdf $F(Q) = 1 - (1 - Q/\overline{Q})^{\lambda}$ for some maximum supply $\overline{Q}$ and concentration λ.

bids, but is net of the changes in bidding behavior induced by discriminatory auction incentives. In all but the (unmixed) discriminatory auction, linear equilibria are not seller-optimal, and I also compare expected revenues in seller-optimal, "maximum bid" equilibria of the discriminatory and uniform-price auctions. As it turns out, the discriminatory auction generates greater expected revenue than even the seller-optimal equilibrium of the uniform-price auction. However, the effect of mechanism selection may be dominated by the effect of enticing an additional bidder to participate the auction.

These results have implications for the implementation of divisible-good auctions. The smoothness of equilibrium multiplicity in the extent of price discrimination contrasts significantly with results for single-unit auctions (Plum, 1992; Lizzeri and Persico, 2000), where even a small proportion of discriminatory implementation implies equilibrium uniqueness.[13] Smoothness of equilibrium multiplicity in divisible-good auctions has been observed by Marszalec et al. (2020), who find that equilibrium collusion is easier to sustain the less discriminatory is the auction format. Uniqueness of equilibrium strategies is related to the range of equilibrium prices, which I show is increasing as the auction becomes less discriminatory. This is at odds with conventional belief that uniform-price prices are easy to predict (Friedman, 1991; Lotfi and Sarkar, 2016), and that bidders should simply "bid their value." Finally, in large markets mechanism selection has little effect on expected revenues, at least within the space of mixed-price auctions. This is in line with earlier large-market analyses (Swinkels, 2001; Jackson and Kremer, 2006).[14]

Beyond revenue equivalence in large markets and revenue near-equivalence under concentrated supply, my explicit revenue comparisons shed light on the ambiguous revenue comparison obtained by empirical analyses of multi-unit auctions.[15] For example, Février et al. (2002), Kang and Puller (2008), Marszalec (2017), and Mariño and Marszalec (2020) find discriminatory auctions raise more revenue than uniform price auctions, Armantier and Sbaï (2006), Castellanos and Oviedo (2008), and Armantier and Sbaï (2009) find the opposite, and Hortaçsu and McAdams (2010), Hortaçsu et al. (2018), and Barbosa et al. (2021) find no statistically significant difference. My results in the linear-Lomax model suggest that discriminatory auctions raise greater revenue than uniform-price auctions, but also suggest that the effect of an additional bidder can outweigh the effect of mechanism selection. Thus in markets which are fairly competitive the selection of auction format will have little effect on equilibrium revenue, and there is little downside to implementing a moderate tax on apparent surplus.[16]

[13] When supply is deterministic, as in single-unit auctions, I also find a discontinuity in equilibrium multiplicity at the uniform-price auction.

[14] Jackson and Kremer (2006)'s findings of revenue equivalence depend on the rate at which supply increases with the number of bidders. In my analysis, I consider markets which grow while per capita supply is held constant.

[15] Theoretically ambiguous revenue and welfare comparisons are also obtained by Jackson and Kremer (2006), Ausubel et al. (2014), and Pycia and Woodward (2021).

[16] In treasury and electricity auctions, the set of participants consists mostly of prequalified institutional bidders. There is thus little variation in the number of bidders participating in these auctions (Armantier and Sbaï, 2009), and accordingly little evidence regarding the effect of an additional bidder on equilibrium outcomes. Hortaçsu et al. (2018) find that bids in (uniform-price) U.S. Treasury auctions are fairly flat. In my model, flat bids in uniform-price auctions correspond to flat marginal values, in which case an additional bidder will yield little additional revenue.

This paper proceeds by formally defining the model of divisible-good auctions with a mixed-price payment rule: Section 2 introduces the divisible-good mixed-price auction model and establishes the basic form of equilibrium. Section 3 establishes results on uniqueness, supply concentration, the range of equilibrium prices, and large-market outcomes, and Section 4 provides comparative statics in the polynomial-Lomax model. Section 5 concludes. Most proofs appear in the appendix.

2 Model and equilibrium

There are $n \geq 2$ bidders, $i \in \{1, ..., n\}$, participating in an auction for a perfectly-divisible good.[17] Bidders share a common signal s with support $\mathcal{S}$, and bidder i has marginal value function $v^i \equiv v$, where her marginal value for quantity q is $v(q; s)$. v is weakly positive and Lipschitz continuous in q, and is strictly decreasing and differentiable in q wherever $v(q; s) > 0$. The available market quantity Q is stochastic and drawn according to the cumulative distribution function F, independent of s, with support $[0, \overline{Q}]$ and density f which is bounded away from zero. Denote per capita maximum supply by $Q^\mu \equiv \overline{Q}/n$, and let the per capita distribution of supply be F^μ, so that $F^\mu(q) = F(nq)$. Each bidder's utility is quasilinear in the transfer she makes to the seller: if bidder i obtains quantity q_i and makes transfer t_i, her utility is

$$\tilde{u}\left(q_i, t_i; s\right) = \int_0^{q_i} v\left(x; s\right) dx - t_i. \tag{1}$$

I denote the (efficient) aggregate marginal value for quantity Q by $\hat{v}(Q; s) = v(Q/n; s)$.

Each agent i submits a decreasing bid function $b^i : [0, \overline{Q}] \to \mathbb{R}_+$ to the auctioneer; b^i is strictly decreasing wherever it is strictly positive, and admits an absolutely continuous inverse φ^i.[18] The inverse bids functions $(\varphi^i)_{i=1}^n$ implicitly define the market-clearing price $p^\star : [0, \overline{Q}] \to \mathbb{R}_+$,

$$\sum_{i=1}^n \varphi^i\left(p^\star\left(Q\right)\right) = Q.$$

After bid functions are submitted, random quantity Q is realized and the seller computes the market-clearing price $p^\star(Q)$ and associated quantities $q_i = \varphi^i(p^\star(Q))$.[19] Each bidder is awarded her market-clearing quantity, and pays the seller an α-convexification of the payments she would make in discriminatory and uniform-price auctions. In a discriminatory auction each bidder would pay her bid for each unit she obtained, and in a uniform-price auction each bidder would pay the (constant) market-clearing price for each unit she obtained. In the mixed-price auction the

[17] In a (pure) uniform price auction with two bidders and linear marginal values, there may exist no equilibrium in linear bids, nonetheless there will exist an equilibrium in nonlinear bids. In practice these auctions generally have at least three participants. I generalize the at-least-three-bidders condition following Theorem 7.

[18] In Appendix A I show that even if discontinuous and weakly decreasing bids are permitted, all equilibria are in continuous, strictly decreasing bids; thus in any equilibrium bidder i's bid b^i admits a continuous inverse φ^i. Since the inverse φ^i is monotone it is differentiable almost everywhere, and absolute continuity serves only to ensure that inverse bids may be "integrated up." Since all equilibria are symmetric, $\varphi_p^i < 0$ everywhere it is defined and thus b^i is also absolutely continuous.

[19] By convention, superscripts indicate functions and subscripts indicate values.

bidder pays proportion α of her discriminatory payment, and proportion $1-\alpha$ of her uniform-price payment,

$$t_i = \alpha \int_0^{q_i} b^i(x)\, dx + (1-\alpha)\, p^\star q_i. \tag{2}$$

When $\alpha = 0$, the mixed-price auction is an unmixed uniform-price auction, and when $\alpha = 1$, the mixed-price auction is an unmixed discriminatory auction. Where useful, I refer to mixed-price auctions with $\alpha > 0$ as partially discriminatory.

When bids are continuous (see footnote 18), if bidder i submits bid b^i and obtains quantity q_i the market-clearing price is $p^\star = b^i(q_i)$. Substituting (2) into (1), the bidder's utility from obtaining quantity q at market price $p^\star = b^i(q_i)$ is

$$u\left(q_i; b^i, s\right) = \int_0^{q_i} v(x; s)\, dx - \left[\alpha \int_0^{q_i} b^i(x)\, dx + (1-\alpha)\, q_i b^i(q_i)\right] dx.$$

I constrain attention to pure-strategy Nash equilibria, in which the equilibrium bid profile $(b^i)_{i=1}^n$ satisfies, for each bidder i and all common signals s,

$$b^i(\cdot; s) \in \operatorname*{argmax}_b \mathbb{E}_{q_i}\left[u\left(q_i; b, s\right)\right].$$

2.1 Equilibrium

Because bidders observe a common signal, equilibrium bids can be analyzed signal-by-signal. To economize on notation I therefore drop the conditioning on the bidders' information (e.g., $v \equiv v(\cdot; s)$).

Bidder i's objective function is

$$\max_b \mathbb{E}_{q_i}\left[\int_0^{q_i} v(x) - \alpha b(x)\, dx - (1-\alpha)\, q_i b(q_i) \middle| b\right].$$

This optimization problem is simplified by integrating by parts (Février et al., 2002; Hortaçsu, 2002; Pycia and Woodward, 2021). Letting G^i be the equilibrium distribution of bidder i's allocation conditional on her bid b, $G^i(q; b) = \Pr(q_i \leq q | b)$, her objective function is

$$\max_b \int_0^{\overline{Q}} \left(v(q) - \alpha b(q) - (1-\alpha)\left(b(q) + q b_q(q)\right)\right)\left(1 - G^i(q; b)\right) dq. \tag{3}$$

The incentives corresponding to the bid for quantity q follow from application of the calculus of variations to the maximization problem in (3).

Lemma 1 (Convexified incentives)**.** *Bidding incentives in the mixed-price auction are the weighted* *sum of incentives in the uniform-price and discriminatory auctions,*

$$\underbrace{\left(\left(v\left(q\right)-b^{i}\left(q\right)\right)+\left(\frac{1-G^{i}\left(q;b^{i}\right)}{\left(n-1\right)G_{q}^{i}\left(q;b\right)}\right)b_{q}^{i}\left(q\right)\right)}_{\text{discriminatory incentives}}\alpha+\underbrace{\left(\left(v\left(q\right)-b^{i}\left(q\right)\right)+\left(\frac{q}{n-1}\right)b_{q}^{i}\left(q\right)\right)}_{\text{uniform price incentives}}\left(1-\alpha\right)=0.$$

Mixed-price auctions not only specify transfers which are convex combinations of discriminatory and uniform-price transfers, but also induce incentives which are convex combinations of discriminatory and uniform-price incentives. Convex incentives do not imply that equilibrium bids are convex combinations of discriminatory and uniform-price bids, and I show in Theorem 2 that equilibrium bids have a nontrivial dependence on the extent of price discrimination.

In equilibrium, each bidder is best responding to the distribution of residual supply generated by her opponents' bidding strategies, taking into account her own market power. This implies that solving for equilibrium bid profiles amounts to solving an n-dimensional differential system. This manifests in Lemma 1, where the bidder's first order conditions depend on her own bid, through b^i and b^i_q, and on her opponents' bids, through G^i. The first step toward obtaining a closed-form expression for equilibrium bids is to show that all equilibria are symmetric, reducing the differential system to a single dimension.

Theorem 1 (Equilibrium symmetry)**.** *All pure-strategy equilibria are symmetric.*

Equilibrium symmetry must be analyzed separately in the case of the unmixed uniform-price auction ($\alpha = 0$), and in all partially-discriminatory auctions ($\alpha > 0$). The analysis of the uniform-price auction is substantially similar to the analysis of supply function equilibrium in Klemperer and Meyer (1989). This case must be analyzed separately because bidding incentives are scaled by the bid-for quantity, and the fundamental theorem of differential equations cannot be applied at $q = 0$. However, because bids must equal marginal values when $q = 0$, standard differential arguments are sufficient to show equilibrium symmetry.

In any partially-discriminatory auction, showing symmetry begins by noting that in any equilibrium, at least two bidders submit the same maximum bid $\overline{b} \equiv b^i(0)$, otherwise at least one of these bidders can reduce her bid without affecting her allocation. Because the optimization problem faced by bidders reduces to solving a differential system, any two bidders with identical maximum bids must be submitting the same bid function. Then if equilibrium is asymmetric, there are at least two distinct maximum bids. Since bidding incentives are smooth in quantity, "high" bidders' bid curves must have a kink at "low" bidders' maximum bid. Then at her maximum bid, the low bidder perceives a discontinuous increase in the elasticity of residual supply, and she can improve her expected utility by slightly increasing her bid at this point; it follows that she is not best responding.

Remark 1. *Equilibrium symmetry relies on the maintained assumption of full support of supply,* $\operatorname{Supp} F = [0, \overline{Q}]$. *By contrast, asymmetric equilibria are known to exist in all partially-uniform* *auctions ($\alpha < 1$) with deterministic supply. For a standard "crank-handle" construction of asymmetric equilibria, see, e.g., Milgrom (2004), Burkett and Woodward (2020a), and Marszalec et al. (2020), among others.*

Equilibrium symmetry implies that upon realization of aggregate supply Q, each bidder recieves the same quantity $q = Q/n$. Then because bidders have identical marginal values, equilibrium outcomes are efficient, conditional on the distribution of supply. In all equilibria the market price is determined by the (symmetric) bid for per-capita quantity, $p(Q) = b(Q/n)$, and any statement about bids has a simple translation in terms of prices, and vice-versa. Additionally, the probability that a bidder receives less than quantity q, $G^i(q;b)$, is identified with the probability that the market quantity is at least nq, $G^i(q;b) = F(nq)$. Then the first-order conditions in Lemma 1 can be rewritten as

$$(v(q) - b(q)) + \left(\frac{1 - F(nq)}{(n-1) f(nq)}\right) \alpha b_q(q) + \left(\frac{q}{n-1}\right)(1-\alpha) b_q(q) = 0. \tag{4}$$

Solving the differential equation (4) leads to the equilibrium representation in Theorem 2.

Theorem 2 (Equilibrium prices and bids)**.** *In any pure-strategy equilibrium, there is a constant $C \geq 0$ such that market-clearing prices are*

$$p(Q) = \hat{v}(Q) + \int_Q^{\overline{Q}} \exp\left(-\int_Q^x \tilde{H}(y)\,dy\right) \hat{v}_Q(x)\,dx - C \exp\left(-\int_Q^{\overline{Q}} \tilde{H}(x)\,dx\right), \tag{5}$$
$$\tilde{H}(z) = \frac{(n-1) f(z)}{n\alpha(1 - F(z)) + (1-\alpha) z f(z)}.$$

Equilibrium bids are

$$b(q) = v(q) + \int_q^{Q^\mu} \exp\left(-\int_q^x \tilde{H}^\mu(y)\,dy\right) v_q(x)\,dx - C \exp\left(-\int_q^{Q^\mu} \tilde{H}^\mu(x)\,dx\right), \tag{6}$$
$$\tilde{H}^\mu(z) = \frac{(n-1) f^\mu(z)}{n\alpha(1 - F^\mu(z)) + (1-\alpha) z f^\mu(z)}.$$

The trailing exponential term in equilibrium bids and prices arises from the homogeneous solution to the differential equation implied by (4), and the multiplier C appears in other analyses of divisible-good auctions. In Wang and Zender (2002), C corresponds to the degree of competition in the uniform-price auction; Wang and Zender (2002) and Pycia and Woodward (2021) separately show that it is uniquely determined ($C = 0$) in the discriminatory auction. Note that the trailing term depends only on the distribution of quantity, and not on marginal values. Because equilibrium nonuniqueness is related to freedom in the multiplier C, one interpretation is that nonuniqueness is a fundamental feature of non-discriminatory auctions, independent of marginal values.

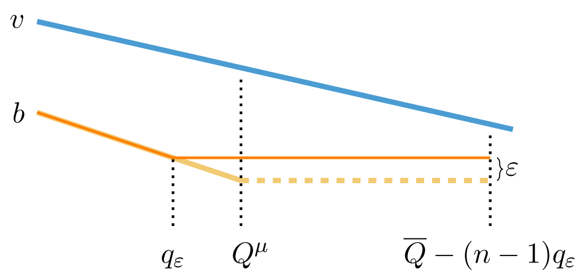

Figure 1: The deviation used to establish the necessary condition for a particular bid function to be sustainable in equilibrium increases the maximum obtainable quantity, and cannot be analyzed by the calculus of variations. This deviation cannot be profitable, establishing the upper bound on $v(Q^\mu) - b(Q^\mu)$ given in Lemma 2.

3 Equilibrium properties

In this section I use the equilibrium expressions developed in Section 2 to derive qualitative features of equilibrium behavior. First, I derive comparative statics with regard to the extent of price discrimination α. I show that equilibrium is "increasingly unique" as the mixed-price auction becomes discriminatory. As a consequence, more equilibrium prices are feasible in less-discriminatory auctions. This effect is particularly stark in auctions with increasingly concentrated supply, where the set of equilibrium prices shrinks to a point in all partially-discriminatory auctions, but not in the uniform-price auction. Finally, I show that when the number of bidders is large, incentives in the mixed-price auction are mixture-scaled versions of discriminatory auction incentives. By corollary, in large markets all mixed-price auctions are revenue-equivalent, irrespective of the distribution of supply, and in auctions with concentrated supply all partially-discriminatory auctions are approximately revenue equivalent, irrespective of the number of bidders.[20]

3.1 Equilibrium multiplicity

Equilibrium uniqueness in discriminatory auctions has been established by Wang and Zender (2002) and Pycia and Woodward (2021). By contrast, Klemperer and Meyer (1989) and Back and Zender (1993) show that the uniform-price auction can sustain a broad multiplicity of equilibria. Because equilibrium bid functions are the solution to a differential equation, equilibrium nonuniqueness is closely related to the set of economically feasible initial conditions I now give a natural constraint on initial conditions which is easier to satisfy the lower is the extent of price discrimination. It follows that uniform-price auctions may admit more equilibria than mixed-price auctions, which may admit more equilibria than discriminatory auctions.

The feasibility condition is derived from the analysis of small upward deviations in bid near the maximum per-capita quantity Q^μ (Figure 1). In particular, a bidder could impose a floor on her own bid, adjusting it so it is never less than $b(Q^\mu) + \varepsilon$.[21] Doing so will increase her allocation

[20] For related results with indivisible goods, see Swinkels (2001) and Jackson and Kremer (2006).

[21] The model in the main text specifies that bid functions are strictly decreasing, for ease of exposition. In the

when the realization of Q is large, and will also increase her payment. This deviation cannot be profitable in equilibrium, implying a bound on the distance between bid and marginal value. This places a natural limit on the set of permissible equilibrium market-clearing price functions.

