Equilibrium with Monotone Actions

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Abstract

I show that pure-strategy equilibria exist in discontinuous games with private information when actions are monotone functions, provided monotone Bayesian Nash equilibria exist in nearby discrete games, payoffs are uniformly upper semicontinuous, players can guarantee ε best responses near a limit of opponent strategies, and the game involves a shared surplus. The proof of equilibrium existence builds equilibrium strategies as the limit of equilibria of nearby discretized models by first establishing that a limit exists, then showing that interim utilities must converge. A direct implication of this approach is that when observables are continuous in action, outcomes in the base model can approximate outcomes in the discrete models. I apply these results to prove the existence of pure-strategy equilibria in a class divisible-good auctions with private information, including discriminatory, uniform price, and hybrid formats. Equilibrium approximation implies that the distribution of observed allocations and seller revenue in discrete auctions can be close to the distribution of these outcomes predicted by the divisible-good model.

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1 Introduction

I consider the question of equilibrium existence in games where agents' actions are monotone functions. In Bayesian games with independent private values, best-response strategies are frequently monotone — an agent's best response is increasing in her private type; in this article actions themselves are monotone functions. For example, in divisible-good auctions where bidders submit demand curves, actions are decreasing functions specifying the willingness to pay for a particular quantity. In games of entry timing mixed strategies are given by monotone functions from time to the net probability of entry by this time. And in classical single-unit auctions, bids are equivalent to monotone functions from a degenerate domain to the space of feasible bids.

I prove the existence of monotone pure-strategy equilibria in this context. The proof of equilibrium existence is based on the intuition that equilibria in discrete settings should, as the discretization becomes fine, converge to an equilibrium in the limiting, *continuum-action* model. Actions in this article's base model are monotone functions on a compact domain, and are discretized by constraining their domain and range to a grid of points. In the discretization many traditional assumptions for equilibrium existence are satisfied — by finiteness all utility functions are continuous in action, and convergent sequences of actions are eventually constant — hence verifying the existence of equilibrium in the discrete setting is a comparatively simple task.[1] When there is a monotone pure-strategy equilibrium in a refining sequence of discretized models, the monotonicity of strategies and actions enables the construction of a limiting action profile, similar to the approach taken in Reny (2011). The existence of Bayesian Nash equilibrium — in which almost all types are best responding — can proceed in a limit-independent manner, but to prove the existence of pure-strategy equilibrium — in which all types are best responding — a particular limit must be taken.

To prove that limiting strategies are mutual best responses I show that utility converges in the limit. The constructed strategies are therefore relatively close to equilibrium strategies in the sequence of discretizations, providing a useful check on the potential empirical value of the continuum-action model. A further implication of this construction is that if an economic outcome

[1] To this end, rather than placing conditions that ensure existence of a discretized equilibrium my results assume (Condition 7) that there is a discretization that admits a monotone Bayesian Nash equilibrium. This assumption can be verified by referencing the extensive equilibrium existence literature (sampled below) or, as in multi-unit auctions and many other cases, noting that discrete equilibrium existence has already been established.

is utility-relevant — that is, if its nonconvergence implies nonconvergence of utility — then it is converging in probability to its distribution in the continuum-action model.

I apply these results to divisible-good auctions, establishing the existence of monotone pure-strategy equilibria and probabilistic approximation of auction outcomes. While it is known that common multi-unit auction formats admit pure-strategy Bayesian-Nash equilibria (Reny, 2011; McAdams, 2003), general models are viewed as intractible (Hortaçsu and Kastl, 2012) suggesting that a divisible-good approximation might yield fruitful results. In the presence of private information, equilibrium existence in divisible-good auctions has gone largely unaddressed. When goods are divisible results which assume finite action spaces do not directly apply, and the payoff discontinuities inherent to auctions impede application of results which assume utility is continuous. I also show that in multi-unit auctions the distributions of quantity allocations and seller revenue are converging to their distributions in the divisible-good model. Since empirical investigations of auctions frequently address questions of efficiency and revenue, these results provide justification for application of the divisible-good model.

Generally, the results in this paper attempt to unify the approach taken toward equilibrium existence in continuum-action models. In these models it is often difficult to directly prove equilibrium existence, so equilibrium is established in nearby models and then a limit is taken (see Jackson et al. (2002), Reny (2011), among others). Viewed from this angle the results in this paper provide a set of conditions under which such methods are valid. The conditions for equilibrium existence presented in this article are similar to those found in the literature on equilibrium existence, but they place particular emphasis on behavior relative to sequences of strategies.

First, utility is uniformly upper semicontinuous in an agent's own action: an infinitesimal increase in action cannot yield a discontinuous drop in utility. To satisfy this condition I allow for type-dependent action spaces, similar to constructions found in Jackson and Swinkels (2005) and elsewhere. For example, this condition is easily verified in a first-price auction under the assumption that bidders bid below their true valuations: a slight increase in bid can yield a discontinuous increase in the probability of winning the object, but it cannot yield a discontinuous increase in the payment for the object (conditional on winning).[2] Provided infeasible actions outside the

[2]Without type-dependent action spaces the first-price auction violates this condition: if an agent is bidding above her value, a slight increase from a tie yields a discontinuous downward drop in utility.

type-dependent action space aren't too much better than feasible actions, type-dependent action spaces can be removed. When small upward deviations cannot negatively affect utility, profitable deviations can be approximated in a discretized model.

Second, given a convergent sequence of their opponents' strategies agents must be able to ensure approximately limiting utility with some nearby action. Roughly, this is payoff security constrained to local deviations. Consider an auction: if, given a sequence of her opponents' bids, an agent's utility is better at the limit than in the limit, there must be a nearby action that allows her to guarantee herself, in the limit, most of her utility at the limit in the other action. In auctions, this situation normally manifests as tiebreaking occurring at the limit, while being unnecessary near the limit. An agent can break a tie in her favor by increasing her bid slightly, at nominal cost. This implies that an agent whose utility improves at a limit of actions could have obtained nearly this utility at some point in the sequence of actions. Together with the first condition, if an agent's utility improves at a limit of discrete equilibria she could attain most of this utility in one of the discretizations by employing a small upward deviation.

Third, models must involve a shared surplus. If, given a sequence of all agents' strategies, one agent's utility is worse at the limit than in the limit, there is another agent whose utility is better at the limit than in the limit, with positive probability. In an auction context an agent's utility dropping at the limit is commonly related to tiebreaking; if one agent begins losing a tie, then another agent begins winning a tie. Care must be taken to ensure that gains occur with positive probability. Together with the first two conditions, this is used to derive an interim form of better-reply security. If an agent's utility jumps upward with positive probability, she can ensure these gains near a limit using a small upward deviation.

Discrete equilibrium strategies converge for almost all types. Under the given conditions, along a sequence of equilibrium strategy profiles utility is converging for all these types and hence agents are best responding with probability one. Then limiting strategies constitute a Bayesian Nash equilibrium. The proof that the constructed equilibrium implies best responses for all signal realizations of all agents involves a further condition: any action in the frontier of an agent's type-dependent feasible action space has an action in the interior of low-types' feasible action spaces which generates almost as much utility. I assume utility is left-continuous in type, so the supremum of lower-type actions is a best response, and the type-dependent action space is irrelevant. Then the constructed strategies are an equilibrium even in the unconstrained model.

Because equilibrium in the continuum-action model is built as a convergent limit of a sequence of equilibria in nearby models, it follows almost immediately that any observed outcome that is continuous in actions will be converging along the same path of equilibria. This can be leveraged to provide a useful empirical statement: the distribution of outcomes in the continuum-action model can approximate the distribution of observed outcomes in the "real world" model it is meant to approximate. I generalize the notion of continuity of observables slightly to utility-relevance, which requires that observables are at least as continuous as utility functions. Inasmuch as many auction observables are discontinuous even while (I show) utility is converging in the limit of strategies I construct, this presents significantly greater value than simple continuity.

1.1 Related literature

This paper follows neatly from two threads of equilibrium existence literature. The first establishes (potentially mixed-strategy) equilibrium existence in models with discontinuous payoffs (as in Athey (2001)), and the second looks at the same question in models with private information and continuous payoffs (as in Reny (1999)). McAdams (2003) extends Athey's result to include multidimensional private information, and Van Zandt and Vives (2007) and Reny (2011) generalize to the case of arbitrary lattices. These results cannot be directly applied because, as is common in auction models, payoff discontinuities cannot be ruled out ex ante. Reny (1999) allows for discontinuous utility functions, but does not permit private information; his results have been extended by McLennan et al. (2011) and Barelli and Meneghel (2013).

The approach of establishing equilibrium as a limit of nearby discretized equilibria has been used by, among others, Simon (1987), Reny and Zamir (2004), Bagh (2010), and Kastl (2012). In contrast to my pure-strategy existence result under private information, Simon (1987) establishes existence in mixed strategies, without private information.[3] The limiting approach of Reny and Zamir (2004) uses a similar method to prove the existence of pure-strategy equilibria in first-price auctions; like my results here, it relies on convergence of utility, but unlike my results actions are point bids rather

[3] In later work, McAdams (2006) shows that these mixed strategies can be rendered into monotone pure strategies without affecting best-responseness. Reny (1999) shows a related result, that in a particular multi-unit auction model the mixed strategies predicted are in fact pure strategies. Other results regarding equilibrium in mixed or distributional strategies include Milgrom and Weber (1985), Kastl (2012), and He and Yannelis (2016).

than generic monotone functions. Kastl (2012) provides equilibrium in distributional strategies with finite bid points, and uses this to suggest the same when bids can be arbitrary nonincreasing functions of quantity.

The condition most directly related to the ability of my conditions to extend existence results to divisible-good auctions with private information is a weakened form of reciprocal upper semi-continuity. Similar conditions have been examined by Bagh and Jofre (2006), Bagh (2010), Allison and Lepore (2014), and He and Yannelis (2016). Bagh and Jofre (2006) examines weak reciprocal upper semicontinuity, which is not necessarily satisfied by divisible-good auctions; Condition 5 requires only that most of the limiting utility can be obtained, and not that it can be dominated. Bagh (2010) employs variational convergence, which invokes dominating sequences of actions; Condition 5 can be weakened to require only that the dominating sequence of actions dominate the original sequence (Condition 6), entirely avoiding behavior at the limit. Lastly, Allison and Lepore (2014) and He and Yannelis (2016) introduce (random) disjoint payoff matching, again requiring dominance in the limit. None of these conditions is obviously satisfied in divisible-good pay-as-bid auctions.

Relatively little is known about bidder behavior in multi-unit auctions with private information. Beyond the apparent theoretical difficulty of computing fully general revenue and efficiency rankings, progress in the analysis of parameterized models has been hampered by the inability to efficiently compute equilibrium strategies in the case where goods, as in practice, are imperfectly divisible. Meaningful results have been obtained in certain settings — see, e.g., Engelbrecht-Wiggans and Kahn (2002), Ausubel et al. (2014), Lotfi and Sarkar (2016), and Burkett and Woodward (2018) — but the general state of the art is best captured by Hortaçsu and Kastl (2012), who state, "Unfortunately, computing equilibrium strategies in (asymmetric) discriminatory multi-unit auctions is still an open question."[4] Häfner (2015) demonstrates the existence of an equilibrium in distributional strategies in a discriminatory auction with constrained bids, but does not obtain a pure-strategy existence result.

Where discrete problems appear intractible, continuous approximations may offer sound and available economic insights. For example, the literature on single-unit auctions frequently employs

[4] There has been work in building approximate equilibria for multi-unit auctions; see, e.g., Armantier and Sbaï (2006), Armantier et al. (2008), and Armantier and Sbaï (2009).

the assumption that the set of available prices is dense. In the case of multi-unit auctions, bids may be approximated as objects determined on a dense domain of quantities, as well; there is no counterpart to this possibility in single-unit auctions, or even in combinatorial auctions. Wilson (1979) was the first to apply this approximation method in the context of multi-unit auctions, and this approximation has been used to establish results for parameterized models such as Back and Zender (1993), Wang and Zender (2002), Ausubel et al. (2014), and Pycia and Woodward (2017), but in the general case it has not even been known if an equilibrium exists. Without a sound basis for the existence of equilibrium strategies, it has been difficult to meaningfully apply the divisible-good model to policy debates.

Section 3 lays out the main results of the model. Section 4 takes these results to the divisible-good pay-as-bid auction model and proves the existence of pure-strategy equilibria as well as equilibrium approximation, and Section 5 concludes.

2 Model

A game is given by $\mathcal{M} = (n, u, X, A, F)$. There is a finite set of agents $i \in \{1, \ldots, n\}$. Agent i has private information $s_i \sim F^i$, with bounded support Supp $s_i = (0,1)^{m_s}$. For all $j \neq i$, s_i is independent of s_j. Since I make no assumptions on utility's relation to signal other than weak monotonicity and left continuity (Condition 1 below), it is without loss of generality to assume that s_i is the product of m indepdendent uniform distributions, $s_{ik} \sim \mathcal{U}(0,1)$ and for all $k' \neq k$, s_{ik} is independent of $s_{ik'}$. For all agents $j \neq i$, s_i and s_j are independent.

Agents' actions are functions from domain $X_D \subset \mathbb{R}_+^{m_D}$ to range $X_R \subset \mathbb{R}_+^{m_R}$; let $X = (X_D, X_R)$. Both X_D and X_R are compact and convex.[5] Let Y be the set of monotone decreasing functions from X_D to X_R.[6] When agent i has signal s_i, her feasible action space is $A^i(s_i) \subseteq Y$; A is the profile of feasible action spaces, $A = (A^i)_{i=1}^n$.[7] A strategy $\alpha^i : (0,1)^m \to y$ is *feasible* for agent i if for all s_i, $\alpha^i(s_i) \in A^i(s_i)$. A strategy profile $\alpha = (\alpha^1, \ldots, \alpha^n)$ is feasible if α^i is feasible for all i.

Agent i's utility function is $u^i : Y \times Y^{n-1} \times (0,1)^{m_s} \to \mathbb{R}$; u is the profile of utility functions,

[5]The assumption that X_D and X_R lie in the positive orthant is made for simplicity, and is without loss of generality conditional on compactness.

[6]All results extend to the case in which different types play actions which are differently monotone, provided it is known in advance which types will employ monotone increasing actions and which will employ monotone decreasing actions. Allowing for this variation introduces additional technical overhead but little additional intuition.

[7]Type-dependent action spaces are not essential but greatly simplify the exposition.

$u = (u^i)_{i=1}^n$. Interim expected utility for agent i is defined with respect to her own information and her opponents' strategies,

$$U^i\left(a_i, \alpha^{-i}; s_i\right) = \mathbb{E}_{s_{-i}}\left[u^i\left(a_i, \alpha^{-i}\left(s_{-i}\right); s_i\right) \middle| s_i\right].$$

Unless otherwise stated, all norms are taken to be L^1 on the relevant domain, $||a|| = \int ||a(x)||dx$ for functions and $||a|| = \sum_{i=1}^m |a_i|$ for vectors.

2.1 ε-discrete model

The proof of equilibrium existence in a model $\mathcal{M}$ uses a sequence of approximating discrete games in which equilibrium is known to exist. As the approximation becomes perfect — that is, as the discretization becomes fine — equilibrium strategies are shown to converge. The first step in this process is defining the discrete approximations.

Given a base model $\mathcal{M}$ and $\varepsilon > 0$, an ε-approximation $\mathcal{M}^\varepsilon = (n, u, X^\varepsilon, A^\varepsilon, F)$ is derived from the base model $\mathcal{M}$ by discretizing its feasible action spaces.

Definition 1 (Finite ε-approximation)**.** *Let $Z \subseteq \mathbb{R}^m$ be compact. $Z^\varepsilon \subseteq Z$ is a* finite ε-approximation *of Z if the following conditions hold:*

1. *Z^ε is finite;*

2. *For any $z \in Z$, there exists $\mathsf{z} \in Z^\varepsilon$ such that $||z - \mathsf{z}|| < \varepsilon$.*

Let $X^\varepsilon = (X_D^\varepsilon, X_R^\varepsilon)$, where X_D^ε and X_R^ε are finite ε-approximations of X_D and X_R, respectively. Let Y^ε be the set of monotone decreasing functions from X_R^ε to X_D^ε. For each agent i and each signal s_i, $A^{i,\varepsilon}(s_i) \subseteq Y^\varepsilon$.

A natural case of interest is when the approximating sets are regular grids with equal spacing, so that $X_R^\varepsilon = \mathbb{Z}\varepsilon \cap X_R$ and $X_D^\varepsilon = \mathbb{Z}\varepsilon \cap X_D$, and action spaces are the closest available approximations of feasible action spaces A^i,

$$A^{i,\varepsilon}\left(s_i\right) = \left\{\mathsf{a} \in Y^\varepsilon : \ \exists a \in A^i\left(s_i\right), \mathsf{a} \in \underset{\mathsf{a}' \in Y^\varepsilon}{\operatorname{argmin}} \left|\left|\mathsf{a}' - a\right|\right|\right\}.$$

The role of the ε-discrete model is to ensure equilibrium existence in a sequence of models approaching the underlying model $\mathcal{M}$. Whether a particular discretization is appropriate is a matter of ease of satisfying the conditions below and ensuring equilibrium existence in the discretized model; see the discussion of divisible-good auctions in Section 4.

2.2 Equilibrium

I obtain existence results for two types of equilibrium. Taking a limit of equilibrium strategies in ε-discrete models gives a Bayesian Nash equilibrium in the base model.

Definition 2 (Constrained Bayesian Nash equilibrium)**.** *A strategy profile* $(\alpha^i)_{i=1}^n$ *is a* constrained Bayesian Nash equilibrium *if for all agents* i*,*

$$\mathbb{E}_{s_i}\left[U^i\left(\alpha^i\left(s_i\right), \alpha^{-i}; s_i\right)\right] = \sup_{\alpha} \mathbb{E}_{s_i}\left[U^i\left(\alpha\left(s_i\right), \alpha^{-i}; s_i\right)\right] \text{ s.t. } \alpha^i\left(s_i\right) \in A^i\left(s_i\right) \ \forall s_i.$$

A constrained Bayesian Nash equilibrium is a Bayesian Nash equilibrium in which agents are constrained to implement feasible strategies. If the type-dependent action space is irrelevant, so that $A^i(s_i) = Y$ for all agents i and types s_i, a constrained Bayesian Nash equilibrium is a (standard) Bayesian Nash equilibrium.

For a strategy profile to be a constrained Bayesian Nash equilibrium it is only necessary that each agent is ex ante best responding, or is best responding with probability 1. Under the Conditions set forth in Section 3 it can be shown that there is a natural limit of ε-discrete equilibrium in which agents are certainly best responding.[8] To capture this I define a *constrained pure-strategy equilibrium* in the sense of McAdams (2003).

Definition 3 (Constrained pure-strategy equilibrium)**.** *A strategy profile* $(\alpha^i)_{i=1}^n$ *is a* constrained pure-strategy equilibrium *if for all agents* i *and type realizations* s_i*,*

$$U^i\left(\alpha^i\left(s_i\right), \alpha^{-i}; s_i\right) = \sup_{a} U^i\left(a, \alpha^{-i}; s_i\right) \text{ s.t. } a \in A^i\left(s_i\right).$$

As with the relation between constrained Bayesian Nash equilibrium and Bayesian Nash equilibrum, when the type-dependent action space is irrelevant, so that $A^i(s_i) = Y$ for all agents i and

[8] Proving the existence of an equilibrium in which all types are best responding relies on left-continuity of utility in signal and the construction of a particular limit of ε-discrete equilibrium. Nonetheless the intuitive reasons behind the existence of the two different kinds of equilibria are fundamentally the same.

types s_i, a constrained pure-strategy equilibrium is a (standard) pure-strategy equilibrium.

3 Results

3.1 Conditions for equilibrium existence

I now state the conditions used to establish the existence of a pure strategy equilibrium in the model $\mathcal{M}$. These conditions are mostly stated in interim utility to capture the intuition behind equilibrium existence, but have related ex post formulations which can be easier to work with; see Lemmas 11 and 13 in Appendix A.

Condition 1 (Utility and actions). *For each agent i, u^i is bounded, and increasing and left-continuous*[9] *in own signal s_i. For all s_i, $A^i(s_i)$ is a complete semilattice.*

Bounded utility ensures the existence of convergent subsequences,[10] and continuity ensures that agents who are close in type should have elements in their best responses which are near one another.

Condition 2 (Imitability). *For each agent i, $s_i < s_i'$ implies $A^i(s_i) \subseteq A^i(s_i')$.*

A natural construction of A^i is the set of actions which generate utility above some outside option. For example, bids in a first-price auction should fall below the highest feasible value for the item, conditional on an agent's information. Since utility is increasing in signal, under this interpretation A^i is weakly increasing in set inclusion order. Subject to this constraint higher-type agents can always imitate lower-type agents, but not vice-versa. Condition 2 is trivially satisfied when action spaces are type-independent.

Condition 3 (Uniform upper semicontinuity). *There is a continuous function $g : \mathbb{R}_+ \times (0,1)^m \to \mathbb{R}_+$, $g(0;\cdot) = 0$, such that for all agents i, all types s_i, all monotone strategy profiles $(\alpha^j)_{j \neq i}$, all $a_i \in A^i(s_i)$, and all $\overline{a}_i \in Y$ with $a_i \leq \overline{a}_i$,*

$$U^i\left(a_i, \alpha^{-i}; s_i\right) \leq U^i\left(\overline{a}_i, \alpha^{-i}; s_i\right) + g\left(||\overline{a}_i - a_i||\,; s_i\right).$$

[9] A multidimensional function is left-continuous if it is coordinatewise left continuous in each argument.

[10] Because it is not of techical importance, I frequently assume that sequences converge. Formally, all arguments go through when applied to convergent subsequences.

Condition 3 implies that agent i's utility is upper semicontinuous in her own action, and that the modulus of semicontinuity is uniform acrosss all actions, stategy profiles, and signals. Absent type-dependent action spaces this condition is not necessarily satisfied in many cases of interest. For example, in a first price auction a bidder can receive zero utility by submitting a bid above her value but below all of her opponents' bids. If her opponents' bids are massed at her own bid, a small increase in her bid can have a disproportionately negative effect on utility. Restricting attention to a particular region of the action space — for example, undominated strategies — permits satisfaction of Condition 3.

Condition 4 (Local utility security). *Let $\langle(\alpha^{j,t})_{j\neq i}\rangle_{t=1}^{\infty}$ be a sequence of monotone strategies for agents $j \neq i$, converging to the strategy profile $(\alpha^{j,\star})_{j\neq i}$. For any $a_i \in A^i(s_i)$ and any $\lambda > 0$, there is $a_i' \in A^i(s_i)$ such that $||a_i' - a_i|| \leq \lambda$, and*

$$\lim_{t\nearrow\infty} U^i\left(a_i', \alpha^{-i,t}; s_i\right) > U^i\left(a_i, \alpha^{-i,\star}; s_i\right) - \lambda.$$

Given an action for agent i and the limit of a sequence of her opponents' strategies, Condition 4 requires that there is a nearby feasible action for agent i which yields, in the limit, nearly as much utility. Local utility security is similar to payoff security, except that the sequence of opponent strategies $\langle(\alpha^{j,t})_{j\neq i}\rangle_{t=1}^{\infty}$ is not a free variable: the specific a_i' can depend on the sequence. In light of Condition 3, Condition 4 is frequently straightforward to satisfy by considering small upward deviations, but as this is a condition on limits of *opponents'* strategies it is a standalone property.

Condition 5 (Surplus splitting). *Let $\langle(\alpha^{k,t})_{k=1}^{n}\rangle_{t=1}^{\infty}$ be a sequence of monotone strategies converging to the feasible strategy profile $(\alpha^{k,\star})_{k=1}^{n}$. Suppose that there is an agent i such that*

$$\Pr{}_s\left(\lim_{t\nearrow\infty} u^i\left(\alpha^{i,t}(s_i), \alpha^{-i,t}(s_{-i}); s_i\right) > u^i\left(\alpha^{i,\star}(s_i), \alpha^{-i,\star}(s_{-i}); s_i\right)\right) > 0.$$

Then there is an agent j such that

$$\Pr{}_s\left(\lim_{t\nearrow\infty} u^j\left(\alpha^{j,t}(s_j), \alpha^{-j,t}(s_{-j}); s_j\right) < u^j\left(\alpha^{j,\star}(s_j), \alpha^{-j,\star}(s_{-j}); s_j\right)\right) > 0.$$

Surplus splitting is related to reciprocal upper semicontinuity (Bagh and Jofre, 2006), but allows for the possibility that *all* agents' utilities face a simultaneous downward discontinuity. Condition 5 reflects the fact that many games have winners and losers, and a discontinuous loss in utility by one agent reflects a gain by one of her opponents when facing her. That her opponents do not necessarily face a discrete increase in interim utility allows for the fact that, globally, they might also lose interim utility, hence Condition 5 is weaker than interim reciprocal upper semicontinuity.

While surplus splitting is sufficient to establish the existence of constrained Bayesian Nash equilibria, it is stronger than is necessary. Under Condition 3 it implies Condition 6 below.

Condition 6 (Limit surplus splitting). *Let $\langle(\alpha^{k,t})_{k=1}^{n}\rangle_{t=1}^{\infty}$ be a sequence of monotone strategies converging to the feasible strategy profile $(\alpha^{k,\star})_{k=1}^{n}$. Suppose that there is an agent i such that*

$$\Pr{}_s\left(\lim_{t\nearrow\infty} u^i\left(\alpha^{i,t}\left(s_i\right), \alpha^{-i,t}\left(s_{-i}\right); s_i\right) > u^i\left(\alpha^{i,\star}\left(s_i\right), \alpha^{-i,\star}\left(s_{-i}\right); s_i\right)\right) > 0.$$

Then there is an agent j and for any $\lambda > 0$ a sequence of feasible strategies $\langle\hat{\alpha}^{j,t}\rangle_{t=1}^{\infty}$, where for all t sufficiently large $||\hat{\alpha}^{j,t}(s_j) - \alpha^{j,t}(s_j)|| < \lambda$ for all types s_j, such that

$$\Pr{}_s\left(\lim_{t\nearrow\infty} u^j\left(\alpha^{j,t}\left(s_j\right), \alpha^{-j,t}\left(s_{-j}\right); s_j\right) < \lim_{t\nearrow\infty} u^j\left(\hat{\alpha}^{j,t}\left(s_j\right), \alpha^{-j,t}\left(s_{-j}\right); s_j\right)\right) > 0.$$

To obtain equilibrium existence I will construct a sequence of equilibria in ε-discrete models; the following conditions place some restrictions on equilibrium existence and action structure in the discretized models. For these conditions, let $\langle\varepsilon_t\rangle_{t=1}^{\infty}$ be a monotone strictly decreasing sequence converging to 0.[11]

Condition 7 (Existence of discrete equilibrium). *There is T such that, for all $t \geq T$, $\mathcal{M}^{\varepsilon_t}$ admits a monotone Bayesian Nash equilibrium.*

Condition 7 is necessary for the proof of existence in the continuum-action case since equilibrium is constructed as a limit of equilibria of the discretized models, hence there must be equilibria in the discretized models. Satisfaction of Condition 7 can be verified with techniques from Athey (2001), McAdams (2003), and Reny (2011), among others.

Condition 8 (Approximating action spaces). *For all agents i and signals s_i:*

[11] Conditions 7 and 8 can be stated as "for $\varepsilon > 0$ sufficiently small," but this is less general. All that is necessary is that the conditions are satisfied on a particular path to 0, not all paths; nonetheless, in many contexts there is little practical difference.

1. *For all* $a_i \in A^i(s_i)$*, there exists a monotone decreasing sequence* $\langle a_i^t \rangle_{t=1}^{\infty}$ *converging to* a_i*, such that* $a_i^t \in A^{i,\varepsilon_t}(s_i)$ *and* $a_i^t \geq a_i$ *for all* t*;*

2. *For all sequences* $\langle a_i^t \rangle_{t=1}^{\infty}$*,* $a_i^t \in A^{i,\varepsilon_t}(s_i)$*, and any convergent subsequence* $\langle a_i^{t_k} \rangle_{k=1}^{\infty}$*, there is* $a_i^\star \in A^i(s_i)$ *such that* $a_i^{t_k} \to a_i^\star$*.*

The first point of Condition 8 requires that any action in $A^i(s_i)$ can be approximated from above arbirarily closely, as the discretization becomes fine; approximation from above is closely related to Condition 3. The second point requires that actions in the discretized models cannot be too far away from actions in the continuum model; together with the first point, this can be viewed as any discretized action must approximate some action in the base model, and any action in the base model can be approximated in sufficiently fine discretized models.

The final condition is not necessary for the existence of constrained Bayesian Nash equilibrium, where each agent is best-responding with probability 1. It is used to demonstrate that there is a constrained pure-strategy equilibrium, where all signal realizations of all agents are best-responding.

Condition 9 (Type insensitivity)**.** *Let* $\underline{A}^i(s_i) = \cup_{s_i' < s_i} A^i(s_i)$*. For all agents* i*, all signals* s_i*, all* $\lambda > 0$*, and all* $a_i \in A^i(s_i) \setminus \underline{A}^i(s_i)$*, there is* $a_i' \in \underline{A}^i(s_i)$ *such that*

$$U^i\left(a_i', \alpha^{-i}; s_i\right) > U^i\left(a_i, \alpha^{-i}; s_i\right) - \lambda.$$

Equilibrium actions will be built as the supremum over the actions of lower types. When Condition 9 is satisfied, lower types can achieve almost as much utility within their constrained action spaces as a given type can achieve in her feasible action space. Left-continuity of utility functions then guarantees that constructed strategies constitute best responses, and the feasibility constraint is not binding.

3.2 Equilibrium existence

Proofs are found in Appendix A.

Let $\langle \varepsilon_t \rangle_{t=1}^{\infty}$ be a decreasing sequence converging to zero, assume that the model $\mathcal{M}$ and its ε_t-discretizations $\mathcal{M}^{\varepsilon_t}$ satisfy Conditions 1-8. For any t, let $(\alpha^{i,t})_{i=1}^n$ be a monotone Bayesian-Nash equilibrium in $\mathcal{M}^{\varepsilon_t}$. Since each $\alpha^{i,t}$ is bounded, selection results (c.f. Widder (1941)) imply that there is a pointwise limit on any countable set of points.[12] It is useful that this set of points be dense, hence let $\mathcal{X}_D = X_D \cap \mathbb{Q}^{m_D}$ and $\mathcal{S} = (0,1)^{m_s} \cap \mathbb{Q}^{m_s}$.

Lemma 1 (Pointwise convergence on countable set)**.** *There is a strategy profile* $(\alpha^{i,\square})$ *such that for all agents* i*, all* $x \in \mathcal{X}_D$*, and all* $s \in \mathcal{S}$*,*

$$\lim_{t \nearrow \infty} \left[\alpha^{i,t}(s)\right](x) = \left[\alpha^{i,\square}(s)\right](x).$$

The sets $\mathcal{X}_D$ and $\mathcal{S}$ are countable while X_D and $(0,1)^{m_s}$ are uncountable, hence the strategy profile $(\alpha^{i,\square})_{i=1}^{n}$ comprised of functions on X_D may have significant "holes." Monotone functions on compact domains are continuous almost everywhere (Lavrič, 1993), thus any monotone function that coincides with $\alpha^{i,\square}$ on $\mathcal{X}_D \times \mathcal{S}$ is L^1-equivalent to $\alpha^{i,\square}$.