Lemma 2 (Equilibrium necessary condition)**.** *A necessary condition for a market price function p to represent an equilibrium is*

$$\frac{1-2\alpha}{1-\alpha}\left(p\left(\overline{Q}\right)-\hat{v}\left(\overline{Q}\right)\right)+\overline{Q}\hat{v}_Q\left(\overline{Q}\right)\leq 0. \tag{7}$$

Remark 2. *The endpoint condition considered in Lemma 2 is one of many possible conditions for equilibrium existence, but upward deviations near $q = Q^\mu$ are particularly focal. Such deviations provide additional analytical power beyond the established first-order conditions, because optimization using the calculus of variations presumes that endpoint conditions are fixed, where in divisible-good auctions they are flexible and must be derived from the theory. A related condition, evaluated at $q = 0$, is established in Section 3.2 below.*

Because market prices are strictly decreasing in quantity Q, the necessary condition for the price at the maximum quantity, $p(\overline{Q})$, can be framed as the minimum price feasible in equilibrium.

Corollary 1 (Equilibrium range of minimum prices)**.** *In equilibrium, when the auction mixture is sufficiently discriminatory ($\alpha > 1/2$), the minimum market clearing price must satisfy*

$$p\left(\overline{Q}\right)\in\left[\hat{v}\left(\overline{Q}\right)-\frac{1-\alpha}{1-2\alpha}\overline{Q}\hat{v}_Q\left(\overline{Q}\right),\hat{v}\left(\overline{Q}\right)\right]_+.$$

When the price mixture is sufficiently non-discriminatory ($\alpha \leq 1/2$), the minimum market clearing price must satisfy $p(\overline{Q}) \in [0, \hat{v}(\overline{Q})]$.

Lemma 2 places a weak restriction on the equilibrium price function. Satisfaction of condition (7) does not guarantee that a particular solution is an equilibrium, only that the response cannot be locally improved-upon from deviations near Q^μ. Note that, in the discriminatory auction ($\alpha = 1$), Lemma 2 implies $p(\overline{Q}) = \hat{v}(\overline{Q})$, as identified in Pycia and Woodward (2021). Lemma 2 is sufficient to derive monotonicity of the set of feasible initial conditions, and therefore on the set of equilibrium bid functions.

Lemma 3 (α-monotonicity of necessary condition)**.** *Suppose that p is a solution to the equilibrium market-price equation for mixing term α, and satisfies the necessary condition of Lemma 2. Then for any $\alpha' < \alpha$, there is a solution to the equilibrium market-price equation p' with $p'(\overline{Q}) = p(\overline{Q})$ which satisfies the necessary condition of Lemma 2.*

Lemma 3 follows from the observation that, holding fixed $p(\overline{Q}) < \hat{v}(\overline{Q})$, the left-hand side of inequality (7) is monotonically increasing in α. Then a large enough increase in price discrimination

formal treatment in the appendix, I allow for locally-constant and discontinuous bids. The constructed deviations are therefore consistent with the formal analysis in the appendix.

α will move the left-hand side from negative to positive, violating inequality (7). Given auction mixture α, let $\mathrm{P}(\alpha)$ be the set of endpoints to valid solutions to the market-clearing equation, so that price is everywhere-positive, its implied bids are below the agents' value functions, and satisfies the inequality of Lemma 2.[22] This construction provides the following result.

Corollary 2 (Decreasing multiplicity). *Let* $\mathrm{P}(\alpha)$ *be the set of endpoints to valid solutions to the market-clearing equation, so that (i) price is weakly positive, (ii) implied bids are below agents' marginal values, and (iii) inequality* (7) *is satisfied. Then* P *is ordered by reverse inclusion:* $\alpha' < \alpha$ *implies* $\mathrm{P}(\alpha') \supseteq \mathrm{P}(\alpha)$.

The set of permissible market price functions, measured by endpoint conditions, is increasing as α falls, in the sense that the set of available endpoint prices is growing. Inclusion-monotonicity of P with respect to α suggests that the multiplicity of equilibria in the uniform-price auction and the uniqueness of equilibrium in the discriminatory auction are not knife-edge cases obtained by mechanism degeneracy, but are part of a continuum of feasible minimum market prices. Indeed, the set of permissible minimum market prices shrinks smoothly as the mechanism moves from uniform-price to discriminatory.

Perhaps surprisingly, the equilibrium validity constraint (7) does not depend on the number of bidders n: while equilibrium bids depend on the number of bidders (holding fixed F^{μ}), the set of feasible endpoint conditions does not. Fixing an initial condition $p(\overline{Q})$, the market-clearing price equation (4) implies that $p_Q(\overline{Q})$ grows without bound as n increases. As the number of bidders grows large, bids with nontrivial homogeneous terms become inelastic at Q^{μ}, and the incentive to deviate upward is lessened. In equilibrium, falling per-bidder elasticity exactly offsets the increase in the number of bidders. I explore this effect further in Section 3.4 below.

3.2 Concentrated supply

In practice, supply distributions are frequently concentrated around their upper bound, and the probability that realized supply is far below the upper bound is quite small. I model supply concentration as the existence of $\varepsilon > 0$ such that $\Pr(Q \leq \overline{Q} - \varepsilon) < \varepsilon$; when ε is close to zero, nearly all probability is placed on supply realizations close to $\overline{Q}$.

The calculus of variations gives pointwise optimality conditions for equilibrium bids which in turn yield the closed-form expressions given in Theorem 2. As noted earlier, the calculus of variations assumes endpoint conditions of the problem analyzed, while in the mixed-price auction context the endpoint conditions must be consistent with economic incentives. The analysis of uniqueness in Lemma 2 considered a deviation in which a bidder extends their bid via a flat interval just above the minimum price. I now consider a deviation in which the bidder submits a flat bid at the maximum price, depicted in Figure 2; this deviation affects the endpoint condition of the calculus of variations and therefore is not accounted for by the analysis.

[22] Prices are weakly positive since bids are weakly positive. Lemma 7 in Appendix A establishes that equilibrium bids must be below marginal values.

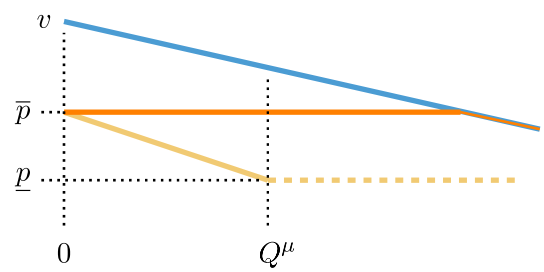

Figure 2: The equilibrium maximum price $\overline{p}$ cannot be too far above the equilibrium minimum price $\underline{p}$, or a bidder will have an incentive to bid $\overline{p}$ for all units.

Lemma 4 (Equilibrium necessary condition). *Let $\overline{p}$ and $\underline{p}$ be the maximum and minimum (respectively) market-clearing prices arising from a symmetric strategy profile $(b^i)_{i=1}^n$. If $(b^i)_{i=1}^n$ is an equilibrium strategy profile, then*

$$\frac{1}{n}\int_0^{\overline{Q}} \left(\hat{v}\left(Q\right) - \underline{p}\right)\left(1 - F\left(Q\right)\right) dQ \geq \int_0^{v^{-1}(\overline{p})} \left(v\left(Q\right) - \overline{p}\right)\left(1 - F\left(Q\right)\right) dQ. \tag{8}$$

By submitting a bid equal to the maximum market-clearing price, bounded above by value, the bidder ensures herself the entire aggregate quantity when its realization is small. For this deviation to not be profitable, it must be that the stochastic improvement in allocation is dominated by the cost of increasing the submitted bid. The bound in Lemma 4 considers a weak upper bound on the utility of non-deviation, and does not depend on the extent of price discrimination α.

Supply concentration implies that bids are relatively flat in any equilibrium of a partially-discriminatory auction. As supply becomes concentrated, buyers become increasingly certain that they will receive allocations near the per-capita maximum supply Q^{μ}. Since bids for lower quantities are partially paid in any mixed-price auction with $\alpha > 0$, and these bids are never marginal, buyers have a strong incentive to reduce their bids for small quantities. In the limit, bids will be flat.[23]

Lemma 5 (Flat bids). *For any mixture $\alpha > 0$, and any $\delta > 0$, there is $\varepsilon > 0$ such that if supply is ε-concentrated, then in all equilibria of the α-mixed price auction the difference between the maximum and minimum market clearing prices is less than δ.*

As supply becomes concentrated and equilibrium bids flatten, the cost of the deviation analyzed in Lemma 4 becomes arbitrarily small, while the benefits of the deviation increase. Then for all non-uniform pricing rules, the necessary condition (8) implies that as supply becomes concentrated at $\overline{Q}$, the set of feasible equilibrium minimum bids converges to the minimum per capita marginal value.

Proposition 1 (Concentrated supply). *Let $\mathrm{P}(\alpha;\varepsilon)$ be the set of endpoints of valid solutions to the market-clearing equation when supply is ε-concentrated, so that (i) price is weakly positive, (ii)*

[23]For discriminatory auctions, the bid-flattening effect of supply concentration has been observed by Pycia and Woodward (2021). Lemma 5 generalizes their result not only by considering mixed-price auctions, but also by relating the flattening of bids to the lowest possible market-clearing price.

implied bids are below agents' marginal values, and (iii) inequality (8) *is satisfied. For all* $\alpha > 0$, $\liminf_{\varepsilon \searrow 0} \mathrm{P}(\alpha; \varepsilon) = v^{-1}(Q^{\mu})$.

Since equilibrium bids flatten as supply becomes concentrated, residual supply becomes highly elastic and each bidder has a strong incentive to just outbid her opponents. This ensures that with high probability she wins all units for which her marginal value is strictly above the maximum market-clearing price; such a deviation is unprofitable only when her minimum bid is sufficiently close to her marginal value for her maximum quantity.

Furthermore, the flattening of bids and convergence of minimum prices to aggregate marginal value for quantity $\overline{Q}$ together imply that equilibrium revenue is converging in supply concentration. I say that two auction formats are *asymptotically revenue equivalent* if, as aggregate supply becomes concentrated at $\overline{Q}$, expected revenues under the two formats converge to the same point.[24]

Corollary 3 (Revenue equivalence under concentrated supply)**.** *All partially-discriminatory auctions are asymptotically revenue equivalent, and asymptotic equilibrium revenue as supply becomes concentrated at* $\overline{Q}$ *equals equilibrium revenue under deterministic supply* $\overline{Q}$*. Moreover, this asymptotic revenue is weakly higher than the expected revenue in any equilibrium of the uniform-price auction with deterministic supply.*

Not only is equilibrium revenue converging in all non-uniform auctions, its limit is identical to the revenue obtained in an auction with deterministic supply. In any pure-strategy equilibrium under deterministic supply, every bidder knows exactly the quantity they will receive. Since the auction is partially discriminatory, it is optimal to submit a bid which is flat from quantity zero up to the allocated quantity. Then, as when aggregate supply is highly concentrated, residual supply is highly elastic and equilibrium bids must intersect marginal values at the allocated quantity.

On the other hand, equilibrium in uniform-price auctions is ex post, and neither supply concentration nor deterministic supply have any effect on the profitability of the deviation analyzed in the proof of Proposition 1.[25] Since in practice supply is frequently concentrated, a natural interpretation is that the introduction of slight price discrimination can rule out seller-suboptimal equilibria with extreme underpricing.

3.3 Observable prices

Given an equilibrium market-clearing price curve, the expression for equilibrium prices obtained in Theorem 2 implies that the minimum market-clearing price is $p(\overline{Q}) = \hat{v}(\overline{Q}) - C$. Because bids are bounded above by marginal values and below by the expression in Lemma 2, we can immediately bound the set of market-clearing prices which may arise in equilibrium.

[24]Formally, it is necessary to consider equilibrium expected revenues for any sequence of equilibria corresponding to increasingly concentrated supply. However, bids are below values for all realized quantities, and Proposition 1 implies that bids cannot be too far below the marginal value for per capita maximum supply Q^{μ}. Then equilibrium expected revenue under any sequence of equilibria will converge to a limit independent of the equilibrium sequence.

[25]As noted in Pycia and Woodward (2021), the uniform-price equilibrium set grows as the support of aggregate quantity shrinks.

Theorem 3 (Extreme-price equilibria)**.** *Let $p_{\min} = \min \mathrm{P}(\alpha)$. In any equilibrium of the mixed-price auction, for any quantity Q the market-clearing price $p^\star(Q)$ is between $\underline{p}(Q)$ and $\overline{p}(Q)$, where*

$$\overline{p}(Q) = \hat{v}(Q) + \int_Q^{\overline{Q}} \exp\left(-\int_Q^x \tilde{H}(y)\,dy\right) \hat{v}_Q(x)\,dx,$$
$$\underline{p}(Q) = \hat{v}(Q) + \int_Q^{\overline{Q}} \exp\left(-\int_Q^x \tilde{H}(y)\,dy\right) \hat{v}_Q(x)\,dx - \left(\hat{v}\left(\overline{Q}\right) - p_{\min}\right)\exp\left(-\int_Q^{\overline{Q}} \tilde{H}(x)\,dx\right).$$

Where Corollary 2 shows that the set of feasible minimum prices $p(\overline{Q})$ is shrinking in α, the bounds in Theorem 3 imply that the range of feasible prices for any quantity Q, measured by $\overline{p}(Q) - \underline{p}(Q)$, is increasing in Q. That is, for a fixed auction format the dispersion of prices due to equilibrium selection is increasing in quantity. To observe this, let $R^n(Q) = \overline{p}(Q) - \underline{p}(Q)$ be the range of feasible equilibrium prices for quantity Q when there are n bidders. Although the endpoint condition identified in Lemma 2 does not depend on the number of bidders n, the range of feasible equilibrium prices for any particular quantity Q shrinks as the market becomes large.

Proposition 2 (Feasible price range)**.** *The range of feasible equilibrium prices, $R^n(Q)$, is weakly increasing in Q and weakly decreasing in n.*

Proposition 2 relates equilibrium uniqueness to the range of observable market prices, and is illustrated in Figure 3. When the market quantity Q is small, the range of feasible market prices will be small; when the market quantity Q is large, the range of feasible market prices will be (comparatively) large. This is particularly apparent at the extreme, unmixed auction implementations. In a discriminatory auction, there is a unique feasible initial condition $p(\overline{Q})$; Proposition 2 then implies that there is a unique equilibrium. At the other extreme, the set of feasible initial conditions P is maximized in a uniform-price auction (Corollary 2), but in a uniform-price auction the zero-quantity price is uniquely determined, $p(0) = v(0)$.

It is straightforward to see that there is no analogue of Proposition 2 with regard to changes in α: when $\alpha = 1$, there is a unique equilibrium, and when $\alpha = 0$ there is a unique price for quantity $Q = 0$. Because strictly mixed-price auctions, $\alpha \in (0, 1)$, admit a range of prices for quantity $Q = 0$, it follows that the range of equilibrium prices is nonmonotone in α. This is visible in Figure 3, where for Q small the range of feasible prices in the mixed-price auction strictly exceeds the range of feasible prices in discriminatory and uniform-price auctions. It is, however, possible to derive comparative statics on the whole set of observable prices. Let $\mathbf{P}(\alpha; n)$ be the set of equilibrium market-clearing prices in mixture α when there are n bidders,

$$\mathbf{P}(\alpha; n) \equiv \left\{p(Q) : Q \in \left[0, \overline{Q}\right],\ p \text{ solves (5), and } p\left(\overline{Q}\right) \in \mathrm{P}(\alpha)\right\}.$$

The set of equilibrium market-clearing prices is shrinking in α.

Theorem 4 (Feasible price space)**.** *Let $\alpha \le \alpha'$ and $n \le n'$. Then $\mathbf{P}(\alpha; n) \subseteq \mathbf{P}(\alpha'; n)$ and $\mathbf{P}(\alpha; n) \subseteq \mathbf{P}(\alpha; n')$.*

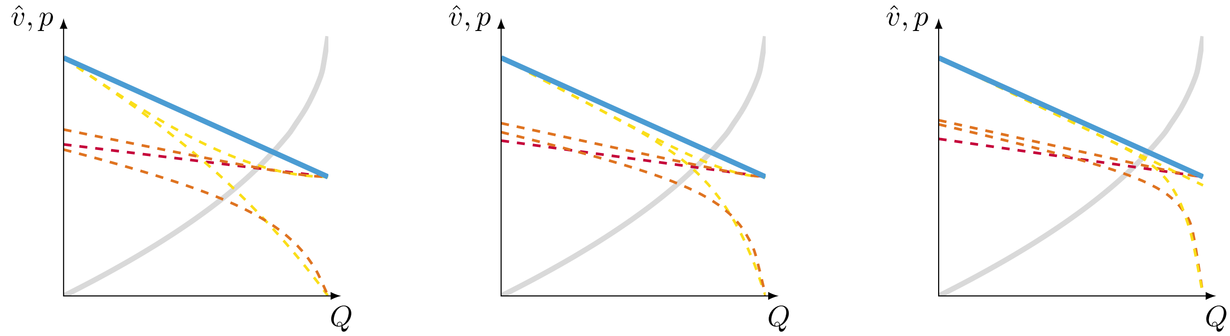

Figure 3: Upper and lower bounds (from Theorem 3) on prices with linear marginal values for $n \in \{4, 8, 16\}$ (left to right). Uniform-price prices are in yellow, discriminatory prices are in red, and mixture prices ($\alpha = 0.5$) are in orange.

Theorem 4 suggests that sellers' outcomes are less certain in uniform-price auctions than in discriminatory auctions. The range of feasible equilibrium prices is decreasing in the extent of price discrimination α, and therefore there is a wider range of observed transfers in uniform-price auctions than in discriminatory auctions. This occurs both because bids are less elastic in uniform-price auctions, and because there is a broader range of equilibria.

While the range of feasible prices for the given quantity is decreasing in the number of bidders n, the space of feasible prices overall is increasing in n. Per-quantity price uniqueness increases with competition, but so too does the slope of equilibrium bids. Since the endpoint condition identified in Lemma 2 is independent of the number of bidders (holding fixed F^μ), it follows that the range of initial prices $p(0)$ is increasing in n, and therefore so is the space of feasible prices.

The results of this and the preceding subsections offer three perspectives on nonuniqueness. First, the set of feasible equilibrium initial conditions—and therefore the size of the equilibrium set—is decreasing in both price discrimination and supply concentration. Second, the range of feasible prices is highest for large quantities. Since equilibrium bids for the maximum per capita quantity are less elastic the less discriminatory is the auction, when the distribution of supply is skewed towards large quantities the set of observed prices will be larger in less-discriminatory auctions. Third, the entire set of feasible equilibrium prices is shrinking in price discrimination. In total, I observe that equilibrium outcomes are more certain when the mixed-price auction is more discriminatory, or when aggregate supply is more certain.

3.4 Large markets

I now consider the case of an auction with a large number of bidders. If quantity is held constant while the number of participants increases, in the limit no agent can receive a strictly positive quantity. Then the unique equilibrium prediction is truthful reporting at $q = 0$, independent of the auction implemented. It is natural then to consider the large-market limit as per-capita distribution of quantity, F^μ, is held constant. In this limit, I show that equilibrium incentives are simply scaled versions of discriminatory auction incentives, regardless of the auction implemented.

When the number of bidders is large, uniform-price demand-reduction incentives (proportional to $q/(n-1)$) go to zero, and discriminatory auction demand-reduction incentives dominate. With a large number of bidders and elastic demand, a small deviation in bid will have a dramatic effect on resulting allocations. Since, in a uniform-price auction, the bid for a particular quantity affects payment only for this quantity, bids must equal marginal values. In a discriminatory auction it remains true that small deviations will have dramatic effects on allocations, but the bid for quantity q is paid whenever $Q > nq$. The-first order conditions given in Lemma 1 relate the margin $v(q)-b(q)$ to $b_q(q)$, the slope of the bid function at q. In the large-market limit, there is no incentive to bid a positive margin in a uniform-price auction, while such an incentive remains in a discriminatory auction, and the resulting first order conditions take the form of a scaled discriminatory auction.

Lemma 6 (Scaled discriminatory incentives in large markets)**.** *In the large-market limit, equilibrium first-order conditions are given by*

$$-\left(v\left(q\right)-b\left(q\right)\right)=\alpha\frac{1-F^{\mu}\left(q\right)}{f^{\mu}\left(q\right)}b_{q}\left(q\right). \tag{9}$$

There is a unique solution to the differential equation (9), thus Proposition 2 may be interpreted as indicating a smooth transition from nonuniqueness to uniqueness as markets become large.

Theorem 5 (Equilibrium in large markets)**.** *In the large-market limit, equilibrium bids are given by*

$$b\left(q\right)=\int_{q}^{Q^{\mu}}v\left(x\right)dF^{\alpha,q}\left(x\right),\quad F^{\alpha,q}\left(x\right)=1-\left(\frac{1-F^{\mu}\left(x\right)}{1-F^{\mu}\left(q\right)}\right)^{\frac{1}{\alpha}}.$$

This equilibrium is unique.

Price discrimination, supply concentration, and market size have similar effects on equilibrium uniqueness: price discrimination and supply concentration reduce the set of initial conditions $p(\overline{Q})$, while market size reduces the set of feasible elasticities, conditional on $p(Q) \leq \hat{v}(Q)$. In either case, equilibrium uniqueness obtains in the limit, either $\alpha = 1$ or $n \nearrow \infty$. When $\alpha < 1$, large-market equilibrium nonuniqueness arises from the homogeneous solution to the market clearing equation. The homogeneous term is derived from uniform-price incentives. Then as the market grows large and uniform-price incentives are diminished (Lemma 6), equilibrium becomes unique.

In relation to Section 3.3's results on price dispersion, the space of feasible prices is increasing in n but as n increases low prices become less and less likely. Low prices arise from equilibrium selection, and low-price equilibria have low elasticities when prices are low. In the limit, $n \nearrow \infty$, these low-probability prices become zero-probability prices, and the feasible price space discontinuously shrinks. Nonetheless, for any fixed quantity $Q < \overline{Q}$, the range of feasible prices (hence, for any $q < Q^{\mu}$, the range of feasible bids) is smoothly decreasing in the number of bidders.

The equilibrium bid representation in Theorem 5 demonstrates a number of interesting features. First, bids are truthful for the maximum per-capita allocation Q^{μ}, $b(Q^{\mu}) = v(Q^{\mu})$. Lemma 7 in Appendix A shows that bids must be below values, and thus no bid function $b' > b$ is sustainable in equilibrium. As mentioned above, uniqueness follows technically from showing that the homogeneous solution to the first-order conditions diverges as $n \nearrow \infty$.

Second, bids are truthful in the uniform-price auction, $\alpha = 0$.[26] In all cases, the bid for quantity q is the expected value of marginal values for larger quantities, taken with respect to a reweighted distribution $F^{\alpha,q}$; when $\alpha = 0$, this distribution is degenerate at q. With a large number of opponents, bidder i can affect her allocation but not the market-clearing price. If she is bidding truthfully she can obtain a larger quantity only at a negative margin, and if she obtains a smaller quantity she loses positive marginal gains. Neither is utility-improving and thus truthful reporting emerges.

Third, for all $q < Q^\mu$, bids are strictly decreasing in α,

$$\frac{d}{d\alpha} b(q) = \int_q^{Q^\mu} \underbrace{v_q(q)}_{<0} \underbrace{\left(-\frac{1}{\alpha^2}\right)}_{<0} \underbrace{\left(\frac{1 - F^\mu(x)}{1 - F^\mu(q)}\right)^{\frac{1}{\alpha}}}_{\geq 0} \underbrace{\ln \frac{1 - F^\mu(x)}{1 - F^\mu(q)}}_{\leq 0} dx.$$

This contrasts the small-market case, where equilibrium bids for different extents of price discrimination may cross. Since bids in the large-market uniform-price auction are truthful, it follows that the more a mixed-price auction resembles a uniform-price auction, the more truthful are its equilibrium bids. Figure 5 illustrates all three effects.

The closed-form expression for equilibrium bids in a large market makes the following result immediate.

Corollary 4 (Large market revenue equivalence)**.** *In large markets, equilibrium per-capita revenue is independent of the extent of price discrimination α.*

Revenue equivalence is a standard property of single-unit auctions, but does not generalize to the multi-unit context.[27] Swinkels (2001) shows that in large multi-unit auctions, equilibrium revenue approaches the expected maginal value in an efficient allocation, and Pycia and Woodward (2021) show that divisible-good auctions are revenue-equivalent when the seller has discretion over the distribution of supply. Although mixed-price auctions in large markets generate identical expected revenue, independent of the extent of price discrimination, in markets with a finite number of bidders expected revenue will depend on the extent of price discrimination α. I explore this dependence in Section 4 below.

4 Conjugate equilibrium

Analysis of the effect of partial discrimination on equilibrium revenue is hampered by the complex dependence of market-clearing prices on the extent of price discrimination α. In this section

[26] Swinkels (2001) obtains large-market truthfulness in multi-unit auctions.

[27] Engelbrecht-Wiggans (1988) shows that the "efficient allocation" condition for revenue equivalence identified by Myerson (1981) and Riley and Samuelson (1981) generalizes to the multi-unit context. However, this condition is not generally satisfied by standard multi-unit auction formats (Ausubel et al., 2014).

I conduct an explicit analysis of equilibrium in the *polynomial-Lomax model*, a generalization of the linear-Lomax model applied elsewhere (cf. Ausubel et al. (2014)). This model has favorable tractability properties, can approximate general marginal values, and allows for quantity distributions which qualitatively match the concentrated distributions observed in practice (Holmberg, 2009).

In the polynomial-Lomax model, marginal values are piecewise polynomial in quantity and the distribution of quantity is negative Lomax with parameter $\lambda > 0$. Given coefficients $\mathbf{v} = (v_k)_{k=0}^{\bar{k}}$, let $\bar{q}_v$ be the smallest positive root of the polynomial $\sum_{k=0}^{\bar{k}} v_k q^k$. Then marginal values and the distribution of quantity are given by

$$v(q) = \begin{cases} \sum_{k=0}^{\bar{k}} v_k q^k & \text{if } q \leq \bar{q}_v, \\ 0 & \text{otherwise;} \end{cases} \qquad F(Q) = 1 - \left(\frac{\overline{Q} - Q}{\overline{Q}} \right)^{\lambda}.$$

I place no restriction on the order of the polynomial $\bar{k}$, so polynomial marginal values are fairly general: the only constraint is that v must be decreasing in quantity.

For initial equilibrium analysis I consider *conjugate equilibria*, a natural class of equilibria in which bids have the same functional form as marginal values. I show later that, in general, equilibria in the polynomial-Lomax model are translations of conjugate equilibria.

Definition 1. *Let marginal values be as in the polynomial-Lomax model with coefficients $(v_k)_{k=0}^{\bar{k}}$. The bid function b is a* conjugate bid function *if there are b_k, $k \in \{0, \ldots, \bar{k}\}$, such that*

$$b(q) = \begin{cases} \sum_{k=0}^{\bar{k}} b_k q^k & \textit{if } q \leq \bar{q}_b, \\ 0 & \textit{otherwise,} \end{cases}$$

where $\bar{q}_b$ is the smallest positive root of $\sum_{k=0}^{\bar{k}} b_k q^k$. The equilibrium bid profile $(b^i)_{i=1}^n$ is a conjugate equilibrium *if each b^i is a conjugate bid function.*

For example, in a model with linear marginal values, equilibrium bids are conjugate if they are linear in quantity. As I show in Theorem 7, when the distribution of supply is Lomax and conjugate price coefficients are well-defined, all equilibria are in conjugate bids plus a potentially degenerate homogeneous term.

Theorem 6 (Conjugate prices and bids)**.** *Conjugate price coefficients in the polynomial-Lomax model are given by*

$$\check{p}_k = \frac{(n-1)\lambda \hat{v}_k + (k+1) n \alpha \overline{Q} \check{p}_{k+1}}{(n-1)\lambda - ((1-\alpha)\lambda - n\alpha) k}, \ \textit{where } \check{p}_{\bar{k}+1} = 0.$$

Conjugate bid coefficients are $\check{b}_k = n^k \check{p}_k$.

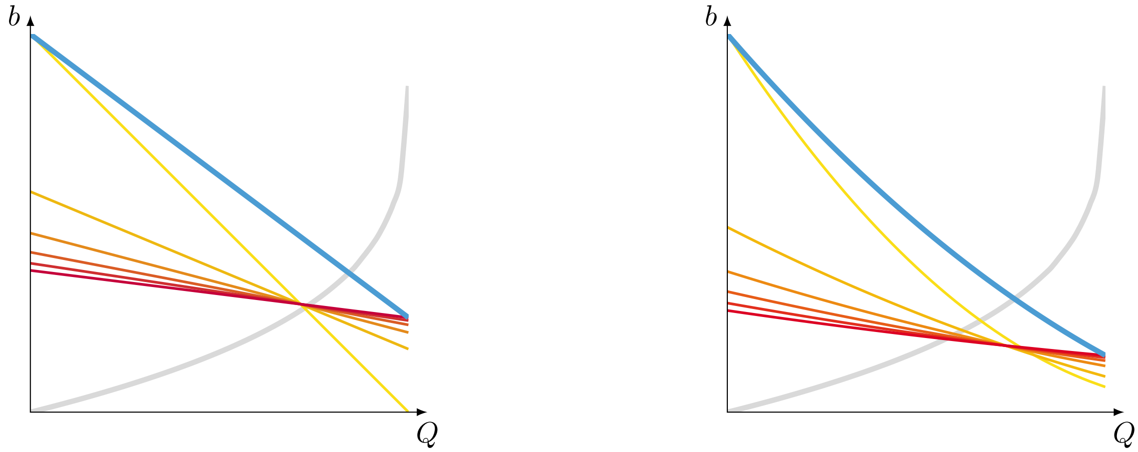

Figure 4: Conjugate bid functions when $\bar{k} = 1$ (left panel) and $\bar{k} = 2$ (right panel). Marginal values are in blue and the CDF of aggregate supply is in gray. Bids in the unmixed discriminatory auction are in red and bids in the unmixed uniform-price auction are in yellow. Bids in strictly mixed-price auctions, $\alpha \in \{0.2, 0.4, 0.6, 0.8\}$, are colored according to the extent of price discrimination.[29]

When marginal values are linear, $v(q) = v_0 + v_1 q$, conjugate equilibrium bid coefficients are

$$\check{b}_0 = v_0 + \frac{\alpha \overline{Q} v_1}{(n-2)\lambda + (n+\lambda)\alpha}, \quad \check{b}_1 = \frac{(n-1)\lambda v_1}{(n-2)\lambda + (n+\lambda)\alpha}$$
$$\implies \quad \check{b}^{\text{PAB}}(q) = \left[v_0 + \frac{\overline{Q} v_1}{(n-1)\lambda + n}\right] + \left[\frac{n-1}{(n-1)\lambda + n}\right]\lambda v_1 q, \quad \check{b}^{\text{UPA}}(q) = v_0 + \left[\frac{n-1}{n-2}\right]\lambda v_1 q.$$

As noted earlier, the price coefficients of the mixed-price auction are not (consistent) convex combinations of the price coefficients in the discriminatory and uniform-price auctions.

In the polynomial-Lomax model conjugate bid functions are the basis for all equilibrium behavior, whenever they exist. The existence of conjugate bid functions is quite general, and fails only when one of the price coefficients in Theorem 6 has a zero denominator. Since best-response bids are solutions to a differential equation, equilibrium consists of a conjugate bid curve together with a (potentially degenerate) non-conjugate bid curve, representing the differential equation's homogeneous solution.

Theorem 7 (Conjugate basis of equilibrium)**.** *Let $r = (n-1)\lambda/((1-\alpha)\lambda - n\alpha)$. If $r \notin \{0, \ldots, \bar{k}\}$, then in any equilibrium of the polynomial-Lomax model there is a constant $C \in \mathbb{R}$ such that the market-clearing price is given by*

$$p(Q) = \check{p}(Q) + \left(1 + \left(\frac{(1-\alpha)\lambda - n\alpha}{n\alpha\overline{Q}}\right) Q\right)^{\frac{(n-1)\lambda}{(1-\alpha)\lambda - n\alpha}} C,$$

where $\check{p}$ is the conjugate market clearing price. If $r \in \{0, \ldots, \bar{k}\}$, there is no conjugate equilibrium

[29] In the case of the linear-Lomax model ($\bar{k} = 1$), equilibrium bids intersect at a common $q^{\perp}$. This is particular to the linear-Lomax model, and does not generalize.

in the polynomial-Lomax model.

The antecedent of Theorem 7 follows from analyzing the denominators of the conjugate price coefficients, given in Theorem 6. Its condition expands known results on the existence of linear pure-strategy equilibria in uniform-price auctions (Klemperer and Meyer, 1989; Ausubel et al., 2014): with linear marginal values a uniform-price auction must have at least three bidders to admit a linear pure-strategy equilibrium. This is an immediate consequence of Theorem 7, since in the uniform-price auction ($\alpha = 0$) the condition reduces to $n - 1 \in \{0, \ldots, \bar{k}\}$. In the uniform-price auction more generally, when the order of the polynomial marginal value function, $\bar{k}$, is weakly greater than $n - 1$, conjugate price coefficients will not be defined.

Corollary 5 (Nonexistence of conjugate equilibrium in uniform-price auction)**.** *If $\bar{k}$, the order of the marginal value polynomial, is at least $n-1$, there is no conjugate equilibrium of the uniform-price auction.*

Remark 3. *Corollary 5 implies only the nonexistence of equilibria with a conjugate basis, and not the nonexistence of equilibria more generally. For example, in the linear-Lomax model with $v(q) = v_0 + v_1 q$ and $n = 2$ bidders, equilibrium bids are*

$$b(q) = v_0 - \frac{q}{Q^{\mu}} \left(v_0 - b\left(Q^{\mu}\right)\right) + v_1 q \ln \frac{q}{Q^{\mu}}.$$

This expression may be derived directly from Theorem 2.

When marginal values are nonlinear it may be that $\check{p}(\overline{Q}) > \hat{v}(\overline{Q})$, and therefore $\overline{p}(Q) < \check{p}(Q)$. In this case there does not exist a conjugate equilibrium, but conjugate bid functions are still the basis for equilibrium behavior (Theorem 7): the scaling term C simply needs to be sufficiently negative.

Since uniform-price equilibria are ex post[30] and the differential system (4) does not depend on F, Corollary 5 applies to all uniform-price auctions with random supply supported on $[0, \overline{Q}]$, and not only those with a negative Lomax distribution of supply. Thus conjugate bids represent the basis of equilibrium behavior in any uniform-price auction with sufficiently-many bidders. This intuition extends to mixed-price auctions more generally, although the required number of bidders may depend on the concentration λ of aggregate supply.

Corollary 6 (Conjugate basis with sufficiently-many bidders)**.** *In the polynomial-Lomax model with sufficiently many bidders, all equilibria in the mixed-price auction are in conjugate bids plus a homogenous term.*

Finally, as noted in Theorem 3, the maximum-bid equilibrium is determined by setting the market price for the maximum quantity to (aggregate) marginal value for the maximum quantity, $\overline{p}(\overline{Q}) = \hat{v}(\overline{Q})$.

[30] This has been observed by, e.g., Klemperer and Meyer (1989) and Ausubel et al. (2014). Here, it follows from the fact that uniform-price bids do not depend on the distribution of supply, since $\tilde{H}(Q) = (n-1)/Q$.

Corollary 7 (Maximum-bid equilibrium′)**.** *When $n > \bar{k}+1$, the highest equilibrium market clearing price function in the polynomial-Lomax model is given by*

$$\overline{p}\left(Q\right) = \check{p}\left(Q\right) + \left(\frac{n\alpha}{\left(1-\alpha\right)\lambda} + \left(1 - \frac{n\alpha}{\left(1-\alpha\right)\lambda}\right)\frac{Q}{\overline{Q}}\right)^{\frac{(n-1)\lambda}{(1-\alpha)\lambda - n\alpha}} \left(\hat{v}\left(\overline{Q}\right) - \check{p}\left(\overline{Q}\right)\right),$$

where $\check{p}$ is the conjugate market clearing price.

4.1 Equilibrium revenue

Revenue comparison of mixed-price auction formats is hampered by the complex effect of the extent of price discrimination α on equilibrium bids. In this section I show that equilibrium revenue is strictly increasing in the price mixture α in conjugate equilibria of the linear-Lomax model. Since revenue is strictly increasing as the auction becomes increasingly discriminatory, the resulting revenue ranking will hold in a neighborhood of the linear-Lomax model.

Proposition 3 (Revenue increasing in α)**.** *In conjugate equilibrium of the linear-Lomax model, per capita revenue is strictly increasing in α.*

Proposition 3 shows a strictly positive revenue difference between the discriminatory auction and the uniform-price auction, for any finite n. However, the per capita revenue difference goes to 0 as n becomes large; thus Proposition 3 does not contradict the large-market revenue equivalence established in Proposition 4.