Lemma 2 (Convergence to limit)**.**[13] *For all agents* i*,* $||\hat{\alpha}^i - \alpha^{i,\square}|| = 0$ *implies* $\lim_{t \nearrow \infty} ||\alpha^{i,t} - \hat{\alpha}^i|| = 0$*. Furthermore, with* s_i*-probability one,*

$$\lim_{t \nearrow \infty} \left|\left|\alpha^{i,t}(s_i) - \hat{\alpha}^i(s_i)\right|\right| = 0.$$

This permits the construction of *supremum-limit strategies* $\overline{\alpha}^i$. These are strategies at which each type realization s_i is playing an action that is the least upper bound of actions for all lower type realizations $s_i' < s_i$. Condition 1 requires that utility is left-continuous in signal, thus an action which is the supremum of lower types' best responses is a natural action to examine as a type's own best response.

Definition 4 (Supremum-limit strategy)**.** $\overline{\alpha}^i$ *is a* supremum-limit *strategy for agent* i *if for all* $s_i \in (0,1)^m$*,*

$$\overline{\alpha}^i(s_i) = \sup_{s_i' < s_i} \alpha^{i,\square}(s_i').$$

As strategies converge, so too does utility for almost all agents. This is the bulk of the proof of equilibrium existence.

[12] In a single dimension, Helly's selection theorem guarantees that any sequence of bounded monotone functions on a compact domain admits a convergent subsequence. In multiple dimensions these results appeal to total boundedness, which is not exogenously guaranteed in many game theoretic models. Instead, pointwise convergence and monotonicity are used to derive L^1 convergence.

[13] In Appendix A this is proved as Lemmas 7 and 8.

Lemma 3 (Utility convergence almost everywhere)**.** *For all agents i,*

$$\Pr_{s_i}\left(\lim_{t\nearrow\infty} U^i\left(\alpha^{i,t}\left(s_i\right), \alpha^{-i,t}; s_i\right) \neq U^i\left(\overline{\alpha}^i\left(s_i\right), \overline{\alpha}^{-i}; s_i\right)\right) = 0.$$

Lemma 3 is established by combining two sides: in the first case, there is zero probability that utility jumps up at the limit, and in the second case there is zero probability that utility jumps down at the limit. One side of Lemma 3 is a direct consequence of Conditions 3 and 4: if the value in the limit is below the value at the limit, a slight upward deviation will yield discretely more utility in some discretized $\mathcal{M}^{\varepsilon_t}$. The other side follows from Condition 5: if agent i's utility falls at the limit, one of her opponents has an available deviation which will yield discretely greater utility in some discretized $\mathcal{M}^{\varepsilon_t}$.

Returning to Condition 4 gives that supremum-limit strategies are mutual best responses.

Theorem 1 (Constrained Bayesian Nash equilibrium)**.** *Suppose that Conditions 1-4 and 6-8 are satisfied. Then the supremum-limit strategy profile $(\overline{\alpha}^i)_{i=1}^n$ forms a monotone constrained Bayesian Nash equilibrium in the model $\mathcal{M}$. For all agents i,*

$$\mathbb{E}_{s_i}\left[U^i\left(\overline{\alpha}^i\left(s_i\right), \overline{\alpha}^{-i}; s_i\right)\right] \geq \sup_{a_i\in A^i(s_i)} \mathbb{E}_{s_i}\left[U^i\left(a_i, \overline{\alpha}^{-i}; s_i\right)\right].$$

Corollary 1 (Symmetric equilibrium)**.** *Suppose that Conditions 1-4 and 6-8 are satisfied. If $A^i = \hat{A}$ and $u^i = \hat{u}$ for all agents i, then $\mathcal{M}$ admits a symmetric monotone constrained Bayesian Nash equilibrium $(\hat{\alpha})_{i=1}^n$.*

If agent i has an action a_i which discretely improves on $\overline{\alpha}^i(s_i)$, she has an action close to a_i which is an improvement over some $\alpha^{i,t}$ against $(\alpha^{j,t})_{j\neq i}$. For ε_t sufficiently small, a_i can be approximated into $A^{i,\varepsilon_t}(s_i)$ at a loss that is of order $g(C\varepsilon_t; s_i)$. Since $(\alpha^{j,t})_{j=1}^n$ is a Bayesian Nash equilibrium, in which almost all signal realizations are best-responding, this is a contradiction if a positive mass of agents have utility-improving actions.

Supremum-limit strategies are not necessary to obtain a Bayesian Nash equilibrium as the limit of discrete equilibria. Lemma 2 establishes that actions are converging with probability one (with respect to type realization), and the proof of Lemma 3 can be adapted to show that these agents are best responding. To obtain a pure-strategy equilibrium requires the satisfaction of Condition 9 and the construction of supremum-limit strategies.

Theorem 2 (Constrained pure-strategy equilibrium). *Let $(\overline{\alpha}^i)_{i=1}^n$ be a monotone constrained Bayesian Nash equilibrium in supremum-limit strategies. If Condition 9 is satisfied, the strategy profile $(\overline{\alpha}^i)_{i=1}^n$ is a monotone constrained pure-strategy equilibrium: for each agent i and all signal realizations s_i,*

$$U^i\left(\overline{\alpha}^i\left(s_i\right),\overline{\alpha}^{-i};s_i\right) \geq \sup_{a_i\in A^i(s_i)} U^i\left(a_i,\overline{\alpha}^{-i};s_i\right).$$

Finally, the preceding results establish existence only in the game with type-dependent action spaces A^i. With a further condition on the nature of type-dependent action spaces, equilibrium existence obtains in the unconstrained game in which $A^i \equiv Y$ for all agents i.

Theorem 3 (Unconstrained equilibrium existence). *Suppose that for all $y \in Y$, all opponent strategy profiles $\alpha^{-i} = (\alpha^j)_{j\neq i}$, and all $\lambda > 0$, there exists $a_i \in A^i(s_i)$ such that*

$$U^i\left(a_i,\alpha^{-i};s_i\right) > U^i\left(y,\alpha^{-i};s_i\right) - \lambda.$$

Let $(\overline{\alpha}^i)_{i=1}^n$ be a monotone pure-strategy equilibrium in supremum-limit strategies of the model $\mathcal{M}$. Then the strategy profile $(\overline{\alpha}^i)_{i=1}^n$ is a monotone pure-strategy equilibrium in the model $\mathcal{M}^Y = (n,u,X,A^Y,F)$, where $A^{Yi}(s_i) = Y$.

3.3 Equilibrium approximation

The construction of equilibrium in $\mathcal{M}$ as a profile supremum-limit strategies suggests that equilibrium in $\mathcal{M}$ may be near equilibrium in the ε_t-discretization $\mathcal{M}^{\varepsilon_t}$.[14] In these results, I assume that $\langle(\alpha^{i,t})_{i=1}^n\rangle_{t=1}^\infty$ is a sequence of monotone constrained Bayesian-Nash equilibria of the ε_t-discrete models $\mathcal{M}^{\varepsilon_t}$ converging to the supremum-limit strategy profile $(\overline{\alpha}^i)_{i=1}^n$.

Definition 5 (Utility-relevant function). *Let (W,T_W) be a topological space, and let $\langle\alpha^t\rangle_{t=1}^\infty$ be a convergent sequence of strategy profiles, $\alpha^t \to \alpha^\star$. A function $o: Y^n \to W$ is* utility-relevant *if $o(\alpha^t(s)) \not\to o(\alpha^\star(s))$ implies that there is an agent i such that $u^i(\alpha^t(s);s_i) \not\to u^i(\alpha^\star(s);s_i)$.*

[14] In models in which equilibrum is unique this approximation is strict, in the sense that all equilibria converge to the unique equilibrium in $\mathcal{M}$; however without further results on uniqueness the strongest statement possible is, "The sequence of equilibria in $\mathcal{M}^{\varepsilon_t}$ contains a subsequence which converges to an equilibrium of $\mathcal{M}$."

A utility-relevant function is a mapping from actions to a set W such that its own discontinuities imply discontinuities in some agent's utility. For example, in many auction models quantity allocations are utility relevant: a discontinuous change in utility in general represents a discontinuous change in quantity.[15]

Theorem 4 (Equilibrium approximation)**.** *Let* (W, T_W) *be a topological space, and suppose that* $o : Y^n \to W$ *is utility-relevant. Then for almost all type profiles* s*,*

$$\lim_{t \nearrow \infty} o\left(\alpha^t(s)\right) = o\left(\overline{\alpha}(s)\right).$$

Proof. This is an immediate consequence of utility-relevance of o and the construction of $\overline{\alpha}^i$ as a limit of $\alpha^{i,t}$ at which almost all utilities converge. □

Corollary 2 (Probabilistic approximation)**.** *Let* $o : Y^n \to W$ *be utility-relevant. Then*

$$o\left(\alpha^t(s)\right) \xrightarrow{P} o\left(\overline{\alpha}(s)\right).$$

Corollary 2 establishes that the base model may be empirically near its discretizations. Since models in which the underlying action space represents a continuum — or, in this case, mappings from one continuum to another — are frequently meant as approximations of discrete realities, Corollary 2 suggests that auction models with actions from a continuum may be empirically useful; in Section 4 I use this result to show that in many multi-unit auctions equilibrium allocations and revenues converge.

It is straightforward extend the model to explicitly include exogenous independent randomness Z, such as might be necessary for an anonymous tiebreaking rule in an auction. Theorem 4 and Corollary 2 naturally extend to this setting, when utility relevance is adjusted to account for the exogenous randomness Z. This extension necessitates additional notation without offering additional insight, so I do not pursue it here.

[15] In auction models without private information it is straightforward to construct sequences of actions at the limit of which quantity is discontinuous but utility is not (for related examples, see Reny (1999) and Jackson et al. (2002), among others). With massless private information and strictly monotone private values, these constructions go away.

4 Application: divisible-good auctions

I now apply the equilibrium existence results from Section 3 to prove that equilibrium exists in divisible-good auctions with private information. An auctioneer is selling $\hat{Q}$ units of a perfectly divisible commodity to $n \geq 2$ bidders, $i \in \{1, \ldots, n\}$. $\hat{Q}$ is determined by a random realization $z_Q \sim \mathcal{U}(0,1)$. Bidder i's private information is $s_i \sim \mathcal{U}(0,1)$. For all agents $j \neq i$, s_i and s_j are independent. Bidder i has marginal value function $v^i : [0, \bar{Q}] \times (0,1) \to \mathbb{R}_+$, where $v^i(q; s_i)$ is her marginal value for the q^{th} unit of the good when her private information is s_i. v^i is bounded, decreasing in q, and strictly increasing and continuous in s_i.

Bidders compete for shares of the aggregate quantity $\hat{Q}$. Bidder i submits a weakly positive, weakly decreasing bid function b_i to the auctioneer, expressing her willingness to pay for the q^{th} (infinitesimal) unit. Bidder i's bidding strategy is β^i, so that when her type is s_i she submits bid function $b_i = \beta^i(\cdot; s_i)$. I will denote the bidder's implicit demand functions by $\overline{\varphi}^i$ and $\underline{\varphi}^i$,

$$\overline{\varphi}^i (p; s_i) = \sup \left\{ q : \ \beta^i (q; s_i) \geq p \right\}, \quad \underline{\varphi}^i (p; s_i) = \inf \left\{ q : \ \beta^i (q; s_i) \leq p \right\}.$$

If there is no q such that $\beta^i(q; s_i) \geq p$, then $\overline{\varphi}^i(p; s) = 0$, and if there is no q such that $\beta^i(q; s_i) \leq p$, then $\underline{\varphi}^i(p; s) = \bar{Q}$. Because bids are defined only on the domain of available quantities, $\overline{\varphi}^i(0; s_i) = \bar{Q}$.[16] Conditional on the random shock z_Q, the auctioneer aggregates the submitted bid functions and computes the market-clearing price $p^\star$,

$$p^\star = \inf \left\{ p : \sum_{i=1}^{n} \underline{\varphi}^i (p; s_i) \leq \hat{Q} \leq \sum_{i=1}^{n} \overline{\varphi}^i (p; s_i) \right\}.^{17}$$

Given this price, the auctioneer allocates to each agent her demand at this price. If $\underline{\varphi}^i(p^\star; s_i) = \overline{\varphi}^i(p^\star; s_i)$ for all i (roughly, if $\beta^i(\cdot; s_i)$ is strictly decreasing for each agent), then $q^i(s_1, \ldots, s_n) = \underline{\varphi}^i(p^\star; s_i)$. Otherwise, the auctioneer employs a random priority tiebreaking rule. Let z_q be a random permutation of agents $\{1, \ldots, n\}$, let $\iota(i)$ be such that $z_{q\iota(i)} = i$, and let $T(p) = \sum_{i=1}^{n} \overline{\varphi}^i(p; s_i) -$

[16] That $\overline{\varphi}^i(0; s_i) = \bar{Q}$ for all i and all s ensures that all acceptable bid functions will generate well-defined market outcomes. In particular, there is no issue with determining the proper rationed quantities if all bidders "bid c everywhere," for some constant c.

[17] Although $p^\star$ is a function of $(\beta^i)_{i=1}^n$ and z_Q, for simplicity of notation I write it as its own random variable. The dependence of $p^\star$ on its inputs will be treated properly where necessary.

$\underline{\varphi}^i(p;s_i)$. Bidder i's allocation is

$$q^i\left(b^i,b^{-i};p,z_q\right)=\underline{\varphi}^i\left(p;s_i\right)+\min\left\{T\left(p\right)-\sum_{\iota(k)<\iota(i)}\overline{\varphi}^k\left(p;s_k\right)-\underline{\varphi}^k\left(p;s_k\right),\overline{\varphi}^i\left(p;s_i\right)-\underline{\varphi}^i\left(p;s_i\right)\right\}_+.$$

Henceforth let $z=(z_Q,z_q)$. The tiebreaking rule is not essential to the existence of a pure-strategy equilibrium in the multi-unit discretization of the divisible-good model.[18] Corollary 5 shows that this carries over to the divisible-good model itself.

Once allocations are determined the auctioneer computes transfers from each of the bidders. To capture many common auction formats I define a *standard transfer rule.*

Definition 6 (Standard transfer rule)**.** *Let Y be the set of monotone functions from $[0,\bar{Q}]\to\mathbb{R}_+$. The transfer rule $\tau^i:[0,\bar{Q}]\times Y\times\mathbb{R}_+\times Y^{n-1}\times SuppZ$ is* standard *if*

i. $\tau^i\equiv\tau$ is symmetric across agents;

ii. τ is increasing and uniformly continuous in the bidder's allocation q, the bidder's submitted demand b_i, the market-clearing price p, and opponent bids b_{-i};

iii. $d^+\tau/dq\in[p,b_i(q)]$ for all q such that $b_i(q)>p$;

iv. Interim expected transfers $\mathbb{E}_{s_{-i}}[\tau(q_i;b_i,p,b_{-i},z)]$ are submodular in bid.

Many common auction formats employ standard transfer rules:

- When $\tau(q;b_i,p,b_{-i},z)=\int_0^q b_i(x)dx$, the mechanism is a discriminatory auction;
- When $\tau(q;b_i,p,b_{-i},z)=pq$, the mechanism is a uniform-price auction;[19]
- When $\tau(q;b_i,p,b_{-i},z)=\lambda\int_0^q b_i(x)dx+(1-\lambda)pq$, the mechanism is a random-payment auction;
- When $\tau(q;b_i,p,b_{-i},z)=\bar{p}^\alpha\bar{q}+\int_{\bar{q}}^q b_i(x)dx$, and $\bar{p}^\alpha$ is the α bid percentile and $\bar{q}=\underline{\varphi}^i(\bar{p}^\alpha)$, the mechanism is a quantile-hybrid auction.