Conjugate equilibrium revenue is strictly increasing in the extent of price discrimination, but there may exist no conjugate equilibrium. And, even when a conjugate equilibrium does exist, it will not in general be revenue-maximizing. This is particularly apparent in the uniform-price auction, where $b(0) = v(0)$ in any equilibrium. Because $b(Q^{\mu}) = v(Q^{\mu})$ in the maximum-bid equilibrium, if the conjugate equilibrium is revenue-maximizing it must be that bids are truthful, which is an equilibrium only in large markets. Constraining attention to the maximum-bid equilibrium of the uniform-price auction (which is revenue-maximizing), I now show that the unique equilibrium of the discriminatory auction strictly revenue-dominates all equilibria of the uniform-price auction.

Proposition 4 (Discriminatory dominates uniform-price)**.** *In the linear-Lomax model, the unique equilibrium of the discriminatory auction strictly revenue-dominates all equilibria of the uniform-price auction.*

Revenue in the discriminatory auction provides an upper bound for equilibrium revenue in the uniform-price auction, suggesting that the discriminatory auction may be preferable for revenue-interested sellers. However, I now show that conjugate equilibrium revenue in the a uniform-price auction with $n+1$ bidders provides an upper bound for equilibrium revenue in the discriminatory auction with n bidders. The importance of competition for revenue is familiar from, e.g., Bulow and Klemperer (1996), but in this divisible-good context the comparison is subtle. Classical revenue comparisons assume that either a bidder may be added, or the sales mechanism may be optimized.

In the divisible-good context, Pycia and Woodward (2021) show that the constrained optimal mechanism offers deterministic supply; and, barring the ability to set the supply distribution, the seller may want to implement a reserve price. By contrast, I implicitly assume that the seller takes the distribution of supply as fixed and can choose only which auction format to implement.[31]

Proposition 5 (Effect of an additional bidder)**.** *There is $\hat{Q}$ such that when $\overline{Q} > \hat{Q}$, all equilibria of the uniform-price auction with $n+1$ bidders raise greater expected revenue than the discriminatory auction with n bidders. Regardless of $\overline{Q}$, conjugate equilibrium in the uniform-price auction with $n+1$ bidders raises greater expected revenue than the discriminatory auction with n bidders.*

Because conjugate equilibrium revenue is in general suboptimal, many equilibria of the uniform-price auction with $n+1$ bidders will raise more revenue than the unique equilibrium of the discriminatory auction with n bidders;[32] in particular, the maximum-bid equilibrium of the uniform-price auction with $n+1$ bidders raises more revenue than the maximum-bid equilibirum of the discriminatory auction with n bidders. Importantly, the former equilibrium always exists, and the latter is the unique equilibrium of the discriminatory auction.

Taken together, Propositions 4 and 5 provide a natural test for the practical relevance of auction format selection. Because equilibrium revenue is increasing in the number of bidders, equilibrium revenue in the discriminatory auction is between uniform-price revenue and uniform-price revenue with an additional bidder.[33] Thus if it is empirically observed that the number of bidders has minimal impact on expected revenue, then switching from a uniform-price to a discriminatory format will have a correspondingly small effect on equilibrium revenue.

5 Conclusion

This paper considers a model of divisible-good auctions in which the allocation rule is an interpolation of discriminatory and uniform-price allocation rules. I show that all equilibria are symmetric, and derive a closed form expression for equilibrium bids and market-clearing prices.

Uniform-price auctions are known to admit multiple equilibria, while discriminatory auctions admit unique equilibria. I show that equilibrium multiplicity is smoothly related to price discrimination: the more discriminatory is an auction, the smaller the range of equilibrium prices it can sustain. This effect is particularly stark when supply is highly concentrated, and the uniform-price auction may admit a significantly larger set of equilibria than nearly all partially-discriminatory

[31] If the seller can fully-optimize the parameters of the auction, the optimal discriminatory auction with n bidders may outperform the (unoptimized) uniform-price auction with $n+1$ bidders. To see this, suppose that supply is set suboptimally so that $\operatorname{Supp} F^{\mu} = [0, \varepsilon]$, while monopoly supply is $Q^{\star} \gg \varepsilon$. Then equilibrium revenue in the uniform-price auction with $n+1$ bidders is bounded above by $(n+1)v(0)\varepsilon$, and revenue under optimal supply will be substantially larger.

[32] The uniform-price auction may also admit equilibria where bids are lower than conjugate equilibrium bids. In these equilibria, the discriminatory auction with n bidders may outperform the uniform-price auction if maximum supply $\overline{Q}$ is small.

[33] In the uniform-price auction equilibrium selection may cause revenue to fall when a bidder is added. Nonetheless, for any equilibrium in a uniform-price auction with n bidders there is an equilibrium in a uniform-price auction with $n+1$ bidders which raises strictly greater revenue.

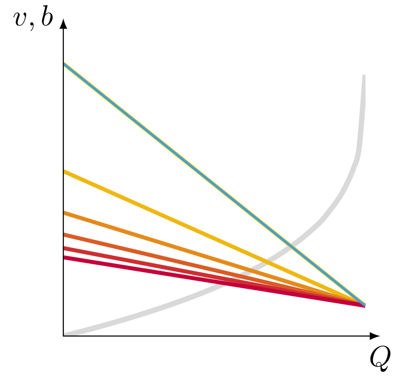

Figure 5: Equilibrium bids in the linear-Lomax model for $n \nearrow \infty$. Because equilibrium is unique, equilibrium bids are conjugate bids, without any homogeneous term. Uniform price auction bids are truthful, and are therefore hidden behind the (blue) marginal value curve.

auctions. Similarly, the range of equilibrium prices is increasing in the number of bidders, regardless of the extent of price discrimination. However, the most extreme of these prices are obtained on a shrinking set of quantities, and in the large-market limit equilibrium is unique. While uncertainty in ex post outcomes due to equilibrium selection vanishes when the number of bidders is large, ex post uncertainty remains higher in the uniform-price auction than in the discriminatory auction as the range of feasible market-clearing prices is larger.

Constraining attention to bidders with polynomial marginal values in auctions with concentrated supply, I show that equilibrium is generally expressable in conjugate (polynomial) bids plus a potentially degenerate homogeneous term. The expression for equilibrium bids clarifies the known "at least three bidders" result for equilibrium existence in uniform-price auctions: conjugate equilibrium in the uniform-price auction relies on the number of bidders exceeding the order of the marginal value polynomial. I show that equilibrium revenue is strictly increasing in the extent of price discrimination, and that the unique equilibrium of the discriminatory auction yields greater expected revenue than all equilibria of the uniform-price auction. However, discriminatory auction revenue is in turn bounded above by revenue in a uniform-price auction with one more bidder, and even though the discriminatory auction outperforms the uniform-price auction the difference may be small.

In total, these results suggest that discriminatory auctions may generate strictly better, more-certain, seller outcomes than uniform-price auctions, especially when supply is highly concentrated. However, when markets are large this difference may be minimal. This is in line with known results on the theory of divisible-good auctions (e.g., Pycia and Woodward (2021) show that when the seller can affect the distribution of supply, the discriminatory auctions outperform uniform-price auctions) as well empirical analyses of multi-unit auctions (e.g., Hortaçsu et al. (2018) show that auction format has a negligible effect on counterfactual outcomes).

References

O. Armantier and E. Sbaï. Estimation and comparison of treasury auction formats when bidders are asymmetric. *Journal of Applied Econometrics*, 21(6):745–779, 2006.

O. Armantier and E. Sbaï. Comparison of alternative payment mechanisms for French Treasury auctions. *Annals of Economics and Statistics/Annales d'Économie et de Statistique*, pages 135–160, 2009.

L. M. Ausubel, P. Cramton, M. Pycia, M. Rostek, and M. Weretka. Demand reduction and inefficiency in multi-unit auctions. *The Review of Economic Studies*, 81(4):1366–1400, 2014.

K. Back and J. F. Zender. Auctions of divisible goods: On the rationale for the Treasury experiment. *Review of Financial Studies*, 6(4):733–764, 1993.

K. Barbosa, D. De Silva, L. Yang, and H. Yoshimoto. Auction mechanisms and Treasury revenue: Evidence from the Chinese experiment. 2021.

M. Brenner, D. Galai, and O. Sade. Sovereign debt auctions: Uniform or discriminatory? *Journal of Monetary Economics*, 56(2):267–274, 2009.

J. Bulow and P. Klemperer. Auctions versus negotiations. *The American Economic Review*, 86(1): 180–194, 1996.

J. Burkett and K. Woodward. Reserve prices eliminate low revenue equilibria in uniform price auctions. *Games and Economic Behavior*, 2020a.

J. Burkett and K. Woodward. Uniform price auctions with a last accepted bid pricing rule. *Journal of Economic Theory*, 185, 2020b.

S. Castellanos and M. Oviedo. Optimal bidding in the Mexican Treasury securities primary auctions: Results of a structural econometric approach. *Cuadernos de Economía*, 45(131):3–28, 2008.

V. Chari and R. J. Weber. How the US Treasury should auction its debt. *Quarterly Review*, (Fall): 3–12, 1992.

P. Del Río. Designing auctions for renewable electricity support. best practices from around the world. *Energy for Sustainable Development*, 41:1–13, 2017.

R. Engelbrecht-Wiggans. Revenue equivalence in multi-object auctions. *Economics Letters*, 26(1): 15–19, 1988.

P. Février, R. Préget, and M. Visser. Econometrics of share auctions. Working paper, 2002.

M. Friedman. Single-price auction with multiple bids. *Wall Street Journal*, 1991.

P. Holmberg. Supply function equilibria of pay-as-bid auctions. *Journal of Regulatory Economics*, 36(2):154–177, 2009.

A. Hortaçsu. Bidding behavior in divisible good auctions: Theory and evidence from the Turkish Treasury auction market. 2002.

A. Hortaçsu and D. McAdams. Mechanism choice and strategic bidding in divisible good auctions: An empirical analysis of the Turkish Treasury auction market. *Journal of Political Economy*, 118(5):833–865, 2010.

A. Hortaçsu, J. Kastl, and A. Zhang. Bid shading and bidder surplus in the US Treasury auction system. *American Economic Review*, 108(1):147–69, 2018.

M. O. Jackson and I. Kremer. The relevance of a choice of auction format in a competitive environment. *The Review of Economic Studies*, 73(4):961–981, 2006.

B.-S. Kang and S. L. Puller. The effect of auction format on efficiency and revenue in divisible goods auctions: A test using Korean Treasury auctions. *The Journal of Industrial Economics*, 56(2):290–332, 2008.

P. D. Klemperer and M. A. Meyer. Supply function equilibria in oligopoly under uncertainty. *Econometrica: Journal of the Econometric Society*, pages 1243–1277, 1989.

A. Lizzeri and N. Persico. Uniqueness and existence of equilibrium in auctions with a reserve price. *Games and Economic Behavior*, 30(1):83–114, 2000.

M. H. Lotfi and S. Sarkar. Uncertain price competition in a duopoly with heterogeneous availability. *IEEE Transactions on Automatic Control*, 61(4):1010–1025, 2016.

E. Mariño and D. Marszalec. Auction performance, strategic supply management, and bidder behavior in Treasury bill auctions: Evidence from the Philippines. 2020.

D. Marszalec. The impact of auction choice on revenue in treasury bill auctions-an empirical evaluation. *International Journal of Industrial Organization*, 2017.

D. Marszalec, A. Teytelboym, and S. Laksá. EPIC fail: How below-bid pricing backfires in multiunit auctions. 2020.

D. McAdams. Adjustable supply in uniform price auctions: Non-commitment as a strategic tool. *Economics Letters*, 95(1):48–53, 2007.

P. Milgrom. *Putting auction theory to work*. Cambridge University Press, 2004.

R. B. Myerson. Optimal auction design. *Mathematics of Operations Research*, 6(1):58–73, 1981.

M. Plum. Characterization and computation of nash-equilibria for auctions with incomplete information. *International Journal of Game Theory*, 20(4):393–418, 1992.

M. Pycia and K. Woodward. Auctions of homogeneous goods: A case for pay-as-bid. Working paper, 2021.

J. G. Riley and W. F. Samuelson. Optimal auctions. *The American Economic Review*, pages 381–392, 1981.

K. Ruddell, A. B. Philpott, and A. Downward. Supply function equilibrium with taxed benefits. *Operations Research*, 65(1):1–18, 2016.

J. M. Swinkels. Asymptotic efficiency for discriminatory private value auctions. *The Review of Economic Studies*, 66(3):509–528, 1999.

J. M. Swinkels. Efficiency of large private value auctions. *Econometrica*, 69(1):37–68, 2001.

S. Viswanathan and J. J. Wang. Market architecture: limit-order books versus dealership markets. *Journal of Financial Markets*, 5(2):127–167, 2002.

J. J. Wang and J. F. Zender. Auctioning divisible goods. *Economic Theory*, 19(4):673–705, 2002.

A Proofs for Section 2 (Model and equilibrium)

A.1 Preparatory results

Lemma 7 (Bids below values)**.** *In any equilibrium, $G^i(q; b^i) < 1$ implies $b^i(q) \leq v(q)$; that is, bids are weakly below marginal values for all achievable quantities.*

Proof. Let $\hat{b} = b \wedge v$ be the pointwise minimum of b and v, and suppose that $\hat{b} < b$ on some set of achievable quantities. Submitting the bid function $\hat{b}$ instead of b yields a weakly lower expected payment even at quantity realizations unaffected by the altered bid. Because this implies that the deviation is weakly utility-improving for quantity realizations unaffected by the deviation, it is sufficient to show that utility is also improved for all quantity realizations directly affected by the deviation.

Suppose that aggregate supply Q leads to the allocation q_i, and that $b(q_i) > v(q_i) = \hat{b}(q_i)$. Because market clearing prices weakly decrease when bids decrease it follows that $p(Q) \geq \hat{p}(Q)$. Since $\hat{b}(q_i) = v(q_i) = p(Q) \geq \hat{p}(Q)$, when she submits the alternate bid function bidder i receives at least as much quantity as where her marginal value equaled the previous market clearing price, $\hat{q}_i \geq v^{-1}(p(Q)) \equiv \check{q}_i$. Utility conditional on this aggregate quantity realization is

$$\underbrace{\int_0^{q_i} v\left(x\right) - \alpha b\left(x\right) dx - \left(1-\alpha\right) q_i b\left(q_i\right)}_{\text{utility under } b} \gtrless \underbrace{\int_0^{\hat{q}_i} v\left(x\right) - \alpha \hat{b}\left(x\right) dx - \left(1-\alpha\right) \hat{q}_i \hat{b}\left(\hat{q}_i\right)}_{\text{utility under } \hat{b}}.$$

Note that the left-hand side can be written as

$$\begin{aligned}
&\int_0^{q_i} v\left(x\right) - \alpha b\left(x\right) dx - \left(1-\alpha\right) q_i b\left(q_i\right) \\
&= \int_0^{\hat{q}_i} v\left(x\right) - \alpha b\left(x\right) - \left(1-\alpha\right) b\left(q_i\right) dx + \int_{\hat{q}_i}^{q_i} v\left(x\right) - \alpha b\left(x\right) - \left(1-\alpha\right) b\left(q_i\right) dx \\
&\leq \int_0^{\hat{q}_i} v\left(x\right) - \alpha \hat{b}\left(x\right) - \left(1-\alpha\right) \hat{b}\left(\hat{q}_i\right) dx + \int_{\hat{q}_i}^{q_i} v\left(x\right) - b\left(q_i\right).
\end{aligned}$$

By assumption, $v(x) < b(q_i)$ for all $x \in (\hat{q}_i, q_i)$. Then as long as $\hat{q}_i < q_i$ this implies the deviation $\hat{b}$ is profitable. The allocation is unaffected by the deviation (conditional on $b(q_i) > v(q_i)$) only if b is discontinuous at q_i. Since b is monotone decreasing and v is continuous, there is some $q_i' < q_i$ such that $b(q_i') > v(q_i')$. Evaluating the above inequality at this $\hat{q}_i$ implies that the deviation $\hat{b}$ is utility-improving. □

Lemma 8 (Bids are strictly below values)**.** *If $\alpha > 0$ and b^i is a best-response, $G^i(q; b^i) < 1$ implies $b^i(q) < v(q)$; that is, in any equilibrium of a partially-discriminatory auction, bids are strictly below values for all achievable quantities.*

Proof. I first show that if $G^i(q; b^i) < 1$ and $b^i(q) = v^i(q)$, then $\limsup_{\varepsilon' \searrow 0}(b^i(q) - b^i(q+\varepsilon'))/\varepsilon' = \infty$.[34] Assume that $b^i(q) = v^i(q)$ and $G^i(q; b^i) < 1$, and for $\varepsilon > 0$ define a deviation b^ε by

$$b^\varepsilon\left(q'\right) = \begin{cases} b^i\left(q'\right) & \text{if } b^i\left(q'\right) \notin \left(b^i\left(q\right) - \varepsilon, b^i\left(q\right)\right), \\ b^i\left(q\right) - \varepsilon & \text{otherwise.} \end{cases}$$

Let $\overline{q}_\varepsilon = \inf\{q' : b^i(q') < b^i(q) - \varepsilon\}$ be the quantity at which the deviation b^ε again equals b^i. When under the original bid function bidder i's allocation q is between q_i and $\overline{q}_\varepsilon$, under the deviation b^ε her allocation is at least q_i. In this interval the deviation yields lost utility of at most

$$\int_q^{\overline{q}_\varepsilon} \int_q^{q'} v\left(x\right) - b^i\left(x\right) dx dG^i\left(q'; b^i\right) = \int_q^{\overline{q}_\varepsilon} \left(v\left(x\right) - b^i\left(x\right)\right)\left(G^i\left(\overline{q}_\varepsilon; b^i\right) - G^i\left(x; b^i\right)\right) dx.$$

The deviation yields cost savings for allocations $q_i \geq \overline{q}_\varepsilon$, given by

$$\alpha \int_{q_i}^{\overline{q}_\varepsilon} b^i\left(x\right) - b^i\left(\overline{q}_\varepsilon\right) dx \left(1 - G^i\left(\overline{q}_\varepsilon; b^i\right)\right).$$

It is thus sufficient to show that, for small $\varepsilon > 0$,

$$\alpha \int_{q_i}^{\overline{q}_\varepsilon} b^i\left(x\right) - b^i\left(\overline{q}_\varepsilon\right) dx \left(1 - G^i\left(\overline{q}_\varepsilon; b^i\right)\right) \geq \int_{q_i}^{\overline{q}_\varepsilon} \left(v\left(x\right) - b\left(\overline{q}_\varepsilon\right)\right)\left(G^i\left(\overline{q}_\varepsilon; b^i\right) - G^i\left(x; b^i\right)\right) dx.$$

Since $G^i(q_i; b^i) < 1$, we may assume that ε is sufficiently small so that $(1 - G^i(\overline{q}_\varepsilon; b^i))\alpha > C >$

[34] For expositional simplicity, I say that an expression equals infinity when the expression is not finite.

0 for some C. And since $G^i(\cdot; b^i)$ is left continuous, for any $K > 0$ there is $\varepsilon > 0$ such that $G^i(\overline{q}_\varepsilon; b^i) - G^i(q; b^i) < K$ for all $q \in [q_i, \overline{q}_\varepsilon]$. Then it is sufficient to show that for $J > 0$ sufficiently small, there is $\varepsilon > 0$ sufficiently small so that

$$\int_{q_i}^{\overline{q}_\varepsilon} b^i(x) - b^i(\overline{q}_\varepsilon)\, dx \geq J \int_{q_i}^{\overline{q}_\varepsilon} v(x) - b^i(\overline{q}_\varepsilon)\, dx.$$

This is satisfied whenever $\limsup_{\varepsilon' \searrow 0}(b^i(q) - b^i(q + \varepsilon'))/\varepsilon'$ is finite.

Retain the supposition that $b^i(q) = v(q)$ and $G^i(q; b^i) < 1$. By the previous argument, $\limsup_{\varepsilon' \searrow 0}(b^i(q) - b^i(q + \varepsilon'))/\varepsilon' = \infty$. For $\varepsilon, \delta > 0$, consider a deviation $b^{\varepsilon\delta}$ given by

$$b^{\varepsilon\delta}(q') = \begin{cases} b^i(q') & \text{if } q' < q - \delta, \\ b^i(q') & \text{if } b^i(q') < b^i(q) - \varepsilon, \\ b^i(q) - \varepsilon & \text{otherwise.} \end{cases}$$

By construction the deviation b^ε saves payment of at least $\alpha\varepsilon\delta$ with probability $1 - G^i(q; b^i)$. Define $P > 0$ so that $1 - G^i(q; b^i) > P > 0$; then a bound on the payment saved is $P\alpha\varepsilon\delta$.

For $q' \in [q - \delta, \overline{q}_\varepsilon]$, the deviation also costs utility. Because marginal values are Lipschitz continuous (with modulus M) and the slope of bids is unboundedly negative at q, for any $K > 0$ there are $\varepsilon, \delta > 0$ sufficiently small so that the utility lost is bounded above by

$$\frac{1}{2}M\delta^2 + K\varepsilon^2 \Delta G(q; \varepsilon), \text{ where } \Delta G(q; \varepsilon) = G^i(\overline{q}_\varepsilon; b^i) - G^i(q; b^i).$$

Then for the deviation $b^{\varepsilon\delta}$ to be profitable, it is sufficient that, for ε and δ sufficiently small,

$$\frac{1}{2}M\delta^2 + K\varepsilon^2 \Delta G(q; \varepsilon) < P\alpha\varepsilon\delta.$$

Let $\delta = P\alpha\varepsilon/M$. Then the desired expression is

$$\frac{P^2\alpha^2}{2M}\varepsilon^2 + K\varepsilon^2 \Delta G(q; \varepsilon) < \frac{P^2\alpha^2}{M}\varepsilon^2 \iff K\Delta G(q; \varepsilon) < \frac{P^2\alpha^2}{2M}.$$

Since G^i is right continuous, $\lim_{\varepsilon \searrow 0} \Delta G(q; \varepsilon) = 0$ and the above inequality is satisfied when $\varepsilon > 0$ is small. Then for small $\varepsilon, \delta > 0$ the deviation $b^{\varepsilon\delta}$ is profitable. It follows that when $\alpha > 0$, $b^i(q) < v(q)$ whenever $G^i(q; b^i) < 1$. □

Lemma 9 (No flat bids)**.** *In any equilibrium bids are strictly decreasing at all quantities q such that $b^i(q) > \underline{p}$.*

Proof. In equilibrium, no two bidders can submit bids which are constant at the same price: otherwise both have a strict incentive to slightly increase their bids near this price. Then it remains only to be shown that no single bidder can submit a flat bid for any quantity for which the bid is strictly above the minimum market-clearing price $\underline{p}$.

Suppose that $b^i(q) = p$ for all $q \in (q_\ell, q_r)$, and that there is a bidder j and a quantity q_j such that $\lim_{q \searrow q_j} b^j(q) = p$. Because $p > \underline{p}$, Lemma 8 implies that $v(q_j) > p$. By bidding just above p in a neighborhood of q_j, bidder j can increase her utility (since bidder i's bid is locally constant, bidder j faces infinitely elastic residual supply). Then in equilibrium there can be no bidder j and quantity q_j such that $\lim_{q \searrow q_j} b^j(q) = p$.

If no bidder j submits a bid which is (in the limit) equal to p, then bidder i can strictly improve her utility by lowering her bid on the flat interval (q_ℓ, q_r), and for slightly higher quantities as necessary. Slightly reducing her bid on this interval never affects her quantity allocation, and reduces her payment with strictly positive probability. Then if $(b^j)_{j=1}^n$ is an equilibrium bid profile and b^i is constant on some (q_ℓ, q_r), there must be some agent j and quantity q_j such that $\lim_{q \searrow q_j} b^j(q) = p$, a contradiction.

□

A.2 Proofs for Section 2.1

Theorem 8 (Equilibrium is symmetric ($\alpha > 0$))**.** *When $\alpha > 0$, all equilibria are symmetric.*

Proof. Assume without loss of generality that bids are right continuous at $q = 0$. There are at least two bidders submitting the highest possible bid, $b^i(0) = b^j(0) = \overline{b}$, $\overline{b} = \max_k b^k(0)$, otherwise the lone high bidder has a strict incentive to reduce her bid. Consider the set of bidders k for whom $b^k(0) = \overline{b}$. By Lemma 9 none of these bids are locally constant at $q = 0$. Suppose that, for two bidders i and j such that $b^i(0) = b^j(0) = \overline{b}$, residual supply curves $S^i = \sum_{\ell \neq i} \varphi^\ell$ and $S^j = \sum_{\ell \neq j} \varphi^\ell$ differ in a neighborhood of 0. If S^i and S^j have different slopes at 0, one of b^i or b^j is nonoptimal, and in equilibrium the slope of residual supply at $q = 0$ is independent of the bidder's identity. From the bidders' equilibrium first-order conditions, standard resuls from differential equations are sufficient to show that bid functions are identical for all bidders who submit the same initial bid $b^i(0)$. Then if equilibrium is asymmetric it must be that there are different classes of bidders, determined by initial bid.

Assume that equilibrium is asymmetric and let $\overline{b}'$ be the next-highest initial bid, $\overline{b}' = \max\{b^\ell(0)\colon b^\ell(0) \neq \overline{b}\}$.[35] For any bidder ℓ with $b^\ell(0) \geq \overline{b}'$, let q_ℓ be such that $b^\ell(q_\ell) = \overline{b}'$.[36] For any bidder j such that q_j is defined, define a deviation $\hat{b}^{j\varepsilon}$ so that

$$\hat{b}^{j\varepsilon}(q) = \begin{cases} \overline{b}' - \varepsilon & \text{if } b^j(q) \in \left[\overline{b}' - \varepsilon, \overline{b}'\right], \\ b^j(q) & \text{otherwise.} \end{cases}$$

[35] The bulk of the ensuing argument is meant to handle the case in which bidders ℓ with $b^\ell(0) = \overline{b}'$ submit bids which become perfectly inelastic as $q \searrow 0$. Otherwise, bidders k with $b^k(0) = \overline{b}$ face a discontinuity in the slope of residual supply at $p = \overline{b}'$, implying that their optimal response is either locally constant, or is kinked. Flat bids are ruled out by Lemma 9, and if their optimal response is kinked then the optimality conditions of bidders k with $b^k(0) = \overline{b}$ and bidders ℓ with $b^\ell(0) = \overline{b}'$ cannot be simultaneously satisfied.

[36] Assuming without loss of generality that bids are right continuous, there is always a bidder ℓ with $q_\ell = 0$; otherwise, $b^j(0) = \overline{b}$ for all bidders j.

For small $\varepsilon > 0$, the deviation $\hat{b}^{j\varepsilon}$ gives bidders cost savings proportional to the slope of their bid functions at q_j, independent of q_j. However, the utility sacrificed is strictly lower for a bidder with $q_k > 0$ than for the bidder with $q_\ell = 0$ (since $v(q_k) < v(0)$), unless the bidder with $q_k > 0$ faces less elastic supply. It follows that for the examined bids to constitute an equilibrium, some bidder with $q_\ell = 0$ is submitting a less elastic bid than some bidder with $q_k > 0$. However, we could also have analyzed the deviation $\check{b}^{j\varepsilon}$,

$$\check{b}^{j\varepsilon}(q) = \begin{cases} \overline{b}' & \text{if } b^j(q) \in \left[\overline{b}' - \varepsilon, \overline{b}'\right], \\ b^j(q) & \text{otherwise.} \end{cases}$$

Under this deviation, the utility gained by the bidder with $q_k > 0$ is strictly below the utility gained by the bidder with $q_\ell = 0$, unless the bidder with $q_k > 0$ faces more elastic supply. This is a contradiction, and it follows that all bidders must have identical upper bids $\overline{b}$. From earlier arguments, equilibrium bids must be symmetric. □

Theorem 9 (Equilibrium is symmetric ($\alpha = 0$))**.** *When $\alpha = 0$, all equilibria are symmetric.*

Proof. I first show that in any equilibrium, $b^i(0) = v(0)$ for all bidders i. For $\varepsilon > 0$, consider a deviation b^ε given by

$$b^\varepsilon(q) = \begin{cases} b^i(0) & \text{if } q \le \varepsilon, \\ b^i(q) & \text{otherwise.} \end{cases}$$

We may assume that when bidder i employs the deviation b^ε and the aggregate quantity realization is $Q \le \varepsilon$, bidder i's allocation is $q_i = Q$ (otherwise, we may consider a slight upward deviation, and let the increase tend toward zero). Letting $D(\varepsilon) = \sum_{j=1}^n \varphi^j(b^i(\varepsilon))$ be aggregate demand at price $b^i(\varepsilon)$, the utility under b^i versus that under b^ε, considering only quantity realizations below $D(\varepsilon)$, is

$$\int_0^\varepsilon \int_0^q v(x) - b(q)\, dx dG^i\left(q; b^i\right) \gtrless \int_0^{D(\varepsilon)} \int_0^{\min\{Q,\varepsilon\}} v(x) - b^i(0)\, dx dF(Q).$$

Note that the right-hand side is a lower bound on the (local) utility under the deviation b^ε, since for $Q > \varepsilon$ the market-clearing price will typically be below $b^i(0)$. When $\varepsilon = 0$, both sides are zero; taking the derivative with respect to ε gives, where D'_ε is defined,

$$\int_0^\varepsilon v(x) - b(\varepsilon)\, dx dG^i\left(\varepsilon; b^i\right) \gtrless \int_0^\varepsilon v(x) - b^i(0)\, dx dF(D(\varepsilon))\, D_\varepsilon(\varepsilon) + \int_\varepsilon^{D(\varepsilon)} \left[v(\varepsilon) - b^i(0)\right] dF(Q). \tag{10}$$

To establish a contradiction, assume that $b^i(0) < v(0)$. Since v is continuous and (without loss of generality) b is right-continuous at 0, for any $\delta > 0$ there are $\overline{\varepsilon}, \overline{\mu}, \underline{\mu} > 0$ such that for all $\varepsilon < \overline{\varepsilon}$,

$$\overline{\mu} > v(0) - b^i(\varepsilon) > v(\varepsilon) - b^i(0) > \underline{\mu} > (1-\delta)\,\overline{\mu}.$$

To show that the deviation b^ε is profitable, it is sufficient to show that for ε small, the left-hand side of (10) is less than its right-hand side. For this to hold, it is sufficient that

$$\overline{\mu}\varepsilon dG^i\left(\varepsilon;b^i\right) < \underline{\mu}\varepsilon dF\left(D\left(\varepsilon\right)\right)D_\varepsilon\left(\varepsilon\right) + \underline{\mu}\left[F\left(D\left(\varepsilon\right)\right) - F\left(\varepsilon\right)\right].$$

Since D is monotone, it is differentiable almost everywhere. Wherever D is differentiable, $dG^i(\varepsilon;b^i) = dF(D(\varepsilon))D_\varepsilon(\varepsilon)$; then it is sufficient to show that, for small $\varepsilon > 0$,

$$dF\left(D\left(\varepsilon\right)\right)D_\varepsilon\left(\varepsilon\right)\delta < \frac{F\left(D\left(\varepsilon\right)\right)}{\varepsilon} - \frac{F\left(\varepsilon\right)}{\varepsilon}.$$

Recalling that δ may be arbitrarily small, letting $\varepsilon \searrow 0$ either this inequality is satisfied for some bidder i, or for all bidders i with $b^i(0) = \max_j b^j(0)$ it is the case that for ε small, $D(\varepsilon) = \varepsilon$; the latter case only applies if there is a unique bidder submitting the highest-possible bid, which cannot be a best response. It follows that in any equilibrium and for all bidders i, $b^i(0) = v(0)$.

We now show that $b^i(0) = v(0)$ for all bidders i is inconsistent with asymmetric equilibrium bids. Since each inverse bid φ^j is decreasing, it is almost everywhere differentiable; since the number of bidders is finite, all φ^j are (simultaneously) differentiable at almost all p. Furthermore, for any given price $p > \underline{p}$ the set $\{q\colon b^i(q) = p\}$ has measure zero (Lemma 9). It follows that in equilibrium, for any bidder i and for almost all q, bidder i's first-order conditions are satisfied:

$$-\left(v\left(q\right) - b^i\left(q\right)\right)\sum_{j\neq i}\varphi^j_p\left(b^i\left(q\right)\right) = q.$$

Now, suppose there are bidders i and j for whom $\varphi^i \neq \varphi^j$. Without loss of generality, assume that there is p such that $\varphi^i(p) > \varphi^j(p)$. Let $\overline{p} = \sup\{p'\colon \varphi^i(p') > \varphi^j(p')\}$; since $b^i(0) = b^j(0)$, $\overline{p}$ is well-defined. Then for $\tilde{p} < \overline{p}$ sufficiently close to $\overline{p}$, it must be that $\varphi^i(\tilde{p}) > \varphi^j(\tilde{p})$ and $\varphi^i_p(\tilde{p}) < \varphi^j_p(\tilde{p})$. Applying the first-order conditions for optimality, we have

$$\begin{aligned}\left(v\left(\varphi^i\left(\tilde{p}\right)\right) - \tilde{p}\right) &= -\frac{\varphi^i\left(\tilde{p}\right)}{\varphi^j_p\left(\tilde{p}\right) + \sum_{k\neq i,j}\varphi^k_p\left(\tilde{p}\right)} \\ &> -\frac{\varphi^j\left(\tilde{p}\right)}{\varphi^i_p\left(\tilde{p}\right) + \sum_{k\neq i,j}\varphi^k_p\left(\tilde{p}\right)} = \left(v\left(\varphi^j\left(\tilde{p}\right)\right) - \tilde{p}\right) > \left(v\left(\varphi^i\left(\tilde{p}\right)\right) - \tilde{p}\right).\end{aligned}$$

The inequality from line to line follows from $\varphi^i(\tilde{p}) > \varphi^j(\tilde{p})$ and $\varphi^i_p(\tilde{p}) < \varphi^j_p(\tilde{p}) \leq 0$ (thus $-\varphi^i_p(\tilde{p}) > -\varphi^j_p(\tilde{p})$); since the inequality results in a contradiction, there cannot exist an asymmetric equilibrium. □

Remark 4. *Since equilibria are symmetric, in equilibrium the slope of the (symmetric) inverse bid function is bounded away from zero for all* $p \in (\underline{p}, \overline{p})$. *Otherwise, since the first-order conditions apply almost everywhere, there is* $q \in (0, Q^\mu)$ *such that* $\lim_{q' \nearrow q} b(q') = v(q)$, *which cannot be the case in equilibrium.*[37] *Then since* φ *is absolutely continuous (by assumption),* $b = \varphi^{-1}$ *is absolutely*

[37] Lemma 8 establishes this for partially-discriminatory auctions. For the uniform-price auction, this is implied by Lemma 9, Theorem 9, and Lipschitz continuity of marginal values.

continuous.