[18] This is noted in Häfner (2015), and elsewhere.

[19] In divisible-good auctions the market price p is perfectly recoverable from b_i, b_{-i}, and z. If p is omitted as an argument to the transfer rule, the uniform-price auction is non-standard since payments are not uniformly continuous in bid — a small change in submitted bids can dramatically affect the market clearing price. A standard transfer rule must have a representation which is uniformly continuous in its parameters, but this representation does not need to be unique.

The Vickrey payment rule is non-standard, in line with its relative unpopularity in multi-unit allocation (Ausubel and Milgrom, 2006; Brenner et al., 2009).

Given the transfer rule t, bidder i's ex post utility is

$$u^i\left(b_i, b_{-i}; s_i\right) = \mathbb{E}_z\left[\int_0^{q^i(b_i,b_{-i};z)} v^i\left(x; s_i\right) dx - \tau\left(q^i\left(b_i, b_{-i}; p^\star, z\right); b_i, p^\star, b_{-i}, z\right)\right].$$

Let $X_D = [0, \bar{Q}]$ and $X_R = [0, \bar{b}]$, where $\bar{b} > \max_i \sup_{s_i} v^i(0; s_i)$. Let $Y^\gamma \subset Y$ be the set of Lipschitz continuous functions, with modulus γ, in Y. For an agent i with signal s_i, the feasible action space is

$$A^i\left(s_i\right) = \left\{y \in Y^\gamma : \ y \leq v^i\left(\cdot; s_i\right)\right\}.^{20}$$

Since values are weakly decreasing in quantity and $v^i(0; s_i) < \bar{b}$ for all agents i and signal realizations s_i, $A^i(s_i)$ contains all weakly positive, Lipschitz γ-continuous decreasing functions that are bounded above by the agent's true marginal value. Lipschitz continuity of bids is inessential (and will be eliminated in Proposition 1), and is used to ensure that the the market-clearing price $p^\star$ is well-behaved in bids. When t can be written indepedent of the market-clearing price — as in the discriminatory and quantile-hybrid auctions — the Lipschitz constraint can be ignored. It is retained so that existence can be proved simultaneously in all auctions with standard transfers.

Lemma 4 (Current transfer continuous in bid)**.** *Let $\varepsilon > 0$, and let $b \in A^i(s_i)$ and $b' \in Y$ be such that $||b - b'|| < \varepsilon$. Then there is $\lambda > 0$ such that for any q, any z, and any bid profile b_{-i} of bidder i's opponents,*

$$\left|\tau\left(q; b, p^\star, b_{-i}, z\right) - \tau\left(q; b', p^{\star\prime}, b_{-i}, z\right)\right| < \lambda.$$

Proof. By construction, $||b - b'|| < \varepsilon$. Since bidder i's allocation must weakly increase, the effect on price can be bounded by $|b(q_i) - b'(q_i)|$. Because bids are Lipschitz g-continuous, it must be that $|b(q_i) - b'(q_i)| \leq \sqrt{2\gamma\varepsilon}$. Uniform continuity of τ in its arguments completes the proof. □

I prove the existence of a pure-strategy equilibrium by verifying the conditions necessary to

[20] Kastl (2012) gives a model in which bidders in a uniform-price auction submit bids that are occasionally above their value functions. Similarly, Bertrand competition without private information involves submitting a constant bid "to infinity." The results here establish the existence of an equilibrium in which all bidders submit bids weakly below their value functions, but do not claim that all equilibria must exhibit this property. In related work, Pycia and Woodward (2017) show in a model without private information that all relevant bids must be weakly below values.

apply Theorem 3. Conditions 1 and 2, on the structure of utility and the type-dependent action spaces, are straightforward to demonstrate, and are omitted. Detailed proofs of the following results are given in Appendix B.

Condition 3 (Uniform upper semicontinuity)**.** The allocation function is monotone in bid, and a slight increase in bid will never negatively affect quantity. If a deviation yields additional quantity, since the original bid function was below the agent's marginal value (by construction of the type-dependent action space $A^i(s_i)$), the deviation can be only slightly above the agent's marginal value function $v^i(\cdot; s_i)$, and any gross utility loss from additional quantity is small. Lemma 4 gives that transfers (for units already won) are continuous in bid. Taken together, this implies that upward deviations cannot be discretely harmful, and similar implications hold with respect to downward deviations.

Condition 4 (Local utility security)**.** Consider any bid function b_i and $\lambda > 0$. When bidder i increases her bid from b_i to $b_i + \lambda$, bounded where appropriate by $v^i(\cdot; s_i)$, her utility in the limit is not discretely worse than her utility at the limit. As opponents' bids converge, a discrete upward shift in agent i's bid function will yield a weak increase in the quantity she is allocated. If this does not hold in the limit, the discreteness of the shift implies that her opponents' actions are not converging. Since the upward shift is near a feasible action profile, any losses caused by increasing her bid are commensurate to the size of the shift, by Condition 3. This intuition is roughly illustrated in Figure 1.

Condition 6 (Limit surplus splitting)**.** Surplus splitting is implied by market clearing. In particular, i's utility can jump down at the limit only if her allocated quantity jumps down or her transfer auctioneer jumps up. Lemma 4 rules out the latter the case. If the former occurs with positive probability, then by market clearing there is an opponent who, with positive probability, witnesses a discrete quantity increase at the limit. This is shown in Lemma 23.

Explicit construction of a sequence $\langle \varepsilon_t \rangle_{t=1}^{\infty}$ and a corresponding sequence of ε-discrete models $\mathcal{M}^{\varepsilon_t}$ is given in Appendix B. Building such a sequence to obain a limit of equilibrium strategies is straightforward, and in the case of the discriminatory and uniform-price auctions can be taken directly from the literature (c.f., Reny (2011)).

Condition 9 (Type insensitivity)**.** If a particular type has a best response that is not available to lower-type bidders, it must be that the best response is occasionally equal to her marginal value.

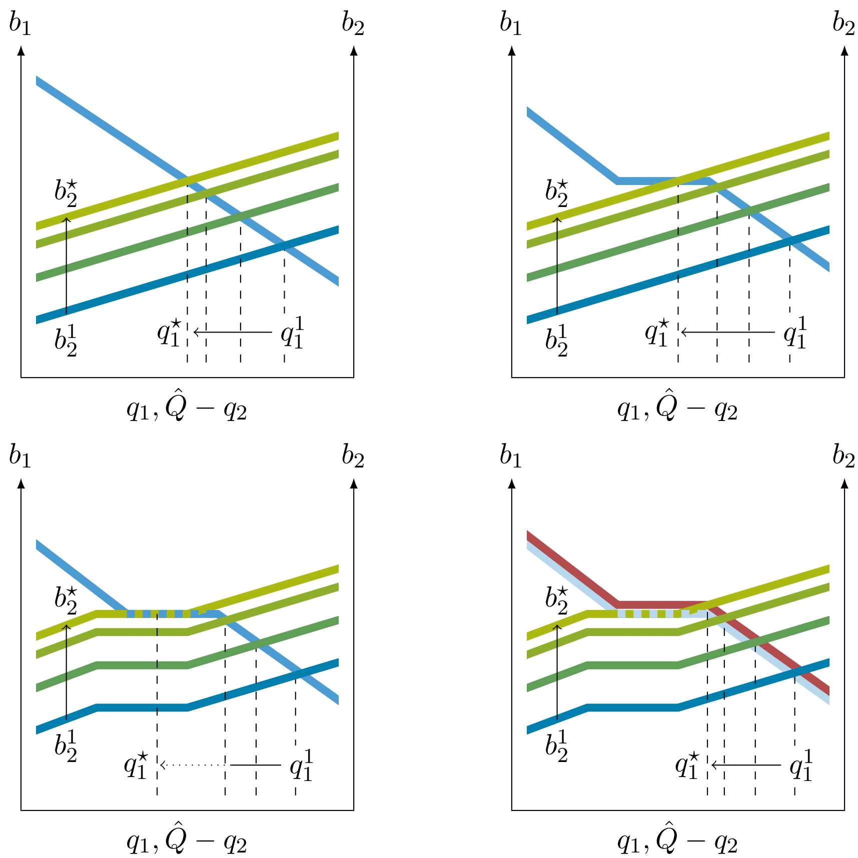

Figure 1: If agent i's utility does not converge in the limit of agent j's actions to her utility at the limit of agent j's actions, it must be that quantity is not converging; if quantity is not converging, it must be that bids are equal and along a common flat. On this interval the tiebreaking rule must be employed, hence a small upward deviation will yield a discretely greater allocation; this deviation is feasible by the assumption that utility is not converging.

Consider an alternate bid function that is at least λ' below the bidder's marginal value. Since standard transfers are monotone in own bid, this will weakly decrease the bidder's payment for any allocation; however, it may also reduce the quantity she receives. If this slight reduction in bid reduces her allocation, her best response must be close to her true marginal value function. The lost quantity does not result in much lost utility, and the bidder can achieve most of her maximum utility by constraining her bid to be within lower types' feasible action spaces.

Lemma 5 (Constrained equilibrium existence in divisible-good auctions)**.** *When transfers are standard and bids are Lipschitz continuous and weakly below values, divisible-good auctions with private information admit monotone constrained pure-strategy equilibria.*

Type-dependent action spaces can be relaxed. If a bid function is infeasible, it is somewhere above the bidder's true marginal value. Consider an alternative bid function, shifted upward slightly from the original and bounded above by the bidder's true value; this alternative bid function is feasible. Vertical shifts cannot discontinuously affect the market-clearing price, so the transfer to the auctioneer varies continuously in this deviation, holding quantity fixed. Then if this deviation is discontinuously unprofitable it must be that quantity is falling, and for quantity to fall discontinuously it must be as a result of bounding the bid function above by marginal value. Anytime a bid above marginal value determines the quantity allocation the market price is above marginal value, and under a standard transfer rule the marginal payment for this unit is above its marginal value. It follows that the alternative bid cannot be discontinuously unprofitable. Then the antecedent of Theorem 3 is satisfied, and there exists an equilibrium in the auction model without type-dependent action spaces.

Corollary 3 (Lipschitz equilibrium existence in divisible-good auctions)**.** *When transfers are standard and bids are Lipschitz continuous, divisible-good auctions with private information admit monotone pure-strategy equilibria.*

The approach taken to establish equilibrium existence applies equally well to convergence as the Lipschitz modulus approaches infinity. Bids and aggregate demand converge, and for any signal profile the market price either converges or jumps discontinuously downward. If all terms are converging, that limiting strategies constitute an equilibrium follows the same argument as why the limit of discrete equilibria is an equilibrium. If market price jumps down at the limit, either no agent's utility is affected (as in a discriminatory auction) or some agent sees a discrete utility improvement at the limit. But as in earlier arguments this agent could have realized this utility improvement near the limit, contradicting the limit being constructed from a sequence of equilibria.

Proposition 1 (Equilibrium existence in divisible-good auctions)**.** *Divisible-good auctions with standard transfers and private information admit isotone pure-strategy equilibria.*

With regard to market outcomes, it can be shown that allocations are utility-relevant; this immediately implies that seller revenues are also utility-relevant. Theorem 2 then implies that quantity and revenue in the ε_t-discrete auctions are approximated by quantity and revenue in the divisible-good auction.

Corollary 4 (Probabilistic convergence of observables)**.** *Let $q : Y \to \mathbb{R}_+^n$ and $\pi : Y \to \mathbb{R}_+$ represent ex post expected allocations and revenue, respectively, in the divisible-good model $\mathcal{M}$. If $\langle \beta^t \rangle_{t=1}^{\infty}$ is a sequence of monotone pure-strategy equilibria in the ε_t-discretized models converging to the supremum-limit strategy profile $\overline{\beta}$, then*

$$q\left(\beta^t\left(s\right)\right) \xrightarrow{P} q\left(\overline{\beta}\left(s\right)\right) \text{ and } \pi\left(\beta^t\left(s\right)\right) \xrightarrow{P} \pi\left(\overline{\beta}\left(s\right)\right).$$

5 Conclusion

This article proves the existence of monotone Bayesian Nash and pure-strategy equilibria in games in which actions are monotone functions. Application of these results to an economic model requires the specification of type-dependent action spaces and a sequence of discretized models such that (Condition 3) slight upward deviations are not discretely harmful, (Condition 4) utility at a limit of actions can be nearly achieved in a limit of actions, (Condition 6) one agent's loss can be transformed into another agent's gain, and (Condition 7) the discretized models each admit a monotone Bayesian Nash equilibrium. Equilibrium existence is established by examining a limit of a sequence of equilibria of discretized models, and showing that utility must converge. This result immediately suggests that equilibrium (and equilibrium outcomes) in these games can provide a natural approximation of equilibria in the nearby discrete games.

I apply these results to a model of divisible-good auctions with private information, under the assumption that transfers to the auctioneer are well-behaved. This allows me to obtain novel existence results. The equilibrium approximation results show that quantity allocations and seller revenue in the divisible-good model are close to their counterparts in nearby multi-unit auctions. This suggests that the divisible-good auction model could be a fruitful approach to understanding multi-unit auctions, which are known to suffer from intractability.

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A Proofs of main results

The proof of equilibrium existence proceeds by defining *limiting strategies*, derived from equilibria of sequential refinements of the ε-discrete model $\mathcal{M}^{\varepsilon}$. Because the proofs below frequently consider sets constrained to the rational numbers, the following shorthands will be used:

$$\mathcal{S} = (0,1)^{m_s} \cap \mathbb{Q}^{m_s}, \qquad \mathcal{S}^C = (0,1)^{m_s} \setminus \mathcal{S};$$
$$\mathcal{X} = X_D \cap \mathbb{Q}^{m_D}, \qquad \mathcal{X}^C = X_D \setminus \mathcal{Q}^{m_D}.$$

Definition 7 (Limiting strategies)**.** *Strategies* $(\alpha^{i,\square})_{i=1}^n$ *are* limiting strategies *if there exists a monotone decreasing sequence* $\langle \varepsilon_t \rangle_{t=1}^{\infty}$*,* $\varepsilon_t \searrow 0$*, and a sequence of equilibria of the* ε_t*-discrete model* $\mathcal{M}^{\varepsilon_t}$*,* $(\alpha^{i,t})_{i=1}^n$ *such that:*

1. $\alpha^{i,\square}$ *is monotone in all arguments;*

2. *For all* $(x, s_i) \in \mathcal{X} \times \mathcal{S}$*,* $[\alpha^{i,t}(s_i)](x) \to [\alpha^{i,\square}(s_i)](x)$*.*

At all rational coordinate pairs, limiting strategies take values equal to the limits of equilibrium strategies in the ε_t-discrete models at these points. When either coordinate is irrational, limiting strategies may take any value which satisfies the stated monotonicity constraints. Monotonicity of $\alpha^{i,\square}(s_i)$, as stated in point 1 above, is guaranteed by monotonicity of functions in $A^i(s_i)$, however monotonicity of $[\alpha^{i,\square}(\cdot)](x)$ must be explicitly stated: although the existence results in Reny (2011) guarantee the existence of a monotone equilibrium in $\mathcal{M}^{\varepsilon_t}$, it is possible that in some contexts a nonmonotone equilibrium will exist. The proof of existence below relies on monotonicity in both dimensions, hence point 1 is crucial.

Lemma 6 establishes that limiting strategies exist.