Proof of Theorem 2. Bidder i's expected utility can be written as

$$\mathbb{E}\left[u\left(b^{i},b^{-i}\right)\right]=\int_{0}^{\overline{Q}}\int_{0}^{q}v\left(x\right)-\alpha b^{i}\left(x\right)dx-\left(1-\alpha\right)qb^{i}\left(q\right)dG^{i}\left(q;b^{i}\right).$$

Integration by parts removes the double-integration, yielding

$$\mathbb{E}\left[u\left(b^{i},b^{-i}\right)\right]=\int_{0}^{\overline{Q}}\left(v\left(q\right)-b^{i}\left(q\right)-\left(1-\alpha\right)qb_{q}^{i}\left(q\right)\right)\left(1-G^{i}\left(q;b^{i}\right)\right)dq.$$

Applying the calculus of variations yields that for all q,

$$-\left(1-G^{i}\left(q;b^{i}\right)\right)-\left(v\left(q\right)-b^{i}\left(q\right)-\left(1-\alpha\right)qb_{q}^{i}\left(q\right)\right)G_{b}^{i}\left(q;b\right)=\frac{d}{dq}\left[-\left(1-\alpha\right)q\left(1-G^{i}\left(q;b^{i}\right)\right)\right].$$

Expanding the derivative yields

$$-\left(v\left(q\right)-b^{i}\left(q\right)\right)G_{b}^{i}\left(q;b^{i}\right)=\alpha\left(1-G\left(q;b^{i}\right)\right)+\left(1-\alpha\right)qG_{q}^{i}\left(q;b^{i}\right). \tag{11}$$

In a symmetric equilibrium all of bidder i's opponents will submit the same bid function b. Since equilibrium bids cannot be flat (see Lemma 9) this bid function has a well-defined inverse φ. By construction, $\varphi_p(b^i(q))=1/b_q^i(q)$. Then

$$G^{i}\left(q;b^{i}\right)=F\left(q+\left(n-1\right)\varphi\left(b^{i}\left(q\right)\right)\right).$$

It follows that

$$\begin{aligned}G_{q}^{i}\left(q;b^{i}\right)&=f\left(q+\left(n-1\right)\varphi\left(b^{i}\left(q\right)\right)\right),\\G_{b}^{i}\left(q;b^{i}\right)&=\left(n-1\right)\varphi_{p}\left(b^{i}\left(q\right)\right)f\left(q+\left(n-1\right)\varphi\left(b^{i}\left(q\right)\right)\right).\end{aligned}$$

In a symmetric equilibrium it must be that $\varphi(b^i(q))=q$ and $b^i=b$. Then

$$G^{i}\left(q;b\right)=F\left(nq\right),\quad G_{q}^{i}\left(q;b\right)=f\left(nq\right),\quad G_{b}^{i}\left(q;b\right)=\frac{n-1}{b_{q}\left(q\right)}f\left(nq\right).$$

Substituting into equation (11) yields

$$-\left(v\left(q\right)-b\left(q\right)\right)f\left(nq\right)=\frac{1}{n-1}\left(\alpha\left(1-F\left(nq\right)\right)+\left(1-\alpha\right)qf\left(nq\right)\right)b_{q}\left(q\right).$$

By market clearing it must be that $p(Q) = b(Q/n)$. Then

$$\begin{aligned} & -\left(v\left(q\right)-p\left(nq\right)\right)f\left(nq\right)=\frac{n}{n-1}\left(\alpha\left(1-F\left(nq\right)\right)+\left(1-\alpha\right)qf\left(nq\right)\right)p_{Q}\left(nq\right) \\ \implies \quad & -\left(\hat{v}\left(Q\right)-p\left(Q\right)\right)f\left(Q\right)=\frac{1}{n-1}\left(n\alpha\left(1-F\left(Q\right)\right)+\left(1-\alpha\right)Qf\left(Q\right)\right)p_{Q}\left(Q\right). \end{aligned}$$

Then $-(\hat{v}(Q)-p(Q))\tilde{H}(Q) = p_Q(Q)$; applying results from differential equations yields the desired expression for market-clearing prices. Since equilibrium is symmetric (Theorem 1), $b(q) = p(nq)$ and $v_q(q) = \hat{v}_Q(nq)$; substituting in to the market-clearing price equation yields the desired expression for equilibrium bids. $\square$

B Proofs for Section 3 (Equilibrium properties)

B.1 Proofs for Section 3.1

Proof of Lemma 2. Suppose that $p(\overline{Q}) < \hat{v}(\overline{Q})$. Then for sufficiently small $\varepsilon > 0$, there is $\delta > 0$ such that for all $q \in (Q^\mu - \delta, Q^\mu)$ it must be that

$$v\left(Q^{\mu}\right)>\lim_{q'\nearrow Q^{\mu}}b\left(q'\right)+\varepsilon>b\left(q\right)\geq\lim_{q'\nearrow Q^{\mu}}b\left(q'\right).$$

Define the limiting price $\overline{b} = \lim_{q'\nearrow Q^\mu} b(q')$, and let $\overline{\varepsilon} = v(Q^\mu) - \overline{b}$. For $\varepsilon \in (0, \overline{\varepsilon})$, define $q_\varepsilon = \inf\{q \colon b(q) \leq \overline{b} + \varepsilon\}$. Define a deviation b^ε such that

$$b^{\varepsilon}\left(q\right)=\begin{cases}b\left(q\right) & \text{if } q<q_{\varepsilon},\\ \overline{b}+\varepsilon & \text{if } q\geq q_{\varepsilon}.\end{cases}$$

Deviating to b^ε affects outcomes only when the realization of supply is $Q > nq_\varepsilon$. In this case, the bidder receives all additional quantity $Q > nq_\varepsilon$, and also pays more for units which would have been won anyway, Q/n. The costs of the deviation must outweigh the benefits, hence

$$\begin{aligned}\Delta u\equiv\int_{nq_{\varepsilon}}^{\overline{Q}} & \underbrace{\int_{\frac{1}{n}Q}^{Q-(n-1)q_{\varepsilon}}v\left(y\right)-b^{\varepsilon}\left(y\right)dy}_{\text{add'l quantity}}-\underbrace{\alpha\int_{q_{\varepsilon}}^{\frac{1}{n}Q}b^{\varepsilon}\left(y\right)-b\left(y\right)dy}_{\text{add'l discriminatory payment}}\\ & -\underbrace{\left(1-\alpha\right)\left(\frac{1}{n}Q\right)\left(b^{\varepsilon}\left(\frac{1}{n}Q\right)-b\left(\frac{1}{n}Q\right)\right)}_{\text{add'l uniform price payment}}dF\left(Q\right)\leq0.\end{aligned}$$

To simplify analysis, rearrange this expression as

$$\Delta u = \int_{nq_\varepsilon}^{\overline{Q}} \int_{\frac{1}{n}Q}^{Q-(n-1)q_\varepsilon} v\left(y\right)dy - Qb\left(q_\varepsilon\right) + \left(n - \left(1-\alpha\right)\right)q_\varepsilon b\left(q_\varepsilon\right)$$
$$+\alpha \int_{q_\varepsilon}^{\frac{1}{n}Q} b\left(y\right)dy + \left(1-\alpha\right)\frac{1}{n}Qb\left(\frac{1}{n}Q\right)dF\left(Q\right) \le 0.$$

When $\varepsilon = 0$, $nq_\varepsilon = \overline{Q}$ and $\Delta u = 0$. I show below that the same is true of both the first and second derivatives. I therefore derive a condition on the third derivative of Δu: when this derivative is negative, Δu will be negative for small $\varepsilon > 0$, a necessary condition for equilibrium.

To simplify analysis, let $b_\varepsilon \equiv \overline{b} + \varepsilon = b(q_\varepsilon)$. The first derivative of Δu is

$$\frac{d\Delta u}{d\varepsilon} = ndq_\varepsilon\left[\int_{q_\varepsilon}^{q_\varepsilon} v\left(y\right)dy - nq_\varepsilon b_\varepsilon + \left(n-\left(1-\alpha\right)\right)q_\varepsilon b_\varepsilon + \alpha\int_{q_\varepsilon}^{q_\varepsilon} b\left(y\right)dy + \left(1-\alpha\right)q_\varepsilon b_\varepsilon\right] f\left(nq_\varepsilon\right)$$
$$+\int_{nq_\varepsilon}^{\overline{Q}} -\left(n-1\right)dq_\varepsilon v\left(Q-\left(n-1\right)q_\varepsilon\right) - Q$$
$$+\left(n-\left(1-\alpha\right)\right)dq_\varepsilon b_\varepsilon + \left(n-\left(1-\alpha\right)\right)q_\varepsilon - \alpha dq_\varepsilon b_\varepsilon dF\left(Q\right)$$
$$= \int_{nq_\varepsilon}^{\overline{Q}} -\left(n-1\right)\left(v\left(Q-\left(n-1\right)q_\varepsilon\right) - b_\varepsilon\right)dq_\varepsilon - Q + \left(n-\left(1-\alpha\right)\right)q_\varepsilon dF\left(Q\right).$$

Since $nq_\varepsilon|_{\varepsilon=0} = \overline{Q}$, it follows that $d\Delta u/d\varepsilon|_{\varepsilon=0} = 0$.

The second derivative of Δu is

$$\frac{d^2\Delta u}{d\varepsilon^2} = -ndq_\varepsilon\left[-\left(n-1\right)\left(v\left(q_\varepsilon\right)-b_\varepsilon\right)dq_\varepsilon - nq_\varepsilon + \left(n-\left(1-\alpha\right)\right)q_\varepsilon\right]f\left(nq_\varepsilon\right)$$
$$+\int_{nq_\varepsilon}^{\overline{Q}} -\left(n-1\right)\left(-\left(n-1\right)v_q\left(Q-\left(n-1\right)q_\varepsilon\right)dq_\varepsilon - 1\right)dq_\varepsilon$$
$$-\left(n-1\right)\left(v\left(Q-\left(n-1\right)q_\varepsilon\right)-b_\varepsilon\right)d^2q_\varepsilon + \left(n-\left(1-\alpha\right)\right)dq_\varepsilon dF\left(Q\right)$$
$$= -ndq_\varepsilon\left[-\left(n-1\right)\left(v\left(q_\varepsilon\right)-b_\varepsilon\right)dq_\varepsilon - \left(1-\alpha\right)q_\varepsilon\right]f\left(nq_\varepsilon\right)$$
$$+\int_{nq_\varepsilon}^{\overline{Q}} -\left(n-1\right)\left(-\left(n-1\right)v_q\left(Q-\left(n-1\right)q_\varepsilon\right)dq_\varepsilon - 1\right)dq_\varepsilon$$
$$-\left(n-1\right)\left(v\left(Q-\left(n-1\right)q_\varepsilon\right)-b_\varepsilon\right)d^2q_\varepsilon + \left(n-\left(1-\alpha\right)\right)dq_\varepsilon dF\left(Q\right).$$

As in the case of the first derivative, the integral term in the second derivative drops out when $\varepsilon = 0$. The leading additive term is proportional to

$$-\left(n-1\right)\left(v\left(Q^\mu\right) - b\left(Q^\mu\right)\right)\frac{q}{b_q\left(Q^\mu\right)} - \left(1-\alpha\right)\frac{1}{n}\overline{Q}.$$

This is exactly the first-order condition at $q = Q^\mu$, and is therefore equal to zero. Then $d^2\Delta u/d\varepsilon^2|_{\varepsilon=0} = 0$.

To avoid unnecessary complications, when taking the third derivative I omit the inner integral derivative, since previous arguments imply that when $\varepsilon = 0$, the integral will evaluate to zero. Since the derivative can be nontrivially signed without this integral term, it can be ignored. Denote by $\tilde{d}^3\Delta u/\tilde{d}\varepsilon^3$ the third derivative of Δu, without this integral term.

$$
\begin{aligned}
\frac{\tilde{d}^3\Delta u}{\tilde{d}\varepsilon^3} &= -nd^2q_\varepsilon \underbrace{\left[-(n-1)(v(q_\varepsilon)-b_\varepsilon)\,dq_\varepsilon - (1-\alpha)\,q_\varepsilon\right]}_{\text{FOC}=0|_{\varepsilon=0}} f(nq_\varepsilon) \\
&\quad - n^2dq_\varepsilon^2 \underbrace{\left[-(n-1)(v(q_\varepsilon)-b_\varepsilon)\,dq_\varepsilon - (1-\alpha)\,q_\varepsilon\right]}_{\text{FOC}=0|_{\varepsilon=0}} df(nq_\varepsilon) \\
&\quad - ndq_\varepsilon\left[-(n-1)(v_q(q_\varepsilon)\,dq_\varepsilon - 1)\,dq_\varepsilon - (n-1)(v(q_\varepsilon)-b_\varepsilon)\,d^2q_\varepsilon - (1-\alpha)\,dq_\varepsilon\right] f(nq_\varepsilon) \\
&\quad - ndq_\varepsilon\left[\begin{array}{l} -(n-1)(-(n-1)\,v_q(q_\varepsilon)\,dq_\varepsilon - 1)\,dq_\varepsilon \\ \qquad -(n-1)(v(q_\varepsilon)-b_\varepsilon)\,d^2q_\varepsilon + (n-(1-\alpha))\,dq_\varepsilon \end{array}\right] f(nq_\varepsilon).
\end{aligned}
$$

Noting that $-dq_\varepsilon \geq 0$, when evaluated at $\varepsilon = 0$ the (adjusted) third derivative is proportional to

$$
\begin{aligned}
\left.\frac{\tilde{d}^3\Delta u}{\tilde{d}\varepsilon^3}\right|_{\varepsilon=0} &\propto -(n-1)(v_q(q_\varepsilon)\,dq_\varepsilon - 1)\,dq_\varepsilon - (n-1)(v(q_\varepsilon)-b_\varepsilon)\,d^2q_\varepsilon - (1-\alpha)\,dq_\varepsilon \\
&\quad + (n-1)^2\,v_q(q_\varepsilon)\,dq_\varepsilon^2 + (n-1)\,dq_\varepsilon + (n-1)(v(q_\varepsilon)-b_\varepsilon)\,d^2q_\varepsilon + (n-(1-\alpha))\,dq_\varepsilon \\
&= (n-1)(n-2)\,v_q(q_\varepsilon)\,dq_\varepsilon^2 - 2(n-1)(v(q_\varepsilon)-b_\varepsilon)\,d^2q_\varepsilon \\
&\quad + \underbrace{\left[2(n-1) - (1-\alpha) + (n-(1-\alpha))\right]}_{3n+2\alpha-4} dq_\varepsilon.
\end{aligned}
$$

Implicit differentiation gives $dq_\varepsilon = 1/b_q$, and $d^2q_\varepsilon = -b_{qq}/b_q^3$. Then the (adjusted) third derivative is weakly negative if and only if

$$
(n-1)(n-2)\,v_q(Q^\mu) + 2(n-1)(v(Q^\mu) - b(Q^\mu))\frac{b_{qq}(Q^\mu)}{b_q(Q^\mu)} + (3n+2\alpha-4)\,b_q(Q^\mu) \leq 0.
$$

In equilibrium,

$$
\begin{aligned}
&-(n-1)(v(q)-b(q))\frac{f(nq)}{b_q(q)} = \alpha(1-F(nq))+(1-\alpha)qf(nq) \\
\implies &-(n-1)\left(v_q(Q^\mu)-b_q(Q^\mu)\right)\frac{f\left(\overline{Q}\right)}{b_q(Q^\mu)} \\
&-n(n-1)\left(v(Q^\mu)-b(Q^\mu)\right)\frac{df\left(\overline{Q}\right)}{b_q(Q^\mu)} \\
&+(n-1)\left(v(Q^\mu)-b(Q^\mu)\right)\frac{f\left(\overline{Q}\right)b_{qq}(Q^\mu)}{b_q(Q^\mu)^2} = -n\alpha f\left(\overline{Q}\right)+(1-\alpha)f\left(\overline{Q}\right)+n(1-\alpha)\overline{Q}df\left(\overline{Q}\right) \\
\implies &-(n-1)\left(v_q(Q^\mu)-b_q(Q^\mu)\right) \\
&+(n-1)\left(v(Q^\mu)-b(Q^\mu)\right)\frac{b_{qq}(Q^\mu)}{b_q(Q^\mu)} = (1-(n+1)\alpha)b_q(Q^\mu).
\end{aligned}
$$

Substituting this into the desired inequality gives

$$
\begin{aligned}
&(n-1)(n-2)v_q(Q^\mu)+2\left[(n-1)\left(v_q(Q^\mu)-b_q(Q^\mu)\right)+(1-(n+1)\alpha)b_q(Q^\mu)\right] \\
&\quad+(3n+2\alpha-4)b_q(Q^\mu) \\
&=n(n-1)v_q(Q^\mu)+(-2(n-1)+2(1-(n+1)\alpha)+(3n+2\alpha-4))b_q(Q^\mu) \\
&=n(n-1)v_q(Q^\mu)+(n-2n\alpha)b_q(Q^\mu) \\
&=n(n-1)v_q(Q^\mu)-(n-2n\alpha)(n-1)\left(v(Q^\mu)-b(Q^\mu)\right)\frac{1}{(1-\alpha)Q^\mu} \\
&\propto Q^\mu v_q(Q^\mu)-\frac{1-2\alpha}{1-\alpha}\left(v(Q^\mu)-b(Q^\mu)\right)\leq 0.
\end{aligned}
$$

Evaluated at $\varepsilon=0$, the third derivative is

$$
\begin{aligned}
&-\left[-2(n-1)\left(\hat{v}\left(\overline{Q}\right)-\overline{b}\right)\frac{d^2q_\varepsilon}{d\varepsilon}+(n-1)(n-2)n\hat{v}_Q\left(\overline{Q}\right)\left(\frac{dq_\varepsilon}{d\varepsilon}\right)^2+(2\alpha+3n-4)\frac{dq_\varepsilon}{d\varepsilon}\right] \\
&\quad\times n\frac{dq_\varepsilon}{d\varepsilon}dF\left(\overline{Q}\right)\leq 0.
\end{aligned}
$$

Noting that $dq_\varepsilon/d\varepsilon = 1/np_Q$, $d^2q_\varepsilon/d\varepsilon^2 = -p_{QQ}/np_Q^3$, and substituting in for the solution to the agent's first-order conditions, this is equivalent to

$$
-2(n-1)\frac{p_{QQ}\left(\overline{Q}\right)}{\tilde{H}\left(\overline{Q}\right)}+(n-1)(n-2)\hat{v}_Q\left(\overline{Q}\right)+(2\alpha+3n-4)p_Q\left(\overline{Q}\right)\leq 0.
$$

Since $p_Q = (p - \hat{v})\tilde{H}$, substitution into this expression gives

$$-2(n-1)\left(p\left(\overline{Q}\right) - \hat{v}\left(\overline{Q}\right)\right)\left(\tilde{H}\left(\overline{Q}\right) + \frac{\tilde{H}_Q\left(\overline{Q}\right)}{\tilde{H}\left(\overline{Q}\right)}\right) + 2(n-1)\hat{v}_Q\left(\overline{Q}\right)$$
$$+ (n-1)(n-2)\hat{v}_Q\left(\overline{Q}\right) + (2\alpha + 3n - 4)\left(p\left(\overline{Q}\right) - \hat{v}\left(\overline{Q}\right)\right)\tilde{H}\left(\overline{Q}\right) \leq 0.$$

Finally, replacing $\tilde{H}$ and $\tilde{H}_Q$ yields

$$\frac{1}{(1-\alpha)\overline{Q}}(n - 2\alpha n)\left(p\left(\overline{Q}\right) - \hat{v}\left(\overline{Q}\right)\right) + n\hat{v}_Q\left(\overline{Q}\right) \leq 0.$$

The desired inequality is immediate. □

B.2 Proofs for Section 3.2

Proof of Lemma 4. Under the symmetric bidding profile $(b^i)_{i=1}^n$, bidder i's expected utility is

$$\begin{aligned}\mathbb{E}[u] &= \int_0^{\overline{Q}} \int_0^{\frac{1}{n}Q} v(x) - \alpha b^i(x)\,dx - (1-\alpha)\frac{1}{n}Qb\left(\frac{1}{n}Q\right) dF(Q) \\ &= \frac{1}{n}\int_0^{\overline{Q}} \left(v\left(\frac{1}{n}Q\right) - \alpha b\left(\frac{1}{n}Q\right)\right)(1 - F(Q)) - (1-\alpha)Qb\left(\frac{1}{n}Q\right) dQ.\end{aligned}$$

By definition, $b^i(Q/n) \geq \underline{p}$ for all $Q \leq \overline{Q}$. Then bidder i's expected utility is bounded above by

$$\begin{aligned}\mathbb{E}[u] &\leq \frac{1}{n}\int_0^{\overline{Q}} \left(v\left(\frac{1}{n}Q\right) - \alpha\underline{p}\right)(1 - F(Q)) + (1-\alpha)Q\underline{p}dQ \\ &= \int_0^{\overline{Q}} \left(\hat{v}(Q) - \underline{p}\right)(1 - F(Q))\,dQ.\end{aligned}$$

Consider the alternative bid $\hat{b}$,

$$\hat{b}(q) = \min\{\overline{p}, v(q)\}.$$

If bidder i submits bid $\hat{b}$ while her opponents employ the symmetric bidding profile $(b^j)_{j\neq i}$, bidder i's expected utility is[38]

$$\begin{aligned}\mathbb{E}[\hat{u}] &= \int_0^{v^{-1}(\overline{p})} \int_0^{Q} (v(x) - \overline{p})\,dx dF(Q) + \left(1 - F\left(v^{-1}(\overline{p})\right)\right)\int_0^{v^{-1}(\overline{p})} v(x) - \overline{p}dx \\ &= \int_0^{v^{-1}(\overline{p})} (v(Q) - \overline{p})(1 - F(Q))\,dQ.\end{aligned}$$

If $(b^i)_{i=1}^n$ is an equilibrium, deviating to $\hat{b}$ cannot be profitable; the result follows. □

[38] If the symmetric bidding function b^j is constant at level $\overline{p}$, this argument fails due to tiebreaking. Allowing an arbitrary small increase of $\hat{b}$ to $\overline{p} + \delta$ and letting $\delta \searrow 0$ will return the desired result.

Proof of Lemma 5. Following the proof of Theorem 3, for a given minimum price $\underline{p}$ the maximum price $\overline{p}$ is

$$\begin{aligned}\overline{p} &= \hat{v}(0) + \int_0^{\overline{Q}} \exp\left(-\int_0^x \tilde{H}(y)\,dy\right) \hat{v}_Q(x)\,dx - \left(\hat{v}\left(\overline{Q}\right) - \underline{p}\right) \exp\left(-\int_0^{\overline{Q}} \tilde{H}(x)\,dx\right) \\ &= \exp\left(-\int_0^{\overline{Q}} \tilde{H}(x)\,dx\right) \underline{p} + \int_0^{\overline{Q}} \exp\left(-\int_0^x \tilde{H}(y)\,dy\right) \tilde{H}(x)\,\hat{v}(x)\,dx.\end{aligned}$$

It is then straightforward to bound $\overline{p}$ by

$$\begin{aligned}\overline{p} \leq \exp\left(-\int_0^{\overline{Q}} \tilde{H}(x)\,dx\right) \underline{p} &+ \int_0^{\overline{Q}-\varepsilon} \exp\left(-\int_0^x \tilde{H}(y)\,dy\right) \tilde{H}(x)\,\hat{v}(0)\,dx \\ &+ \int_{\overline{Q}-\varepsilon}^{\overline{Q}} \exp\left(-\int_0^x \tilde{H}(y)\,dy\right) \tilde{H}(x)\,\hat{v}\left(\overline{Q}-\varepsilon\right)dx.\end{aligned}$$

Noting that $\exp(-\int_0^x \tilde{H}(y)dy)\tilde{H}(x) = -d[\exp(-\int_0^x \tilde{H}(y)dy)]/dx$, we find

$$\begin{aligned}\overline{p} &\leq \exp\left(-\int_0^{\overline{Q}} \tilde{H}(x)\,dx\right) \underline{p} + \left(1 - \exp\left(-\int_0^{\overline{Q}-\varepsilon} \tilde{H}(y)\,dy\right)\right) \hat{v}(0) \\ &\quad + \left(\exp\left(-\int_0^{\overline{Q}-\varepsilon} \tilde{H}(y)\,dy\right) - \exp\left(-\int_0^{\overline{Q}} \tilde{H}(y)\,dy\right)\right) \hat{v}\left(\overline{Q}-\varepsilon\right) \\ &= \left(\underline{p} - \hat{v}\left(\overline{Q}-\varepsilon\right)\right) \exp\left(-\int_0^{\overline{Q}} \tilde{H}(x)\,dx\right) + \hat{v}(0) - \left(\hat{v}(0) - \hat{v}\left(\overline{Q}-\varepsilon\right)\right) \exp\left(-\int_0^{\overline{Q}-\varepsilon} \tilde{H}(y)\,dy\right).\end{aligned}$$

It follows that a further upper bound for $\overline{p}$ can be found by bounding the exponential terms below, which can be achieved by bounding their interior integrals above.

We compute

$$\begin{aligned}\int_0^{\overline{Q}-\varepsilon} \tilde{H}(y)\,dy &= \int_0^{\overline{Q}-\varepsilon} \frac{(n-1)\,f(y)}{n\alpha\,(1-F(y)) + (1-\alpha)\,y f(y)} dy \\ &\leq \frac{n-1}{n\alpha} \int_0^{\overline{Q}-\varepsilon} \frac{f(y)}{1-F(y)} = -\frac{n-1}{n\alpha} \ln(1-\varepsilon).\end{aligned}$$

It follows that when supply is ε-concentrated,

$$\exp\left(-\int_0^{\overline{Q}-\varepsilon} \tilde{H}(y)\,dy\right) \geq (1-\varepsilon)^{\frac{n-1}{n\alpha}}.$$

Similarly, we compute

$$\begin{aligned}\int_0^{\overline{Q}} \tilde{H}(y)\,dy &= \int_0^{\overline{Q}-\varepsilon} \tilde{H}(y)\,dy + \int_{\overline{Q}-\varepsilon}^{\overline{Q}} \tilde{H}(dy)\,dy \\ &\leq -\frac{n-1}{n\alpha}\ln(1-\varepsilon) + \int_{\overline{Q}-\varepsilon}^{\overline{Q}} \frac{n-1}{1-\alpha}dy \\ &= -\frac{n-1}{n\alpha}\ln(1-\varepsilon) + \frac{n-1}{1-\alpha}\varepsilon.\end{aligned}$$

It follows that when supply is ε-concentrated,

$$\exp\left(-\int_0^{\overline{Q}} \tilde{H}(y)\,dy\right) \geq (1-\varepsilon)^{\frac{n-1}{n\alpha}} \exp\left(\frac{n-1}{1-\alpha}\varepsilon\right).$$

Returning to the upper bound on the maximum market-clearing price, this gives

$$\begin{aligned}\overline{p} \leq\ & (1-\varepsilon)^{\frac{n-1}{n\alpha}} \exp\left(\frac{n-1}{1-\alpha}\varepsilon\right)\underline{p} \\ & + \left(1-(1-\varepsilon)^{\frac{n-1}{n\alpha}}\right)\hat{v}(0) + (1-\varepsilon)^{\frac{n-1}{n\alpha}}\left(1-\exp\left(\frac{n-1}{n\alpha}\varepsilon\right)\right)\hat{v}\left(\overline{Q}-\varepsilon\right).\end{aligned}$$

It follows that for any $\alpha \in (0,1)$ and any $\delta > 0$, there is ε such that whenever supply is ε-concentrated, $\overline{p} - \underline{p} < \delta$.

For the specific case of $\alpha = 1$ (the discriminatory auction), a similar analysis proceeds by bounding $\exp(-\int_0^{\overline{Q}} \tilde{H}(x)dx) \geq 0$, which implies that $\overline{p} \leq \hat{v}(\overline{Q}-\varepsilon)$. Since $\underline{p} = \hat{v}(\overline{Q})$ in the unique equilibrium of the discriminatory auction, the result is immediate. □

Proof of Proposition 1. From Lemma 5, for any $\delta > 0$ there is ε such that whenever supply is ε-concentrated, $|\overline{p} - \underline{p}| < \delta$ in all equilibria of the partially-discriminatory auction. Now suppose that $\underline{p} < \hat{v}(\overline{Q}) - \gamma$, and that supply is sufficiently concentrated so that $\overline{p} < \underline{p} + \delta$. From Lemma 4, for $\underline{p}$ to be a valid equilibrium initial condition for the market-price function it must be that

$$\begin{aligned} & \int_0^{Q^\mu} \left(v(q) - \underline{p}\right)\left(1 - F^\mu(q)\right)dq \geq \int_0^{v^{-1}(\overline{p})} \left(v(Q) - \overline{p}\right)\left(1 - F(Q)\right)dQ \\ \implies\quad & \int_0^{Q^\mu} \left(v(q) - \underline{p}\right)\left(1 - F^\mu(q)\right)dq \geq \int_0^{v^{-1}\left(\underline{p}+\delta\right)} \left(v(Q) - \left[\underline{p}+\delta\right]\right)\left(1 - F(Q)\right)dQ \\ \iff\quad & \int_0^{Q^\mu} \left(v(q) - \underline{p}\right)\left(F(q) - F^\mu(q)\right)dq \geq \int_{Q^\mu}^{v^{-1}\left(\underline{p}+\delta\right)} \left(v(Q) - \underline{p}\right)\left(1 - F(Q)\right)dQ \\ & \qquad\qquad - \delta\int_0^{v^{-1}\left(\underline{p}+\delta\right)} 1 - F(Q)\,dQ. \end{aligned}$$

The left-hand side is strictly negative, since $F(q) < F^\mu(q)$, so to derive a contradiction it suffices to show that the right-hand side is weakly positive. The right-hand side is bounded below by

$$\int_{Q^\mu}^{v^{-1}\left(\underline{p}+\delta\right)} \left(v\left(Q\right)-\underline{p}\right)\left(1-F\left(Q\right)\right) dQ - \delta \int_{0}^{v^{-1}\left(\underline{p}+\delta\right)} 1-F\left(Q\right) dQ$$
$$\geq \int_{Q^\mu}^{v^{-1}\left(\underline{p}+\delta\right)} \left(v\left(Q\right)-\underline{p}\right)\left(1-F\left(Q\right)\right) dQ - \delta v^{-1}\left(\underline{p}+\delta\right).$$

Since supply is ε-concentrated, in a sufficiently small neighborhood of Q^μ we have $1-F(Q) \geq 1-\varepsilon$; and since marginal values are Lipschitz continuous (with modulus M), we have

$$v\left(Q\right)-\underline{p} \geq v\left(Q^\mu\right) - M\left(Q-Q^\mu\right) - \underline{p} > \gamma - M\left(Q-Q^\mu\right).$$

Furthermore, $v^{-1}(\underline{p}+\delta) \geq Q^\mu + (\gamma-\delta)/M$. Then

$$\int_{Q^\mu}^{v^{-1}\left(\underline{p}+\delta\right)} \left(v\left(Q\right)-\underline{p}\right)\left(1-F\left(Q\right)\right) dQ - \delta v^{-1}\left(\underline{p}+\delta\right) \geq \frac{1}{2M}\left(1-\varepsilon\right)\left(\gamma-\delta\right)\gamma - \delta v^{-1}\left(\underline{p}+\delta\right).$$

When δ is small, this is positive. Then for $(b^i)_{i=1}^n$ to be an equilibrium it must be that γ is small, implying that $\underline{p}$ cannot be too far from $v(Q^\mu)$. The result follows. $\square$

B.3 Proofs for Section 3.3

Proof of Proposition 2. Lemma 2 implies that $R^n(\overline{Q}) = \overline{p}(\overline{Q}) - \underline{p}(\overline{Q})$ is constant in n. Since both $\overline{p}$ and $\underline{p}$ must solve the equilibrium market-clearing equation,

$$R_Q^n\left(q\right) = \overline{p}_Q\left(Q\right) - \underline{p}_Q\left(Q\right) = \tilde{H}\left(q\right)\left(\overline{p}\left(Q\right) - \underline{p}\left(Q\right)\right) \geq 0.$$

Thus R^n is increasing in Q.

To consider the effect of the number of bidders n, note that $\tilde{H}$ is increasing in n:[39]

$$\frac{d}{dn}\tilde{H}\left(Q\right) \overset{\text{sign}}{=} f\left(Q\right)\left[n\alpha\left(1-F\left(Q\right)\right) + \left(1-\alpha\right)Qf\left(Q\right)\right] - \left(n-1\right)f\left(Q\right)\left(1-F\left(Q\right)\right)\alpha$$
$$= \left[\left(1-F\left(Q\right)\right)\alpha + \left(1-\alpha\right)Qf\left(Q\right)\right]f\left(Q\right) > 0.$$

Then letting $n' > n$, if q is such that $R^{n'}(Q) = R^n(Q)$, then $R_Q^{n'}(Q) > R_Q^n(Q)$. It follows that R^n and $R^{n'}$ may cross only once. Since they meet when $Q = \overline{Q}$, it follows that $R^{n'}(Q) \leq R^n(Q)$ for all $Q \in [0, \overline{Q}]$, and this inequality is strict for all $Q < \overline{Q}$. $\square$

Proof of Theorem 4. Corollary 2 shows that the minimum feasible market clearing price is increasing in α. Since in any equilibrium bids are continuous, it is sufficient to show that in the maximum-bid equilibrium, $\overline{b}^\alpha(0)$ is decreasing in α. Suppose otherwise; then there are α and α', $\alpha < \alpha'$, such

[39] Although n is integer-valued, it is more straightforward to analyze the derivative of $\tilde{H}$ with respect to n. Since this is everywhere-positive, $\tilde{H}$ is also increasing on the integers.

that $\overline{b}^{\alpha}(0) < \overline{b}^{\alpha'}(0)$. For any q such that $\overline{b}^{\alpha}(q) < \overline{b}^{\alpha'}(q)$, it must be that $|\overline{b}^{\alpha}(q)| > |\overline{b}^{\alpha'}(q)|$, and it follows that $\overline{b}^{\alpha}$ and $\overline{b}^{\alpha'}$ cannot cross. This contradicts the construction of maximum-bid equilibrium, where $\overline{b}(Q^{\mu}) = v(Q^{\mu})$.

With regard to changes in the number of bidders n, recall the market clearing first order condition:

$$b_q(q) = -\frac{(n-1)(v(q) - b(q)) f^{\mu}(q)}{n\alpha(1 - F^{\mu}(q)) + (1-\alpha) q f^{\mu}(q)} = H^n(q)(v(q) - b(q)).$$

Since $H^n(q)$ is increasing in n, if $v(q) - b^n(q) = v(q) - b^{n+1}(q)$, it follows that $|b^n_q(q)| < |b^{n'}_q(q)|$, and bid curves can intersect at most once. For all $\alpha < 1$, the fact that $\mathrm{P}(\alpha)$ is independent of n implies that $\mathbf{P}(\alpha; n) \subseteq \mathbf{P}(\alpha; n')$. For the discriminatory auction, $\alpha = 1$, equilibrium is unique, and limiting arguments for $q \nearrow Q^{\mu}$ imply that $b^n(q) < b^{n'}(q)$. $\square$

B.4 Proofs for Section 3.4

Proof of Proposition 4. Equilibrium expected per-capita revenue is

$$\begin{aligned}
&\mathbb{E}_{F^{\mu}}\left[\alpha \int_0^q b(x)\,dx + (1-\alpha) q b(q)\right] \\
&= \int_0^{Q^{\mu}} \alpha b(q)(1 - F^{\mu}(q)) + (1-\alpha) q b(q) f^{\mu}(q)\,dq \\
&= \int_0^{Q^{\mu}} (\alpha(1 - F^{\mu}(q)) + (1-\alpha) q f^{\mu}(q)) \int_q^{Q^{\mu}} v(x)\,dF^{\alpha,q}(x) \\
&= \int_0^{Q^{\mu}} \alpha \frac{d}{dy}\left[y(1 - F^{\mu}(y))^{1-\frac{1}{\alpha}}\right]\Big|_{y=q} (1 - F^{\mu}(q))^{\frac{1}{\alpha}} \int_q^{Q^{\mu}} \frac{1}{\alpha} v(x)\left(\frac{1 - F^{\mu}(x)}{1 - F^{\mu}(Q)}\right)^{\frac{1}{\alpha}-1} \frac{f^{\mu}(x)}{1 - F^{\mu}(q)} dx dq \\
&= \int_0^{Q^{\mu}} q(1 - F^{\mu}(q))^{1-\frac{1}{\alpha}} v(q)(1 - F^{\mu}(q))^{\frac{1}{\alpha}-1} f(q)\,dq = \mathbb{E}_{F^{\mu}}[qv(q)].
\end{aligned}$$

Then in large markets, all mixed-price auctions generate revenue identical to a uniform-price auction with truthful bids. $\square$

C Proofs for Section 4 (Conjugate equilibrium)

Proof of Theorem 6. In equilibrium, market prices solve

$$p_Q(Q) = (p(Q) - \hat{v}(Q))\tilde{H}(Q). \tag{12}$$

In the polynomial-Lomax model

$$\tilde{H}(Q) = \frac{(n-1)\lambda}{n\alpha\overline{Q} + ((1-\alpha)\lambda - n\alpha) Q} \equiv \frac{c_1}{c_2 + c_3 Q}.$$

Then equation (12) can be written

$$(c_2 + c_3 Q)\, p_Q(Q) = (p(Q) - \hat{v}(Q))\, c_1. \tag{13}$$

When bids are conjugate, the market price is a polynomial of the same order as marginal value. Matching coefficients gives a system of equations,

$$\begin{aligned}
\overline{k} c_3 p_{\overline{k}} &= \left(p_{\overline{k}} - \hat{v}_{\overline{k}}\right) c_1 & \implies \quad p_{\overline{k}} &= \frac{c_1 \hat{v}_{\overline{k}}}{c_1 - \overline{k} c_3}; \\
(k+1)\, c_2 p_{k+1} + k c_3 p_k &= (p_k - \hat{v}_k)\, c_1 \quad \left(k < \overline{k}\right) & \implies \quad p_k &= \frac{c_1 \hat{v}_k + (k+1)\, c_2 p_{k+1}}{c_1 - k c_3}.
\end{aligned}$$