Lemma 6 (Existence of limiting strategies)**.** *Given any monotone decreasing sequence* $\langle \varepsilon_t \rangle_{t=1}^{\infty}$*, there is a subsequence* $\langle \varepsilon_{t_k} \rangle_{k=1}^{\infty}$*, that admits limiting strategies* $(\alpha^{i,\square})_{i=1}^n$

Proof. Condition 7 establishes that for all t, there is a pure-strategy equilibrium $(\alpha^{i,t})$ of the ε_t-discrete model $\mathcal{M}^{\varepsilon_t}$. Selection results (Widder (1941), Theorem 16.1) imply that for any countable $\tilde{\mathcal{X}} \times \tilde{\mathcal{S}}$ there is a subsequence $\langle \varepsilon_{t_k} \rangle_{k=1}^{\infty}$ such that $[\alpha^{i,t_k}(s_i)](x) \to [\alpha^{i,\square}(s_i)](x)$ pointwise for all i and all $(x, s_i) \in \tilde{\mathcal{X}} \times \tilde{\mathcal{S}}$. For any such sets monotonicity of $\alpha^{i,\square}$ is guaranteed by the fact that $\alpha^{i,t}$ is monotone for all i and all ε_t. The desired result follows from letting $\tilde{\mathcal{X}} = \mathcal{X}$ and $\tilde{\mathcal{S}} = \mathcal{S}$. □

Proof of Lemma 1. This is a restatement of Lemma 6, under the maintained assumption that all sequences converge. □

Lemma 7 (L^1 convergence on $\mathcal{S}$)**.** *Let* $(\alpha^{i,\square})_{i=1}^n$ *be limiting strategies associated with some sequence* $\langle(\alpha^{i,t})_{i=1}^n\rangle_{t=1}^\infty$ *of* $\mathcal{M}^{\varepsilon_t}$*-equilibria. Then for all* i *and all* $s_i \in \mathcal{S}$, $\alpha^{i,t}(s_i) \to \alpha^{i,\square}(s_i)$.

Proof. Suppose that $\alpha^{i,\square}(s_i)$ is continuous at some $x \in X_D$. Then for all $\lambda > 0$ there is a $\delta > 0$ such that $|[\alpha^{i,\square}(s_i)](x) - [\alpha^{i,\square}(s_i)](x+\delta')| < \lambda$ for all $\delta' \in (-\delta, \delta)$. Since $\mathcal{Q}$ is dense, there are $x_\ell, x_r \in \mathcal{X} \times (x-\delta, x+\delta)$ such that $x_\ell < x < x_r$; by pointwise convergence of $\alpha^{i,t}$ to $\alpha^{i,\square}$ on $\mathcal{X} \times \mathcal{S}$, there is a T such that $|[\alpha^{i,t}(s_i)](x') - [\alpha^{i,\square}(s_i)](x')| < \lambda$ for $x' \in \{x_\ell, x_r\}$ and all $t > T$.

The difference between $[\alpha^{i,\square}(s_i)](x_\ell)$ and $[\alpha^{i,\square}(s_i)](x_r)$ is bounded,

$$\begin{aligned}\left|\left[\alpha^{i,\square}(s_i)\right](x_\ell) - \left[\alpha^{i,\square}(s_i)\right](x_r)\right| &= \left|\left[\alpha^{i,\square}(s_i)\right](x_\ell) - \left[\alpha^{i,\square}(s_i)\right](x)\right| \\ &\quad + \left|\left[\alpha^{i,\square}(s_i)\right](x) - \left[\alpha^{i,\square}(s)\right](x_r)\right| < 2\lambda.\end{aligned}$$

This implies

$$\begin{aligned}\left|\left[\alpha^{i,t}(s_i)\right](x_\ell) - \left[\alpha^{i,t}(s_i)\right](x_r)\right| \leq \Big[&\left|\left[\alpha^{i,t}(s_i)\right](x_\ell) - \left[\alpha^{i,\square}(s_i)\right](x_\ell)\right| \\ &+ \left|\left[\alpha^{i,\square}(s_i)\right](x_\ell) - \left[\alpha^{i,\square}(s_i)\right](x_r)\right| \\ &+ \left|\left[\alpha^{i,\square}(s_i)\right](x_r) - \left[\alpha^{i,t}(s_i)\right](x_r)\right| \Big] < 4\lambda.\end{aligned}$$

Since $\alpha^{i,t}(s_i)$ is monotone, this further implies that $|[\alpha^{i,t}(s_i)](x') - [\alpha^{i,t}(s_i)](x)| < 4\lambda$ for $x' \in \{x_\ell, x_r\}$. Then

$$\begin{aligned}\left|\left[\alpha^{i,t}(s_i)\right](x) - \left[\alpha^{i,\square}(s_i)\right](x)\right| \leq \Big[&\left|\left[\alpha^{i,t}(s_i)\right](x) - \left[\alpha^{i,t}(s_i)\right](x_\ell)\right| \\ &+ \left|\left[\alpha^{i,t}(s_i)\right](x_\ell) - \left[\alpha^{i,\square}(s_i)\right](x_\ell)\right| \\ &+ \left|\left[\alpha^{i,\square}(s_i)\right](x_\ell) - \left[\alpha^{i,\square}(s_i)\right](x)\right| \Big] < 6\lambda.\end{aligned}$$

Since $\lambda > 0$ may be arbitrarily small, it follows that there is T' such that $|[\alpha^{i,t}(s_i)](x) - [\alpha^{i,\square}(s_i)](x)| < \lambda$ for all $t > T'$. Then $[\alpha^{i,t}(s_i)](x) \to [\alpha^{i,\square}(s_i)](x)$ whenever $\alpha^{i,\square}(s_i)$ is continuous at x.

Since $\alpha^{i,\square}(s_i)$ is a monotone bounded function, it has at most a measure-zero set of discontinuities; hence $[\alpha^{i,t}(s_i)](x) \to [\alpha^{i,\square}(s)](x)$ for almost all x, and thus $\alpha^{i,t}(s_i) \to \alpha^{i,\square}(s_i)$. $\square$

Limiting strategies are defined almost nowhere. However, since limiting strategies are monotonic in all dimensions and map into a compact space, they can be used to naturally define functions on all of $(0,1)^{m_s} \times X_D$.

Recall that a strategy $\overline{\alpha}^i$ is a supremum-limit strategy if $\alpha^{i,\square}$ is the pointwise limit of some sequence of equilibrium strategies $\langle \alpha^{i,t} \rangle_{t=1}^{\infty}$ and $\overline{\alpha}^i(s_i) = \sup_{s' < s_i} \overline{\alpha}^i(s')$ for all s_i. The choice of supremum in this construction relates to Condition 3, which ensures that small upward deviations are not discretely unprofitable. In what follows, I will fix a particular convergent sequence of discretized equilibria $\langle (\alpha^{i,t})_{i=1}^n \rangle_{t=1}^{\infty}$, an associated limiting strategy profile $(\alpha^{i,\square})_{i=1}^n$, and an associated supremum-limit strategy profile $(\overline{\alpha}^i)_{i=1}^n$.

Lemma 8 (Almost-sure convergence to supremum-limit strategies)**.** *For all i, $\alpha^{i,t}(s_i) \to \overline{\alpha}^i(s_i)$ with probability one.*

Proof. This proof is made notationally simpler by using measure-theoretic language. Since s_i is distributed as m independent uniform draws, it is without loss of generality to interchange Lebesgue measure and signal probability.

Note that any limiting strategy $\alpha^{i,\square}$ has at most a measure-zero set of discontinuities (Lavrič, 1993), so $\alpha^{i,t} \to \overline{\alpha}^i$. Let $\tilde{\alpha}^i$ be a completion of $\alpha^{i,\square}$ such that $[\tilde{\alpha}^i(s_i)](x) = [\alpha^{i,\square}(s_i)](x)$ whenever $[\alpha^{i,\square}(\cdot)](\cdot)$ is continuous at $(x; s_i)$; then $|[\alpha^{i,\square}(s_i)](x) - [\tilde{\alpha}^i(s_i)](x)| = 0$; adapting arguments from Lemma 7 implies that $\alpha^{i,t}(s_i) \to \tilde{\alpha}^i(s_i)$.

Let S_δ be the set signals s with δ-nonconvergent actions,

$$S_\delta = \left\{ s_i' : \lim_{t \nearrow \infty} \left|\left| \alpha^{i,t}\left(s_i'\right) - \tilde{\alpha}^i\left(s_i'\right) \right|\right| > \delta \right\},$$

$$\rightsquigarrow S_0 = \left\{ s_i' : \lim_{t \nearrow \infty} \left|\left| \alpha^{i,t}\left(s_i'\right) - \tilde{\alpha}^i\left(s_i'\right) \right|\right| > 0 \right\} = \bigcup_{w \in \mathbb{N}} S_{1/2^w}.$$

If $s \in S_0$ with positive probability, then there is $w \in \mathbb{N}$ such that $s \in S_{1/2^w}$ with positive probability.

Let μ^k be the Lebesgue measure on $\mathbb{R}^k$. Consider the measure of all points of nonconvergence,

$$\begin{aligned}
&\mu^{m_D+m_s}\left(\left\{\left(x,s_i'\right):\left|\left[\alpha^{i,t}\left(s_i'\right)\right](x)-\left[\tilde{\alpha}^i\left(s_i'\right)\right](x)\right|\not\to 0\right\}\right)\\
&=\int_{(0,1)^{m_D}}\mu^{m_D}\left(\left\{x:\lim_{t\nearrow\infty}\left|\left[\alpha^{i,t}\left(s_i'\right)\right](x)-\left[\tilde{\alpha}^i\left(s_i'\right)\right](x)\right|>0\right\}\right)d\mu^{m_D}\left(s_i'\right)\\
&\geq\int_{s_i'\in S_{1/2^w}}\mu^{m_D}\left(\left\{x:\lim_{t\nearrow\infty}\left|\left[\alpha^{i,t}\left(s_i'\right)\right](x)-\left[\tilde{\alpha}^i\left(s_i'\right)\right](x)\right|>0\right\}\right)d\mu^{m_D}\left(s_i'\right).
\end{aligned}$$

Let $\overline{x},\underline{x}\in\mathbb{R}^{m_R}$ be upper and lower bounds, respectively, for X_R, and define $\Delta=||\overline{x}-\underline{x}||$. Note that for any $s_i'\in S_{1/2^w}$, the boundedness of A^i is sufficient to imply that[21]

$$\mu^{m_D}\left(\left\{x:\lim_{t\nearrow\infty}\left|\left[\alpha^{i,t}\left(s_i'\right)\right](x)-\left[\tilde{\alpha}^i\left(s_i'\right)\right](x)\right|>0\right\}\right)\geq\frac{1}{2^w\Delta}.$$

Then it follows that

$$\begin{aligned}
&\mu^{m_D+m_S}\left(\left\{\left(x,s_i'\right):\left|\left[\alpha^{i,t}\left(s_i'\right)\right](x)-\left[\tilde{\alpha}^i\left(s_i'\right)\right](x)\right|\not\to 0\right\}\right)\\
&\geq\int_{s_i'\in S_{1/2^w}}\frac{1}{2^w\Delta}d\mu^m\left(s_i'\right)=\frac{\mu^m\left(S_{1/2^w}\right)}{2^w\Delta}>0.
\end{aligned}$$

Then $\mu^{m_D+m_s}(\{(x,s_i'):|[\alpha^{i,t}(s_i')](x)-[\tilde{\alpha}^i(s_i')](x)|>0\})>0$, contradicting the fact that $\alpha^{i,t}\to\tilde{\alpha}^i$. Since $\overline{\alpha}^i$ is a completion of $\alpha^{i,\square}$, it follows that $||\alpha^{i,t}(s_i)-\overline{\alpha}^i(s_i)||\to 0$ for almost all s_i. □

Definition 8 (Upper $\mathcal{M}^\varepsilon$-approximation)**.** *The* upper $\mathcal{M}^\varepsilon$ approximation a^ε *of action* $a\in A^i(s_i)$ *is given by*

$$\mathsf{a}^\varepsilon(x)\in\underset{\mathsf{a}'\in A^{i,\varepsilon}(s_i),\ \mathsf{a}'\geq a}{\operatorname{arginf}}\left|\left|a-\mathsf{a}'\right|\right|.$$

Since A^{i,ε_t} is finite, an upper $\mathcal{M}^{\varepsilon_t}$ approximation of a exists (for t sufficiently large) and is in A^{i,ε_t} so long as Condition 8 is satisfied.

Lemma 9 (Interim utility approximation)**.** *There is a constant* $C\in\mathbb{R}_+$ *such that for any* $a_i\in A^i(s_i)$*,*

$$U^i\left(\mathsf{a}_i^\varepsilon,\alpha^{-i};s_i\right)\geq U^i\left(a,\alpha^{-i};s_i\right)-g\left(C\varepsilon;s_i\right).$$

Proof. Let $x\in X^D$. Then if $x-3\varepsilon\in X_D$, there is $\mathsf{x}\in X_D^\varepsilon$ such that $x-3\varepsilon<\mathsf{x}<x$. By

[21] Technically this must also include a term for the possible difference between $\alpha^{i,t}$ and the nearest element of $A^i(s_i)$. This difference is at most linear, and hence the argument does not change.

construction, $\mathsf{a}_i^\varepsilon(\mathsf{x}) < a_i(\mathsf{x}) + \varepsilon$, and by monotonicity

$$\mathsf{a}_i^\varepsilon(x) \leq \mathsf{a}_i^\varepsilon(\mathsf{x}) < a_i(\mathsf{x}) + \varepsilon \leq a_i(x - 3\varepsilon) + \varepsilon.$$

Let $\overline{x}, \underline{x}$ be upper and lower bounds, respectively, for X_R. Since a_i^ε is monotone, $\mathsf{a}_i^\varepsilon \geq y$, and $X_R \geq 0$,

$$\begin{aligned}
||a_i - \mathsf{a}_i^\varepsilon|| &= \int_{X_D} \left|\mathsf{a}_i^\varepsilon(x) - a_i(x)\right| dx \\
&= \int_{X_D} \left|\mathsf{a}_i^\varepsilon(x)\right| dx - \int_{X_D} |a_i(x)|\, dx \\
&\leq \int_{X_D \setminus (X_D + 3\varepsilon)} |\overline{x}|\, dx + \int_{X_D \cap (X_D + 3\varepsilon)} |a_i(x - 3\varepsilon) + \varepsilon| \\
&\qquad - \int_{X_D \cap (X_D - 3\varepsilon)} |a_i(x)|\, dx - \int_{X_D \setminus (X_D - 3\varepsilon)} |\underline{x}|\, dx \\
&\leq 3\varepsilon\, |\overline{x}| + \int_{X_D \cap (X_D - 3\varepsilon)} |a_i(x) + \varepsilon| - |a_i(x)|\, dx \leq 4\varepsilon\, |\overline{x}|\,.
\end{aligned}$$

By construction, $\mathsf{a}_i^\varepsilon \in A^{i,\varepsilon}(s_i)$ and $a_i \in A^i(s_i)$, hence Condition 3 implies that

$$U^i\left(\mathsf{a}_i^\varepsilon, \alpha^{-i}; s_i\right) \geq U^i\left(a_i, \alpha^{-i}; s_i\right) - g\left(C\varepsilon; s_i\right).$$

□

Corollary 5 (Existence of utility approximation)**.** *For t sufficiently large, given any $a \in A^i(s_i)$, there is $\mathsf{a}^{\varepsilon_t} \in A^{i,\varepsilon_t}(s_i)$ such that*

$$U^i\left(\mathsf{a}^{\varepsilon_t}, \alpha^{-i}; s_i\right) \geq U^i\left(a, \alpha^{-i}; s_i\right) - g\left(C\varepsilon_t; s_i\right).$$

Lemma 10 (Almost no upward jumps at limit)**.** *For all agents i,*

$$\Pr{}_{s_i}\left(\lim_{t \nearrow \infty} U^i\left(\alpha^{i,t}(s_i), \alpha^{-i,t}; s_i\right) \geq U^i\left(\overline{\alpha}^i(s_i), \overline{\alpha}^{-i}; s_i\right)\right) = 1.$$

Proof. This is an application of Lemma 8, Condition 4, and Corollary 5. Lemma 8 implies that with s_i-probability one,

$$\lim_{t \nearrow 0} \left|\left|\alpha^{i,t}\left(s_i\right) - \overline{\alpha}^i\left(s_i\right)\right|\right| = 0.$$

Then it is sufficient to prove the claim of this Lemma under the assumption that agent i's action converges when her signal is s_i.

Assume that there is $\delta > 0$ such that

$$\lim_{t \nearrow \infty} U^i\left(\alpha^{i,t}\left(s_i\right), \alpha^{-i}; s_i\right) < U^i\left(\overline{\alpha}^i\left(s_i\right), \overline{\alpha}^{-i}; s_i\right) - 3\delta.$$

By Condition 4 there is $a \in A^i(s_i)$ such that

$$\lim_{t \nearrow \infty} U^i\left(a, \alpha^{-i}; s_i\right) > U^i\left(\overline{\alpha}^i\left(s_i\right), \overline{\alpha}^{-i}; s_i\right) - \delta > \lim_{t \nearrow \infty} U^i\left(\alpha^{i,t}\left(s_i\right), \alpha^{-i,t}; s_i\right) + 2\delta.$$

For any t, Lemma 9 implies that there is $\mathsf{a}^{\varepsilon_t} \in A^{i,\varepsilon_t}(s_i)$ such that

$$U^i\left(\mathsf{a}^{\varepsilon_t}, \alpha^{-i,t}; s_i\right) \geq U^i\left(a, \alpha^{-i,t}; s_i\right) - g\left(C\varepsilon_t; s_i\right).$$

Putting these inequalities together, it follows that

$$\lim_{t \nearrow \infty} U^i\left(\mathsf{a}^{\varepsilon_t}, \alpha^{-i,t}; s_i\right) + g\left(C\varepsilon_t; s_i\right) > \lim_{t \nearrow \infty} U^i\left(\alpha^{i,t}\left(s_i\right), \alpha^{-i,t}; s_i\right) + 2\delta.$$

Then there is T such that for all $t > T$,

$$U^i\left(\mathsf{a}^{\varepsilon_t}, \alpha^{-i,t}; s_i\right) > U^i\left(\alpha^{i,t}\left(s_i\right), \alpha^{-i,t}; s_i\right) + \delta.$$

Since $\mathsf{a}^{\varepsilon_t}$ is an action available to agent i in $\mathcal{M}^{\varepsilon_t}$, this implies that $\alpha^{i,t}(s_i)$ is not a best response. Since the strategy $\alpha^{i,t}$ is a best response for agent i in model $\mathcal{M}^{\varepsilon_t}$, this can only happen with probability zero. □

Lemma 11 (Ex post uniform upper semicontinuity). *Condition 3 is satisfied if and only if there is a continuous function* $\hat{g} : \mathbb{R}_+ \times (0,1)^m \to \mathbb{R}_+$, $\hat{g}(0; \cdot) = 0$, *such that for all agents* i, *all* $s_i \in (0,1)^m$, *all* $(a_j)_{j\neq i} \in Y^{n-1}$*,* $a_i \in A^i(s_i)$*, and all* $\overline{a}_i \in Y$ *with* $a_i \leq \overline{a}_i$*,*

$$u^i\left(a_i, a_{-i}; s_i\right) \leq u^i\left(\overline{a}_i, a_{-i}; s_i\right) + \hat{g}\left(||\overline{a}_i - a_i||\,; s_i\right).$$

Proof. Given a feasible action profile $(a_j)_{j\neq i}$ for agent i's opponents, let $\alpha^{-i} = (\alpha^j)_{j\neq i}$ be a strategy profile such that $\alpha^j(\cdot) = a_j$ for all $j \neq i$. Then Condition 3 implies the above variant.

Now, fix a strategy profile $\alpha^{-i} = (\alpha^j)_{j\neq i}$. Note that

$$\begin{aligned} U^i\left(\underline{a}_i, \alpha^{-i}; s_i\right) &= \mathbb{E}_{s_{-i}}\left[u^i\left(\overline{a}_i, \alpha^{-i}\left(s_{-i}\right); s_i\right)\right] \\ &\geq \mathbb{E}_{s_{-i}}\left[u^i\left(a_i, \alpha^{-i}\left(s_{-i}\right); s_i\right) + g\left(||\overline{a}_i - a_i||\,; s_i\right)\right] \\ &= U^i\left(a_i, \alpha^{-i}; s_i\right) + g\left(||\overline{a}_i - a_i||\,; s_i\right). \end{aligned}$$

This implies Condition 3. □

Lemma 12 (Convergence of monotone functions)**.** *Let* $\langle f^t\rangle_{t=1}^{\infty}$ *be a sequence of functions,* $f^t : (0,1)^m \times X_D \to X_R$*, such that* $f^t(w;\cdot)$ *is monotone for all* t *and* w*. Suppose that for all* $w \in (0,1)^m$*,* $f^t(w;\cdot) \to f^\star$*. Then* $\sup f^t = \inf\{f \in Y :\ \forall w \in (0,1)^{\hat{m}}\ f \geq f^t(w;\cdot)\} \to f^\star$*.*

Proof. Suppose otherwise, and let $\bar{f}^t = \sup f^t$ and $\bar{f}^\star = \lim_{t\nearrow\infty} \bar{f}^t$. Since each f^t is monotone, $\bar{f}^t$ and hence $\bar{f}^\star$ are monotone and continuous almost everywhere (Lavrič, 1993). Then if $||\bar{f}^\star - f^\star|| \neq 0$, there is $\delta > 0$ and an $x \in X_D$ such that $f^\star$ is continuous at x and $\bar{f}^\star(x) > f^\star(x) + 4\delta$, and there is $\varepsilon > 0$ such that $f^\star(x') < f^\star(x) + \delta$ for all $x' \in [x, x+\varepsilon)$.

Since $\bar{f}^t \to \bar{f}^\star$, for all T there is $t > T$ such that $\bar{f}^t(x) > f^\star(x) + 3\delta$, and thus a signal $w \in (0,1)$ such that $f^t(w;x) > f^\star(x) + 2\delta$. By monotonicity, it follows that $f^t(w;x') > f^\star(x') + \delta$ for all $x' \in [x, x+\varepsilon)$, and $||f^t(s;\cdot) - f^\star|| > \varepsilon\delta$. Since ε and δ are independent of t (provided it is sufficiently large), this contradicts $f^t(w;\cdot) \to f^\star$. □

Lemma 13 (Surplus splitting implies limit surplus splitting)**.** *Let* $(\alpha^{k,\star})_{k=1}^n$ *be a feasible strategy profile, and let* $\langle(\alpha^{k,t})_{k=1}^n\rangle_{t=1}^{\infty}$ *be a sequence of strategies converging to* $(\alpha^{k,\star})_{k=1}^n$*. Suppose that there is an agent* i *and a set* S_i *with* $\Pr(s_i \in S_i) > 0$ *such that for all* $s_i \in S_i$*,*

$$\lim_{t\nearrow\infty} U^i\left(\alpha^{i,t}\left(s_i\right), \alpha^{-i,t}; s_i\right) > U^i\left(\alpha^{i,\star}\left(s_i\right), \alpha^{-i,\star}; s_i\right).$$

If Conditions 3 and 5 are satisfied, then there is an agent j, a set S_j with $\Pr(s_j \in S_j) > 0$, for each $s_j \in S_j$ a set $S_{-j}(s_j)$ with $\Pr(s_{-j} \in S_{-j}(s_j)) > 0$, and for any $\lambda > 0$ a sequence $\langle \hat{\alpha}^{j,t} \rangle_{t=1}^{\infty}$, $\hat{\alpha}^{j,t}(s_j) \in A^j(s_j)$ and $||\hat{\alpha}^{j,t}(s_j) - \alpha^{j,\star}(s_j)|| < \lambda$ for all t sufficiently large, such that for all $s_j \in S_j$ and $s_{-j} \in S_{-j}(s_j)$,

$$\lim_{t \nearrow \infty} u^j \left(\hat{\alpha}^{j,t}(s_j), \alpha^{-j,t}(s_{-j}); s_j \right) > \lim_{t \nearrow \infty} u^j \left(\alpha^{j,t}(s_j), \alpha^{-j,t}(s_{-j}); s_j \right).$$

Proof. Let $\langle (\alpha^{k,t})_{k=1}^{n} \rangle_{t=1}^{\infty}$ be a sequence of strategies converging to $(\alpha^{k,\star})_{k=1}^{n}$, and suppose that there is an agent i and a set S_i with $\Pr(s_i \in S_i) > 0$ such that, for all $s_i \in S_i$,

$$\lim_{t \nearrow \infty} U^i \left(\alpha^{i,t}(s_i), \alpha^{-i,t}; s_i \right) > U^i \left(\alpha^{i,\star}(s_i), \alpha^{-i,\star}; s_i \right).$$

Then for each $s_i \in S_i$ there is a set $S_{-i}(s_i)$ with $\Pr(s_{-i} \in S_{-i}(s_i)) > 0$ such that for all $s_{-i} \in S_{-i}(s_i)$,

$$\lim_{t \nearrow \infty} u^i \left(\alpha^{i,t}(s_i), \alpha^{-i,t}(s_{-i}); s_i \right) > u^i \left(\alpha^{i,\star}(s_i), \alpha^{-i,\star}(s_{-i}); s_i \right).$$

Condition 5 implies that there is an agent j, a set S_j with $\Pr(s_j \in S_j) > 0$, and for each $s_j \in S_j$ a set $S_{-j}(s_j)$ with $\Pr(s_{-j} \in S_{-j}(s_j)) > 0$ such that for all $s_j \in S_j$ and all $s_{-j} \in S_{-j}(s_j)$,

$$\lim_{t \nearrow \infty} u^j \left(\alpha^{j,t}(s_j), \alpha^{-j,t}(s_{-j}); s_j \right) < u^j \left(\alpha^{j,\star}(s_j), \alpha^{-j,\star}(s_{-j}); s_j \right).$$

By Condition 4 and Lemma 11, for each such (s_j, s_{-j}) and any $\lambda > 0$, there is $\hat{a}_j(s_j, s_{-j}; \lambda) \geq \alpha^{j,\star}(s_j)$ such that $||\hat{a}_j(s_j, s_{-j}; \lambda) - \alpha^{j,\star}(s_j)|| < \lambda$ and

$$\lim_{t \nearrow \infty} u^j \left(\hat{a}_j(s_j, s_{-j}; \lambda), \alpha^{-j,t}(s_{-j}); s_j \right) > u^j \left(\alpha^{j,\star}(s_j), \alpha^{-j,\star}(s_{-j}); s_j \right) - \lambda.$$

Let $\overline{a}_j(s_j; \lambda) = \sup_{s_{-j} \in S_{-j}(s_j)} \hat{a}_j(s_j, s_{-j}; \lambda)$; since $A^j(s_j)$ is a complete lattice, $\overline{a}_j(s_j; \lambda) \in A^j(s_j)$. By Lemma 12, $\lambda \searrow 0$ implies $||\overline{a}_j(s_j; \lambda) - \alpha^{j,\star}(s_j)|| \equiv d(s_j, \lambda) \to 0$.

By Lemma 11,

$$u^j \left(\overline{a}_j(s_j; \lambda), \alpha^{-j,t}(s_{-j}); s_j \right) > u^j \left(\alpha^{j,\star}(s_j), \alpha^{-j,\star}(s_{-j}); s_j \right) - \hat{g} \left(\left|\left| \overline{a}_j(s_j; \lambda) - \alpha^{j,\star}(s_j) \right|\right|; s_j \right).$$

Let $\delta(s_j, s_{-j}) = u^j(\alpha^{j,\star}(s_j), \alpha^{-j,\star}(s_{-j}); s_j) - \lim_{t\nearrow\infty} u^j(\alpha^{j,t}(s_j), \alpha^{-j,t}(s_{-j}); s_j)$. For all $s_{-j} \in S_{-j}(s_j)$, $\delta(s_j, s_{-j}) > 0$. Then

$$\begin{aligned}
&\lim_{t\nearrow\infty} u^j\left(\overline{a}_j\left(s_j;\lambda\right), \alpha^{-j,t}\left(s_{-j}\right); s_j\right) \\
&\quad\geq u^j\left(\alpha^{j,\star}\left(s_j\right), \alpha^{-j,\star}\left(s_{-j}\right); s_j\right) - \hat{g}\left(\lambda + d\left(s_j,\lambda\right); s_j\right) \\
&\quad= \lim_{t\nearrow\infty} u^j\left(\alpha^{j,t}\left(s_j\right), \alpha^{-j,t}\left(s_{-j}\right); s_j\right) + \delta\left(s_j, s_{-j}\right) - \hat{g}\left(\lambda + d\left(s_j,\lambda\right); s_j\right).
\end{aligned}$$

Then for λ sufficiently small,

$$\lim_{t\nearrow\infty} u^j\left(\overline{a}_j\left(s_j;\lambda\right), \alpha^{-j,t}\left(s_{-j}\right); s_j\right) > \lim_{t\nearrow\infty} u^j\left(\alpha^{j,t}\left(s_j\right), \alpha^{-j,t}\left(s_{-j}\right); s_j\right).$$

In fact, for λ sufficiently small this relation holds for a set $\hat{S}_j \subseteq S_j$, $\Pr(s_j \in \hat{S}_j) > 0$, and set $\hat{S}_{-j}(s_j) \subseteq S_{-j}(s_j)$, $\Pr(s_{-j} \in \hat{S}_{-j}(s_j)) > 0$, for all $s_j \in \hat{S}_j$. Then the result is shown. □

Lemma 14 (Almost no downward jumps at limit)**.** *For all agents i,*

$$\Pr{}_{s_i}\left(\lim_{t\nearrow\infty} U^i\left(\alpha^{i,t}\left(s_i\right), \alpha^{-i,t}; s_i\right) \leq U^i\left(\overline{\alpha}^i\left(s_i\right), \overline{\alpha}^{-i}; s_i\right)\right) = 1.$$

Proof. This follows from Lemma 10, Conditions 3, 4, and 6, and Lemmas 11 and 13. Let S_i be the set of signals for which agent i's utility in the limit is strictly greater than her utility at the limit,

$$S_i = \left\{\lim_{t\nearrow\infty} U^i\left(\alpha^{i,t}\left(s_i\right), \alpha^{-i,t}; s_i\right) > U^i\left(\overline{\alpha}^i\left(s_i\right), \overline{\alpha}^{-i}; s_i\right)\right\}.$$

Suppose that $\Pr(s_i \in S_i) > 0$, and let $S_{-i} : S_i \rightrightarrows [(0,1)^m]^{n-1}$ be given by

$$S_{-i}\left(s_i\right) = \left\{s_{-i} : \lim_{t\nearrow\infty} u^i\left(\alpha^{i,t}\left(s_i\right), \alpha^{-i,t}\left(s_{-i}\right); s_i\right) > u^i\left(\overline{\alpha}^i\left(s_i\right), \overline{\alpha}^{-i}\left(s_{-i}\right); s_i\right)\right\}.$$

The boundedness of u^i implies that for any $s_i \in S_i$, $\Pr(s_{-i} \in S_{-i}(s_i)) > 0$. Then by Lemma 13 there is a $\delta > 0$, an agent j, a set S_j with $\Pr(s_j \in S_j) > 0$, for each $s_j \in S_j$ a set $S_{-j}(s_j)$ with $\Pr(s_{-j} \in S_{-j}(s_j)) > 0$, and for any $\lambda > 0$ a sequence $\langle\hat{\alpha}^{j,t}\rangle_{t=1}^{\infty}$ with $\hat{\alpha}^{j,t}(s_j) \in A^j(s_j)$,