□

Proof of Theorem 7. If conjugate equilibrium bids exist, any market-clearing price function p can be written as $p(Q) = \check{p}(Q) + r(Q)$, where $\check{p}$ is the conjugate market-clearing price and r is a residual term. Then

$$\begin{aligned}
-\left(\hat{v}(Q) - p(Q)\right) \tilde{H}(Q) = p_Q(Q) \quad &\iff \quad -\left(\hat{v}(Q) - \left(\check{p}(Q) + r(Q)\right)\right) \tilde{H}(Q) = \check{p}_Q(Q) + r_Q(Q) \\
&\iff \quad r(Q)\, \tilde{H}(Q) = r_Q(Q).
\end{aligned}$$

The latter implication follows from the fact that $\check{p}$ solves the market-clearing price equation. Solving the resulting differential equation in r gives

$$r(Q) = r\left(\overline{Q}\right) \exp\left(-\int_Q^{\overline{Q}} \tilde{H}(x)\, dx\right).$$

In the polynomial-Lomax model we have

$$\begin{aligned}
\tilde{H}(Q) &= \frac{(n-1)\lambda}{n\alpha\left(\overline{Q} - Q\right) + (1-\alpha)\lambda Q}, \\
\implies \quad \exp\left(-\int_Q^{\overline{Q}} \tilde{H}(x)\, dx\right) &= \left(\frac{n\alpha\overline{Q} + ((1-\alpha)\lambda - n\alpha)\, Q}{(1-\alpha)\lambda\overline{Q}}\right)^{\frac{(n-1)\lambda}{(1-\alpha)\lambda - n\alpha}}.
\end{aligned}$$

Multiplying the above expression by $(1-\alpha)\lambda/(n\alpha)$ gives the desired result (noting the arbitrary constant C in the expression in the Theorem). □

C.1 Proofs for Section 4.1

Lemma 10. *The moments of the Lomax distribution are given by*

$$\mathbb{E}\left[Q^k\right] = \prod_{t=1}^{k} \frac{t\overline{Q}}{\lambda + t}.$$

Proof. When $k > 0$ direct computation yields

$$\begin{aligned}\mathbb{E}\left[Q^k\right] &= \frac{\lambda}{\overline{Q}}\int_0^{\overline{Q}}\left(1-\frac{Q}{\overline{Q}}\right)^{\lambda-1} Q^k dQ \\ &= -\left(1-\frac{Q}{\overline{Q}}\right)^{\lambda} Q^k\Bigg|_{Q=0}^{\overline{Q}} + k\int_0^{\overline{Q}}\left(1-\frac{Q}{\overline{Q}}\right)^{\lambda} Q^{k-1}dQ \\ &= k\int_0^{\overline{Q}}\left(1-\frac{Q}{\overline{Q}}\right)^{\lambda} Q^{k-1}dQ = \frac{k\overline{Q}}{\lambda}\mathbb{E}\left[Q^{k-1}\right] - \frac{k}{\lambda}\mathbb{E}\left[Q^k\right].\end{aligned}$$

It follows that

$$\mathbb{E}\left[Q^k\right] = \frac{k\overline{Q}}{\lambda+k}\mathbb{E}\left[Q^{k-1}\right].$$

Since $\mathbb{E}[Q^0] = 1$, the result follows. $\square$

Lemma 11 (Equilibrium revenue)**.** *In conjugate equilibrium of the polynomial-Lomax model, expected revenue is*

$$\mathbb{E}\left[\pi\right] = \sum_{k=0}^{\bar{k}}\left\{\left(1-\frac{\alpha k}{k+1}\right)\left(\prod_{t=1}^{k+1}\frac{t\overline{Q}}{\lambda+t}\right)\check{p}_k\right\}.$$

Proof. Expected revenue in the mixed-price auction is given by

$$\begin{aligned}\mathbb{E}\left[\pi\right] &= \mathbb{E}\left[\alpha\int_0^Q p\left(x\right)dx + \left(1-\alpha\right)Qp\left(Q\right)\right] \\ &= \mathbb{E}\left[\alpha\int_0^Q\sum_{k=0}^{\bar{k}}p_k x^k dx + \left(1-\alpha\right)Q\sum_{k=0}^{\bar{k}}p_k Q^k\right] \\ &= \mathbb{E}\left[\alpha\sum_{k=0}^{\bar{k}}\frac{p_k}{k+1}Q^{k+1} + \left(1-\alpha\right)\sum_{k=0}^{\bar{k}}p_k Q^{k+1}\right] \\ &= \sum_{k=0}^{\bar{k}}\left(1-\frac{\alpha k}{k+1}\right)p_k\mathbb{E}\left[Q^{k+1}\right] = \sum_{k=0}^{\bar{k}}\left\{\left(1-\frac{\alpha k}{k+1}\right)\left(\prod_{t=1}^{k+1}\frac{t\overline{Q}}{\lambda+t}\right)p_k\right\}.\end{aligned}$$

$\square$

Proof of Proposition 3. When $\hat{v}_1 = 0$, all conjugate equilibria have $\check{p}_0 = \hat{v}_0$ and $\check{p}_1 = 0$, and revenue is independent of the price mixture α.

By Lemma 11, expected revenue in conjugate equilibrium of the linear-Lomax model is

$$\mathbb{E}\left[\pi\right] = \check{p}_0\frac{\overline{Q}}{\lambda+1} + \left(1-\frac{1}{2}\alpha\right)\check{p}_1\frac{2\overline{Q}^2}{\left(\lambda+1\right)\left(\lambda+2\right)} \propto \frac{\lambda+2}{\overline{Q}}\check{p}_0 + \left(2-\alpha\right)\check{p}_1.$$

Since $\check{p}_0 = \hat{v}_0 + n\alpha\overline{Q}\check{p}_1/((n-1)\lambda)$, the derivative of expected revenue with respect to $\hat{v}_1$ is

$$\frac{d}{d\hat{v}_1}\mathbb{E}\left[\pi\right] \propto \frac{(\lambda+2)\, n\alpha}{(n-1)\,\lambda}\frac{d\check{p}_1}{d\hat{v}_1} + (2-\alpha)\frac{d\check{p}_1}{d\hat{v}_1} \propto \left[(\lambda+2)\, n\alpha + (2-\alpha)(n-1)\,\lambda\right]\frac{d\check{p}_1}{d\hat{v}_1}.$$

Substituting in for $d\check{p}_1/d\hat{v}_1$ and collecting terms gives

$$\begin{aligned}\frac{d}{d\hat{v}_1}\mathbb{E}\left[\pi\right] &= \left[2(n-1)\,\lambda + (2n+\lambda)\,\alpha\right]\frac{(n-1)\,\lambda}{(n-2)\,\lambda + (n+\lambda)\,\alpha}\\ &\propto \frac{2(n-1)\,\lambda + (2n+\lambda)\,\alpha}{(n-2)\,\lambda + (n+\lambda)\,\alpha} = 1 + \underbrace{\frac{\lambda+\alpha}{(n-2)\,\lambda + (n+\lambda)\,\alpha}}_{J(\alpha)} n.\end{aligned}$$

Note that all terms removed in the chain of proportionality are independent of α.

Now see that

$$\frac{dJ}{d\alpha} \overset{\text{sign}}{=} (n-2)\,\lambda + (n+\lambda)\,\alpha - (n+\lambda)(\lambda+\alpha) = -(2+\lambda)\,\lambda < 0.$$

It follows that $d^2\mathbb{E}[\pi]/d\hat{v}_1 d\alpha < 0$, and revenue is increasing more slowly in $\hat{v}_1$ the larger is α. Since all mixed-price auctions raise identical conjugate revenue when $\hat{v}_1 = 0$, it follows that expected revenue is strictly increasing in α whenever $\hat{v}_1 < 0$. $\square$

Proof of Proposition 4. In general, the conjugate equilibrium of the uniform-price auction will not be revenue-maximizing. Then to compare discriminatory auction revenue against uniform-price auction revenue, I analyze the maximal-bid equilibrium of the uniform-price auction, where $b(\overline{Q}/n) = v(\overline{Q}/n)$. Per Theorem 2, equilibrium maximum market-clearing prices are

$$\begin{aligned}\overline{p}\,(Q) &= \hat{v}\left(\overline{Q}\right)\exp\left(-\int_Q^{\overline{Q}}\tilde{H}\,(x)\,dx\right) + \int_Q^{\overline{Q}}\exp\left(-\int_Q^x \tilde{H}\,(y)\,dy\right)\tilde{H}\,(x)\,\hat{v}\,(x)\,dx\\ &= \hat{v}\,(Q) + \int_Q^{\overline{Q}}\exp\left(-\int_Q^x \tilde{H}\,(y)\,dy\right)\hat{v}_Q\,(x)\,dx.\end{aligned}$$

The latter equality follows from integration by parts.

Now, note that

$$\begin{aligned}\tilde{H}^{\text{PAB}}\,(x) &= \frac{n-1}{n}\left(\frac{f\,(x)}{1-F\,(x)}\right) &&\implies \exp\left(-\int_Q^x \tilde{H}^{\text{PAB}}\,(y)\,dy\right) = \left(\frac{1-F\,(x)}{1-F\,(Q)}\right)^{\frac{n-1}{n}},\\ \tilde{H}^{\text{UPA}}\,(x) &= \frac{n-1}{x} &&\implies \exp\left(-\int_Q^x \tilde{H}^{\text{UPA}}\,(y)\,dy\right) = \left(\frac{Q}{x}\right)^{n-1}.\end{aligned}$$

Substituting in for the Lomax distribution, this gives

$$\overline{p}^{\mathrm{PAB}}(Q) = \hat{v}(Q) + \int_{Q}^{\overline{Q}} \left(\frac{\overline{Q} - x}{\overline{Q} - Q} \right)^{\left(\frac{n-1}{n}\right)\lambda} \hat{v}_Q(x)\, dx,$$

$$\overline{p}^{\mathrm{UPA}}(Q) = \hat{v}(Q) + \int_{Q}^{\overline{Q}} \left(\frac{Q}{x} \right)^{n-1} \hat{v}_Q(x)\, dx.$$

In the linear-Lomax model, $\hat{v}(Q) = \hat{v}_0 + \hat{v}_1 Q$, where $\hat{v}_0 > 0$ and $\hat{v}_1 < 0$. Substituting in then gives a closed form for revenue-maximizing equilibrium prices,

$$\begin{aligned}
\overline{p}^{\mathrm{PAB}}(Q) &= \hat{v}_0 + \hat{v}_1 Q + \frac{n}{n + (n-1)\lambda} \hat{v}_1 \left(\overline{Q} - Q\right) \\
&= \left[\hat{v}_0 + \frac{n \hat{v}_1 \overline{Q}}{n + (n-1)\lambda} \right] + \left[\frac{(n-1)\lambda}{n + (n-1)\lambda} \right] \hat{v}_1 Q, \\
\overline{p}^{\mathrm{UPA}}(Q) &= \hat{v}_0 + \hat{v}_1 Q + \frac{1}{2-n} Q^{n-1} \left(\overline{Q}^{2-n} - Q^{2-n} \right) \hat{v}_1 \\
&= \hat{v}_0 + \left[\frac{1-n}{2-n} \right] \hat{v}_1 Q + \left[\frac{\overline{Q}^{2-n}}{2-n} \right] \hat{v}_1 Q^{n-1}.
\end{aligned}$$

In each equilibrium, expected revenue is given by

$$\mathbb{E}\left[\pi^{\mathrm{PAB}}\right] = \int_0^{\overline{Q}} (1 - F(Q))\, \overline{p}^{\mathrm{PAB}}(Q)\, dQ,$$

$$\mathbb{E}\left[\pi^{\mathrm{UPA}}\right] = \int_0^{\overline{Q}} Q \overline{p}^{\mathrm{UPA}}(Q) f(Q)\, dQ = \int_0^{\overline{Q}} (1 - F(Q)) \left(\overline{p}^{\mathrm{UPA}}(Q) + Q \overline{p}_Q^{\mathrm{UPA}}(Q) \right) dQ.$$

From the above, we have

$$\overline{p}^{\mathrm{UPA}}(Q) + Q \overline{p}^{\mathrm{UPA}}(Q) = \hat{v}_0 + 2 \left[\frac{1-n}{2-n} \right] \hat{v}_1 Q + n \left[\frac{\overline{Q}^{2-n}}{2-n} \right] \hat{v}_1 Q^{n-1}.$$

Then the difference in expected revenue is

$$
\begin{aligned}
&\mathbb{E}\left[\pi^{\mathrm{PAB}}-\pi^{\mathrm{UPA}}\right] \\
&=\int_0^{\overline{Q}}(1-F(Q))\left(\left[\frac{n\hat{v}_1\overline{Q}}{n+(n-1)\lambda}\right]+\left[\frac{(n-1)\lambda\hat{v}_1}{n+(n-1)\lambda}-\frac{2(1-n)\hat{v}_1}{2-n}\right]Q-n\left[\frac{\overline{Q}^{2-n}}{2-n}\right]\hat{v}_1Q^{n-1}\right)dQ \\
&\propto\int_0^{\overline{Q}}\left(\overline{Q}-Q\right)^{\lambda}\Big(\left[n(2-n)\overline{Q}\right]+\left[(2-n)(n-1)\lambda-2(1-n)(n+(n-1)\lambda)\right]Q \\
&\qquad\qquad -\left[(n+(n-1)\lambda)n\overline{Q}^{2-n}\right]Q^{n-1}\Big)dQ \\
&\propto\int_0^{\overline{Q}}\left(\overline{Q}-Q\right)^{\lambda}\left([2-n]\overline{Q}+[(n-1)(2+\lambda)]Q-\left[(n+(n-1)\lambda)\overline{Q}^{2-n}\right]Q^{n-1}\right)dQ \\
&=[2-n]\frac{\overline{Q}^{\lambda+2}}{\lambda+1}+[(n-1)(2+\lambda)]\frac{\overline{Q}^{\lambda+2}}{(\lambda+1)(\lambda+2)}-\left[(n+(n-1)\lambda)\overline{Q}^{2-n}\right]\frac{\overline{Q}^{\lambda+n}}{\lambda+1}\prod_{m=2}^{n}\frac{m-1}{\lambda+m} \\
&\propto[2-n]+[n-1]-\left[(n+(n-1)\lambda)\prod_{m=2}^{n}\frac{m-1}{\lambda+m}\right]=1-(n+(n-1)\lambda)\prod_{m=2}^{n}\frac{m-1}{\lambda+m}. \qquad (14)
\end{aligned}
$$

When $n=2$, this is 0;[40] and when $n=3$, this is

$$1-(3+2\lambda)\frac{2}{(\lambda+2)(\lambda+3)}=1-\frac{6+4\lambda}{6+5\lambda+\lambda^2}>0.$$

Let $n'=n+1$. Note that

$$
\begin{aligned}
\frac{(n'+(n'-1)\lambda)\prod_{m=2}^{n'}\frac{m-1}{\lambda+m}}{(n+(n-1)\lambda)\prod_{m=2}^{n}\frac{m-1}{\lambda+m}}<1 \quad &\Longleftrightarrow \quad ((n+1)+n\lambda)n<(n+(n-1)\lambda)(\lambda+n+1) \\
&\Longleftrightarrow \quad 0<(n-1)(1+\lambda). \qquad \checkmark
\end{aligned}
$$

Then the product term in (14) is decreasing in n, for all $n\geq 3$. Since the difference in revenue is strictly positive when $n\in\{2,3\}$, it is strictly positive for all n, and the discriminatory auction generates strictly more revenue than any equilibrium of the uniform-price auction. □

Proof of Proposition 5. Following the proof of Proposition 4, equilibrium price coefficients are (recall that there are n bidders in the discriminatory auction and $n+1$ bidders in the uniform-price auction)

$$
\begin{aligned}
p_0^{\mathrm{PAB}}&=v_0+\frac{v_1\overline{Q}}{n+(n-1)\lambda}, & p_1^{\mathrm{PAB}}&=\frac{1}{n}\left[\frac{(n-1)\lambda}{n+(n-1)\lambda}\right]v_1; \\
p_0^{\mathrm{UPA}}&=v_0, & p_1^{\mathrm{UPA}}&=\frac{1}{n+1}\left[\frac{n}{n-1}\right]v_1.
\end{aligned}
$$

[40] When $n=2$, the uniform-price auction does not admit a conjugate equilibrium, but still admits a maximum-bid equilibrium. See Remark 3 in the main text.

Note that these coefficients have been written in terms of v_0 and v_1, rather than $\hat{v}_0$ and $\hat{v}_1$, since the different numbers of bidders in the two auction formats affects the relationship between v and $\hat{v}$. Following Lemma 11, the difference in equilibrium expected revenues is

$$\begin{aligned}
\mathbb{E}\left[\pi^{\mathrm{PAB}}-\pi^{\mathrm{UPA}}\right] &= \frac{v_1\overline{Q}}{n+(n-1)\lambda}\left[\frac{\overline{Q}}{\lambda+1}\right]+\left(\frac{1}{2n}\left[\frac{(n-1)\lambda}{n+(n-1)\lambda}\right]-\frac{1}{n+1}\left[\frac{n}{n-1}\right]\right)\frac{\overline{Q}^2 v_1}{(\lambda+1)(\lambda+2)} \\
&\mathrel{\bar{\propto}} 2\left(n^2-1\right)(\lambda+2)n+\left(n^2-1\right)(n-1)\lambda-2\left(n+(n-1)\lambda\right)n^2 \\
&= \left(2\left(n^2-1\right)n+\left(n^2-1\right)(n-1)-2(n-1)n^2\right)\lambda+\left(4\left(n^2-1\right)n-2n^3\right) \\
&= \left(n^2+2n-1\right)(n-1)\lambda+2n\left(n^2-2\right).
\end{aligned}$$

This expression is strictly positive for all $n \geq 2$, thus expected revenues are lower in the conjugate equilibrium of the discriminatory auction with n bidders than in the conjugate equilibrium of the uniform-price auction with $n+1$ bidders.

Because conjugate equilibrium in the uniform-price auction is independent of the distribution F, whenever $\hat{v}_0 - n\overline{Q}/(n-1) < 0$ all equilibria will be in bids above conjugate bids, and therefore equilibrium revenue will be above conjugate equilibrium revenue. It follows that when $\overline{Q}$ is sufficiently large, all equilibria of the uniform-price auction with $n+1$ bidders yield greater expected revenue than the unique equilibrium of the discriminatory auction with n bidders.[41] □

[41]The argument from conjugate equilibria is sufficient to establish revenue dominance for large $\overline{Q}$. A tighter bound on $\overline{Q}$ may be obtained by evaluating the expected revenue of the minimum-bid equilibrium in the uniform-price auction, where $\underline{p}(\overline{Q}) = 0$.