$||\hat{\alpha}^{j,t}(s_j) - \alpha^{j,\star}(s_j)|| < \lambda$ for all t sufficiently large, such that for all $s_j \in S_j$ and $s_{-j} \in S_{-j}(s_j)$,

$$\lim_{t \nearrow \infty} u^j \left(\hat{\alpha}^{j,t}(s_j), \alpha^{-j,t}(s_{-j}); s_j \right) > \lim_{t \nearrow \infty} u^j \left(\alpha^{j,t}(s_j), \alpha^{-j,t}(s_{-j}); s_j \right) + 5\delta.$$

Let $\bar{\alpha}^{j,t}(s_j) = \hat{\alpha}^{j,t}(s_j) \vee \alpha^{j,t}(s_j)$. Then by Condition 3 and Lemma 11, for all $s_{-j} \in S_{-j}(s_j)$,

$$\begin{aligned} &\lim_{t \nearrow \infty} u^j \left(\bar{\alpha}^{j,t}(s_j), \alpha^{-j,t}(s_{-j}); s_j \right) \\ &\quad \geq \lim_{t \nearrow \infty} u^j \left(\alpha^{j,t}(s_j), \alpha^{-j,t}(s_{-j}); s_j \right) + 5\delta - g \left(\left|\left| \bar{\alpha}^{j,t}(s_j) - \hat{\alpha}^{j,t}(s_j) \right|\right|; s_j \right) \\ &\quad \geq \lim_{t \nearrow \infty} u^j \left(\alpha^{j,t}(s_j), \alpha^{-j,t}(s_{-j}); s_j \right) + 5\delta - g(\lambda; s_j). \end{aligned}$$

Then for λ sufficiently small,

$$\lim_{t \nearrow \infty} u^j \left(\bar{\alpha}^{j,t}(s_j), \alpha^{-j,t}(s_{-j}); s_j \right) > \lim_{t \nearrow \infty} u^j \left(\alpha^{j,t}(s_j), \alpha^{-j,t}(s_{-j}); s_j \right) + 4\delta.$$

For $s_{-j} \notin S_{-j}(s_j)$,

$$\lim_{t \nearrow \infty} u^j \left(\bar{\alpha}^{j,t}(s_j), \alpha^{-j,t}(s_{-j}); s_j \right) > \lim_{t \nearrow \infty} u^j \left(\alpha^{j,t}(s_j), \alpha^{-j,t}(s_{-j}); s_j \right) - g(\lambda; s_j).$$

By the law of iterated expectations,

$$\begin{aligned} U^j \left(\alpha^{j,t}(s_j), \alpha^{-j,t}; s_j \right) &= \Pr\left(s_{-j} \in S_{-j}(s_j) \right) \mathbb{E}_{s_{-j}} \left[u^j \left(\alpha^{j,t}(s_j), \alpha^{-j,t}(s_{-j}); s_j \right) \middle| s_{-j} \in S_{-j}(s_j) \right] \\ &\quad + \Pr\left(s_{-j} \notin S_{-j}(s_j) \right) \mathbb{E}_{s_{-j}} \left[u^j \left(\alpha^{j,t}(s_j), \alpha^{-j,t}(s_{-j}); s_j \right) \middle| s_{-j} \notin S_{-j}(s_{-j}) \right]. \end{aligned}$$

Then

$$\begin{aligned} &\lim_{t \nearrow \infty} U^j \left(\bar{\alpha}^{j,t}(s_j), \alpha^{-j,t}; s_j \right) \\ &\quad > \lim_{t \nearrow \infty} U^j \left(\alpha^{j,t}(s_j), \alpha^{-j,t}; s_j \right) + 4\delta \Pr\left(s_{-j} \in S_{-j}(s_j) \right) - g(\lambda; s_j) \Pr\left(s_{-j} \notin S_{-j}(s_j) \right). \end{aligned}$$

For λ sufficiently small,

$$\lim_{t \nearrow \infty} U^j \left(\bar{\alpha}^{j,t}(s_j), \alpha^{-j,t}; s_j \right) > \lim_{t \nearrow \infty} U^j \left(\alpha^{j,t}(s_j), \alpha^{-j,t}; s_j \right) + 3\delta.$$

Appealing to Corollary 5, let $\langle \tilde{\alpha}^{j,t} \rangle_{t=1}^{\infty}$ be a sequence of strategies with $\tilde{\alpha}^{j,t}(s_j) \in A^{j,\varepsilon_t}(s_j)$ for all s_j such that, for t sufficiently large,

$$U^j \left(\tilde{\alpha}^{j,t} (s_j), \alpha^{-j,t}; s_j \right) > U^j \left(\bar{\alpha}^{j,t} (s_j), \alpha^{-j,t}; s_j \right) - \delta.$$

Then

$$\lim_{t \nearrow \infty} U^j \left(\tilde{\alpha}^{j,t} (s_j), \alpha^{-j,t}; s_j \right) > \lim_{t \nearrow \infty} U^j \left(\alpha^{j,t} (s_j), \alpha^{-j,t}; s_j \right) + 2\delta.$$

It follows that for t sufficiently large,

$$U^j \left(\tilde{\alpha}^{j,t} (s_j), \alpha^{-j,t}; s_j \right) > U^j \left(\alpha^{j,t} (s_j), \alpha^{-j,t}; s_j \right) + \delta.$$

Then $\alpha^{j,t}(s_j)$ is not a best response for agent j when her type is s_j, against opponent play $\alpha^{-j,t}$ in model $\mathcal{M}^{\varepsilon_t}$. $\Pr(s_j \in S_j) > 0$, contradicting α^t being a constrained Bayesian Nash equilibrium. □

Definition 9 (Convergent agent-types)**.** *Agent i with type s_i is a* convergent agent-type *if*

$$\lim_{t \nearrow \infty} \left| \left| \alpha^{i,t} (s_i) - \overline{\alpha}^i (s_i) \right| \right| = 0, \text{ and}$$
$$\lim_{t \nearrow \infty} U^i \left(\alpha^{i,t} (s_i), \alpha^{-i,t}; s_i \right) = U^i \left(\overline{\alpha}^i (s_i), \overline{\alpha}^{-i}; s_i \right).$$

If either of these equalities does not hold, (i, s_i) is a nonconvergent agent-type*.*

Lemma 15 (Utility convergence)**.** *For all agents i,*

$$\Pr_{s_i} \left((i, s_i) \text{ is a convergent agent-type} \right) = 1.$$

Proof. For each agent i, Lemma 8 establishes that $\Pr_{s_i}(||\alpha^{i,t}(s_i) - \overline{\alpha}^i(s_i)|| \to 0) = 1$. Lemma 10 implies that

$$\Pr_{s_i} \left(\lim_{t \nearrow \infty} U^i \left(\alpha^{i,t} (s_i), \alpha^{-i,t}; s_i \right) \leq U^i \left(\overline{\alpha}^i (s_i), \overline{\alpha}^{-i}; s_i \right) \right) = 1.$$

Lemma 14 establishes that

$$\Pr_{s_i} \left(\lim_{t \nearrow \infty} U^i \left(\alpha^{i,t} (s_i), \alpha^{-i,t}; s_i \right) \geq U^i \left(\overline{\alpha}^i (s_i), \overline{\alpha}^{-i}; s_i \right) \right) = 1.$$

The result is then immediate. $\square$

Lemma 16 (Best responses for convergent agent-types)**.** *For all agents i, $\overline{\alpha}^i(s_i)$ is a best response to $(\overline{\alpha}^j)_{j\neq i}$ with s_i-probability one.*

Proof. Suppose that agent i has a better response a_i when her type is s_i. Then there is $\delta > 0$ with

$$U^i\left(a_i, \overline{\alpha}^{-i}; s_i\right) > U^i\left(\overline{\alpha}^i\left(s_i\right), \overline{\alpha}^{-i}; s_i\right) + 4\delta.$$

By Condition 4 there is $\hat{a}_i \in A^i(s_i)$ such that

$$\lim_{t\nearrow\infty} U^i\left(\hat{a}_i, \alpha^{-i,t}; s_i\right) > U^i\left(a_i, \overline{\alpha}^{-i}; s_i\right) - \delta > \lim_{t\nearrow\infty} U^i\left(\alpha^{i,t}\left(s_i\right), \alpha^{-i,t}; s_i\right) + 3\delta.$$

Then there is T such that for all $t > T$,

$$U^i\left(\hat{a}_i, \alpha^{-i,t}; s_i\right) > U^i\left(\alpha^{i,t}\left(s_i\right), \alpha^{-i,t}; s_i\right) + 2\delta.$$

By Lemma 9, there is $\hat{\mathsf{a}}^{\varepsilon_t} \in A^{i,\varepsilon_t}(s_i)$ such that

$$U^i\left(\hat{\mathsf{a}}^{\varepsilon_t}, \alpha^{-i,t}; s_i\right) > U^i\left(\hat{a}_i, \alpha^{-i,t}; s_i\right) - g\left(C\varepsilon_t; s_i\right).$$

Then there is $T' \geq T$ such that for all $t > T'$,

$$U^i\left(\hat{\mathsf{a}}^{\varepsilon_t}, \alpha^{-i,t}; s_i\right) > U^i\left(\alpha^{i,t}\left(s_i\right), \alpha^{-i,t}; s_i\right) + \delta.$$

If this holds for a positive probability set of signals for agent i, this contradicts the construction of constrained Bayesian Nash equilibrium in $\mathcal{M}^{\varepsilon_t}$. $\square$

Lemma 17 (Best responses for nonconvergent agent-types)**.** *Under Condition 9, $\overline{\alpha}^i(s_i)$ is a best response to $(\overline{\alpha}^j)_{j\neq i}$ for all agents i and signals $s_i \in (0,1)^m$.*

Proof. Suppose otherwise. Then there is $a_i \in A^i(s_i)$ and $\delta > 0$ such that

$$U^i\left(a_i, \overline{\alpha}^{-i}; s_i\right) > U^i\left(\overline{\alpha}^i\left(s_i\right), \overline{\alpha}^{-i}; s_i\right) + 3\delta$$

If $a_i \in \underline{A}^i(s_i)$, then there is $\gamma > 0$ such that for all $s_i' < s_i$ with $||s_i - s_i'|| < \gamma$, $a_i \in A^i(s_i')$. Since utility is increasing in signal and is upper semicontinuous in action,

$$\begin{aligned}U^i\left(a_i, \overline{\alpha}^{-i}; s_i\right) &> U^i\left(\overline{\alpha}^i\left(s_i\right), \overline{\alpha}^{-i}; s_i\right) + 3\delta \\ &\geq U^i\left(\overline{\alpha}^i\left(s_i'\right), \overline{\alpha}^{-i}; s_i\right) + 3\delta \geq U^i\left(\overline{\alpha}^i\left(s_i'\right), \overline{\alpha}^{-i}; s_i'\right) + 3\delta.\end{aligned}$$

Since utility is continuous in signal, for γ sufficiently small it will be the case that whenever $||s_i - s_i'|| < \gamma$,

$$U^i\left(a_i, \overline{\alpha}^{-i}; s_i'\right) > U^i\left(\overline{\alpha}^i\left(s_i'\right), \overline{\alpha}^{-i}; s_i'\right) + 2\delta.$$

This contradicts the fact that type s_i' is almost-surely best-responding. Then $a_i \in A^i(s_i) \setminus \underline{A}^i(s_i)$.

By Condition 9 there is $a_i' \in \underline{A}^i(s_i)$ such that

$$U^i\left(a_i', \overline{\alpha}^{-i}; s_i\right) > U^i\left(a_i, \overline{\alpha}^{-i}; s_i\right) - \delta.$$

Then there is $a_i' \in \underline{A}^i(s_i)$ such that

$$U^i\left(a_i', \overline{\alpha}^{-i}; s_i\right) > U^i\left(\overline{\alpha}^i\left(s_i\right), \overline{\alpha}^{-i}; s_i\right) + \delta.$$

The rest of the proof proceeds identically to the above, implying $a_i' \in A^i(s_i) \setminus \underline{A}^i(s_i)$, a contradiction.

□

Lemma 18 (Implied type independence)**.** *The antecedents of Theorem 3 imply Condition 9.*

Proof. Let $(\alpha^j)_{j \neq i}$ be a strategy profile for agent i's opponents, and let $a_i \in A^i(s_i) \setminus \underline{A}^i(s_i)$; by definition, $a_i \in Y$. Since u^i is continuous in signal, there is $s_i' < s_i$ such that

$$U^i\left(a_i, \alpha^{-i}; s_i'\right) > U^i\left(a_i, \alpha^{-i}; s_i\right) - \frac{1}{2}\lambda.$$

Taking as given the antecedents of Theorem 3, there is $a_i' \in A^i(s_i')$ such that

$$U^i\left(a_i', \alpha^{-i}; s_i'\right) > U^i\left(a_i, \alpha^{-i}; s_i'\right) - \frac{1}{2}\lambda.$$

Since utility is increasing in signal, it follows that

$$U^i\left(a_i', \alpha^{-i}; s_i\right) > U^i\left(a_i', \alpha^{-i}; s_i'\right) > U^i\left(a_i, \alpha^{-i}; s_i\right) - \lambda.$$

By construction, $a_i' \in \underline{A}^i(s_i)$, and Condition 9 is satisfied. □

Proof of Theorem 3. Suppose otherwise. Then there is an agent i, a signal $s_i \in (0,1)^m$, an action $y \in Y$, and a $\delta > 0$ such that

$$U^i\left(y, \overline{\alpha}^{-i}; s_i\right) > U^i\left(\overline{\alpha}^i\left(s_i\right), \overline{\alpha}^{-i}; s_i\right) + 2\delta.$$

By assumption there is $a_i \in A^i(s_i)$ such that

$$U^i\left(a_i, \overline{\alpha}^{-i}; s_i\right) > U^i\left(y, \overline{\alpha}^{-i}; s_i\right) - \delta.$$

It follows that

$$U^i\left(a_i, \overline{\alpha}^{-i}; s_i\right) > U^i\left(\overline{\alpha}^i\left(s_i\right), \overline{\alpha}^{-i}; s_i\right) + \delta.$$

This directly implies that $\overline{\alpha}^i(s_i)$ is not a best response for agent i in the constrained-action game $\mathcal{M}$ when her type is s_i. Lemma 18 demonstrates that Condition 9 is satisfied whenever the antecedents of Theorem 3 hold, and Lemma 17 then implies that $\overline{\alpha}^i(s_i)$ is a best response in $\mathcal{M}$ for agent i when her signal is s_i, a contradiction. □

B Divisible-good auctions

B.1 Standard transfer rules

Lemma 19 (Standard transfer rules)**.** *The example transfer rules given in Section 4 are standard.*

Proof. It is straightforward to see that these transfer rules are symmetric and uniformly continuous in their arguments, and that their derivatives are appropriately bounded. The random payment auction has a submodular payment rule if the discriminatory and uniform-price auctions have submodular payment rules; each of these latter two auctions has a submodular payment rule if the quantile-hybird auction has a submodular payment rule.[22]

Let b, b' be bid functions, and let $b^\vee = b \vee b'$ and $b^\wedge = b \wedge b'$. To prove modularity of expected transfer, it suffices to show that for each z,

$$\tau\left(q^\vee; b^\vee, p^\vee, b^{-i}, z\right) + t\left(q^\wedge; b^\wedge, p^\wedge, b_{-i}, z\right) \leq \tau\left(q; b, p, b_{-i}, z\right) + t\left(q'; b, p, b^{-i}, z\right),$$

where $q^\vee = q_i(b^\vee, b_{-i}; z)$, $p^\vee = p(b^\vee, b_{-i}; z)$, and similarly for the other decorated parameters. Under random priority tiebreaking, $\{q^\vee, q^\wedge\} = \{q, q'\}$ and $\{p^\vee, p^\wedge\} = \{p, p'\}$. Fixing $\alpha \in [0,1]$, it also holds that $\{p^{\alpha\vee}, p^{\alpha\wedge}\} = \{p^\alpha, p^{\alpha\prime}\}$, where p^α is the α-quantile bid when bidder i's bid is b. It is without loss to assume below that $\{q^{\alpha\vee}, q^{\alpha\wedge}\} = \{q^\alpha, q^{\alpha\prime}\}$.[23]

Without loss of generality assume that $p^\vee = p$; then $q^\vee = q$, $p^\wedge = p'$, and $q^\wedge = q'$. For simplicity of notation, suppress function arguments. Then establishing submodularity requires showing

$$p^{\alpha\vee} q^{\alpha\vee} + \int_{q^{\alpha\vee}}^{q^\vee} b^\vee dx + p^{\alpha\wedge} q^{\alpha\wedge} + \int_{q^{\alpha\wedge}}^{q^\wedge} b^\wedge dx \leq p^\alpha q^\alpha + \int_{q^\alpha}^{q} b dx + p^{\alpha\prime} q^{\alpha\prime} + \int_{q^{\alpha\prime}}^{q'} b' dx. \tag{1}$$

If $q^\vee = q$, then $b \geq b'$ for all $x \in (q', q)$. Then (1) can be rewritten as

$$p^{\alpha\vee} q^{\alpha\vee} + p^{\alpha\wedge} q^{\alpha\wedge} + \int_{q^{\alpha\vee}}^{q'} b^\vee - b dx \leq p^\alpha q^\alpha + p^{\alpha\prime} q^{\alpha\prime} + \int_{q^{\alpha\wedge}}^{q'} b' - b^\wedge dx - \int_{q^{\alpha\vee}}^{q^\alpha} b dx - \int_{q^{\alpha\wedge}}^{q^{\alpha\prime}} b' dx.$$

The simplifying assumptions on α-quantile quantities imply $p^{\alpha\vee} q^{\alpha\vee} + p^{\alpha\wedge} q^{\alpha\wedge} = p^\alpha q^\alpha + p^{\alpha\prime} q^{\alpha\prime}$. Then the above is equivalent to

$$\int_{q^{\alpha\vee}}^{q'} b^\vee - b dx \leq \int_{q^{\alpha\wedge}}^{q'} b' - b^\wedge dx - \int_{q^{\alpha\vee}}^{q^\alpha} b dx - \int_{q^{\alpha\wedge}}^{q^{\alpha\prime}} b' dx. \tag{2}$$

Note that $q^{\alpha\wedge} \leq q^{\alpha\vee}$. If $q^{\alpha\vee} = q^\alpha$, inequality (2) is

$$\int_{q^{\alpha\vee}}^{q'} b^\vee - b dx \leq \int_{q^{\alpha\wedge}}^{q'} b^\vee - b dx = \int_{q^{\alpha\wedge}}^{q'} b' - b^\wedge dx.$$

[22] To see this, let $\alpha = 1$ or $\alpha = 0$ to generate the discriminatory and uniform-price auctions, respectively, from a quantile-hybrid auction. In addition, McAdams (2003) establishes modularity of the discriminatory and uniform-price payment rules.

[23] The only effect of this assumption is whether the α-quantile quantity is allocated at the α-quantile bid, or at the submitted bid. Since these are equal it is without loss to make the simplifying assumption.

If instead $q^{\alpha\vee} = q^{\alpha\prime}$, inequality (2) is

$$\int_{q^{\alpha\vee}}^{q'} b^{\vee} - b' dx \leq \int_{q^{\alpha\wedge}}^{q'} b^{\vee} - b' dx = \int_{q^{\alpha\wedge}}^{q'} b - b^{\wedge} dx.$$

Then in either case, transfers are submodular. □

B.2 Proofs for equilibrium existence in divisible-good auctions

The proof of Lemma 4 (in the main text) establishes that the market clearing price is uniformly continuous in bidder i's own bid, and can be easily adapted to show that the market clearing price is uniformly continuous in all submitted bids. In light of this result, for compactness the proofs below generally omit price as an argument to the transfer function τ. When a sequence of bids is converging, so too is the market price, and uniform continuity of standard transfer rules implies that the effect on price is irrelevant to convergence. All proofs can be adapted to explicitly include the effect of a change in price. Henceforth, let $\tau(q; b_i, b_{-i}, z) = \tau(q; b_i, p^{\star}(b_i, b_{-i}; z), b_{-i}, z)$.

Lemma 20 (Satisfaction of Condition 3)**.** *The divisible-good auction with Lipschitz bids and a standard transfer rule satisfies Condition 3.*

Proof. For simplicitly we prove satisfaction of an ex post formulation of Condition 3 for any bid profile, implying the interim formulation given in Condition 3 (see Lemma 11). Let $b_i \in A^i(s_i)$, $b_{-i} = (b_j)_{j \neq i}$ be actions for agent i's opponents, and $\overline{b}_i \in Y$, $\overline{b}_i \geq b_i$. Define $\lambda = ||\overline{b}_i - b_i||$, and for compactness let q_i and $\overline{q}_i$ be allocations under bids b_i and $\overline{b}_i$, respectively, and $\tau(\cdot; b_i)$ and $\tau(\cdot; \overline{b}_i)$ be the associated transfers. Then the difference in ex post utility is given by

$$u^i\left(\overline{b}_i, b_{-i}; s_i\right) - u^i\left(b_i, b_{-i}; s_i\right) = \mathbb{E}_z\left[\int_0^{\overline{q}_i} v^i\left(x; s_i\right) dx - \tau\left(\overline{q}_i; \overline{b}_i\right) - \mathbb{E}_z\left[\int_0^{q_i} v^i\left(x; s_i\right) dx - \tau\left(q_i; b_i\right)\right]\right].$$

Allocations are weakly monotone in bid, so $q^i(\overline{b}_i, b_{-i}; z) \geq q^i(b_i, b_{-i}; z)$. Since the transfer rule is standard,

$$\begin{aligned}
&u^i\left(\bar{b}_i, b_{-i}; s_i\right) - u^i\left(b_i, b_{-i}; s_i\right) \\
&\geq \mathbb{E}_z\left[\int_0^{q_i} v^i\left(x; s_i\right) dx - \tau\left(q_i; \bar{b}_i\right)\right] - \mathbb{E}_z\left[\int_0^{q_i} v^i\left(x; s_i\right) dx - \tau\left(q_i; b_i\right)\right] + \mathbb{E}_z\left[\int_{q_i}^{\overline{q}_i} v^i\left(x; s_i\right) - \bar{b}_i\left(x\right) dx\right] \\
&= \mathbb{E}_z\left[t\left(q_i; b_i\right) - \tau\left(q_i; \bar{b}_i\right)\right] + \mathbb{E}_z\left[\int_{q_i}^{\overline{q}_i} v^i\left(x; s_i\right) - \bar{b}_i\left(x\right) dx\right].
\end{aligned}$$

Lemma 4 implies that the left-hand term is uniformly bounded in λ. Further, since $b_i \leq v^i(\cdot; s_i)$ and $||b_i - \overline{y}_i|| = \lambda$, $\int_{q_i}^{\overline{q}_i} v^i(x; s_i) - \bar{b}_i(x)dx \geq -\lambda$. This completes the proof. □

In the following let $I : \mathbb{R}^2 \rightrightarrows \mathbb{R}$ be given by $I(a, b) = (\min\{a, b\}, \max\{a, b\})$, the open interval between a and b, accounting as necessary for the cases in which $a \leq b$ and $b \leq a$.

Lemma 21 (Discontinuous allocations)**.** *Let $\langle (b_{i,t})_{i=1}^n \rangle_{t=1}^\infty$ be a sequence of bid functions converging to $(b_{i,\star})_{i=1}^n$. Suppose that $\lim_{t\nearrow\infty} q^i(b_{i,t}, b_{-i,t}; z) \neq q^i(b_{i,\star}, b_{-i,\star}; z)$, and that the limit exists. Then*

$$b_{i,\star}\left(q'\right) = b_{i,\star}\left(q''\right),\ \forall q', q'' \in I\left(\lim_{t\nearrow\infty} q^i\left(b_{i,t}, b_{-i,t}; z\right), q^i\left(b_{i,\star}, b_{-i,\star}; z\right)\right).$$

Proof. For any agent j, define quantities

$$q_{j,t} = q^j\left(b_j, b_{-j,t}; z\right),\ q_{j,\star} = q^j\left(b_j, b_{-j,\star}; z\right),\ \overline{q}_j = \lim_{t\nearrow\infty} q_{j,t}.$$

Assume without loss of generality that $\overline{q}_i < q_{i,\star}$; by market clearing there a nonempty set of agents J such that for all $j \in J$, $\overline{q}_j > q_{j,\star}$. Let $\delta > 0$ be such that $\lim_{t\nearrow\infty} |q_{k,t} - q_{k,\star}| > 2\delta$ for all $k \in J \cup \{i\}$.

By market clearing, it must be that for all t sufficiently large and all $j \in J$,

$$b_{j,t}\left(q_{j,\star} + \delta\right) \geq b_{i,t}\left(q_{i,\star} - \delta\right),\ \text{and}\ b_{j,t}\left(\overline{q}_j - \delta\right) \geq b_{i,t}\left(\overline{q}_i + \delta\right).$$

In the limit, it must be that

$$\lim_{t\nearrow\infty} b_{j,t}\left(q_{j,\star} + \delta\right) \geq b_{j,\star}\left(\overline{q}_j - \delta\right),\ \text{and}\ \lim_{t\nearrow\infty} b_{i,t}\left(q_{i,\star} - \delta\right) \leq b_{i,\star}\left(\overline{q}_i + \delta\right).$$

It follows that

$$\lim_{t\nearrow\infty} b_{j,t}\left(q_{j,\star}+\delta\right) \geq \lim_{t\nearrow\infty} b_{j,t}\left(\overline{q}_j-\delta\right) \geq \lim_{t\nearrow\infty} b_{i,t}\left(\overline{q}_i+\delta\right) \geq \lim_{t\nearrow\infty} b_{i,t}\left(q_{i,\star}-\delta\right). \tag{3}$$

Further, it must be the case that $\lim_{t\nearrow\infty} b_{j,t}(q_{j,\star}+\delta) \leq \lim_{t\nearrow\infty} b_{i,t}(q_{i,\star}-\delta)$. Otherwise, monotonicity and convergence together imply that

$$b_{j,\star}\left(q_{j,\star}+\delta'\right) > b_{i,\star}\left(q_{i,\star}-\delta'\right)\ \forall\delta' \in (0,\delta).$$

This contradicts the definition of $q_{j,\star}$ and $q_{i,\star}$. Then the inequalities in (3) hold with equality, and

$$\lim_{t\nearrow\infty} b_{i,t}\left(q_{i,\star}-\delta\right) = b_{i,\star}\left(\overline{q}_i+\delta\right).$$

Since this is true for all $\delta' \in (0,\delta)$, it follows that

$$\lim_{t\nearrow\infty} b_{i,t}\left(\lim_{q'\nearrow q_{i,\star}} q'\right) = b_{i,\star}\left(\lim_{q'\searrow\overline{q}_i} q'\right).$$

Then monotonicity and convergence imply that

$$b_{i,\star}\left(q'\right) = b_{i,\star}\left(q''\right),\ \forall q',q'' \in I\left(\lim_{t\nearrow\infty} q^i\left(b_{i,t},b_{-i,t};z\right), q^i\left(b_{i,\star},b_{-i,\star};z\right)\right).$$

□

In what follows let u^i_z denote realized utility,

$$\begin{aligned} u^i_z\left(\tilde{b}_i,\tilde{b}_{-i};s_i,z\right) &= \int_0^{q^i\left(\tilde{b}_i,\tilde{b}_{-i};z\right)} v^i\left(x;s_i\right)dx - \tau\left(q^i\left(\tilde{b}_i,\tilde{b}_{-i};z\right);\tilde{b}_i,\tilde{b}_{-i},z\right), \\ u^i\left(\tilde{b}_i,\tilde{b}_{-i};s_i\right) &= \mathbb{E}_z\left[u^i_z\left(\tilde{b}_i,\tilde{b}_{-i};s_i,z\right)\right]. \end{aligned}$$

Showing that an inequality holds for u^i_z for all z is sufficient to show that it holds for u^i, and in turn if this holds for all $\tilde{b}_{-i}$ the inequality will hold for interim utility U^i.

Lemma 22 (Utility dominance in limit)**.** *Let $\langle(b_{j,t})_{j\neq i}\rangle_{t=1}^{\infty}$ be bid functions for agent i's opponents, converging to $b_{-i,\star} = (b_{j,\star})_{j\neq i}$. Then there is a continuous function $h : \mathbb{R}_+ \to \mathbb{R}_+$, $h(0) = 0$, such* *that for any $y \in Y$ and all $\lambda > 0$,*

$$\lim_{t \nearrow \infty} u^i \left([y + \lambda] \wedge v^i (\cdot; s_i), b_{-i,t}; s_i \right) \geq u^i (y, b_{-i,\star}; s_i) - h(\lambda).$$

Proof. To establish this result it is sufficient to prove the above inequality with respect to u^i_z for any z. Let $\overline{y}^\lambda = [y + \lambda] \wedge v^i(\cdot; s_i)$. Note that

$$\begin{aligned}
&\lim_{t \nearrow \infty} u^i_z \left(\overline{y}^\lambda, b_{-i,t}; s_i, z \right) \\
&= \lim_{t \nearrow \infty} \int_0^{q^i\left(\overline{y}^\lambda, b_{-i,t}; z\right)} v^i (x; s_i)\, dx - \tau \left(q^i \left(\overline{y}^\lambda, b_{-i,t}; z \right); \overline{y}^\lambda, b_{-i,t}, z \right) \\
&= \lim_{t \nearrow \infty} \int_0^{q^i(y, b_{-i,\star}; z)} v^i (x; s_i)\, dx - \tau \left(q^i (y, b_{-i,\star}; z); y, b_{-i,\star}, z \right) \\
&\qquad + \int_{q^i(y, b_{-i,\star}; z)}^{q^i\left(\overline{y}^\lambda, b_{-i,t}; z\right)} v^i (x; s_i)\, dx - \left[\tau \left(q^i \left(\overline{y}^\lambda, b_{-i,t}; z \right); \overline{y}^\lambda, b_{-i,t}, z \right) - \tau \left(q^i (y, b_{-i,\star}; z); y, b_{-i,\star}, z \right) \right].
\end{aligned}$$

Let $q^\lambda_{i,t} = q^i(\overline{y}^\lambda, b_{-i,t}; z)$ and $q_{i,\star} = q^i(y, b_{-i,\star}; z)$. It will suffice to show that there is a continuous $h : \mathbb{R}_+ \to \mathbb{R}_+$, $h(0) = 0$, such that

$$\lim_{t \nearrow \infty} \left[\tau \left(q^\lambda_{i,t}; \overline{y}^\lambda, b_{-i,t}, z \right) - \tau (q_{i,\star}; y, b_{-i,\star}, z) \right] - \int_{q_{i,\star}}^{q^\lambda_{i,t}} v^i (x; s_i)\, dx \leq h(\lambda). \tag{4}$$

Since $b_{-i,t} \to b_{-i,\star}$, uniform continuity of transfers in opponent demand implies that for any q, $t(q; \overline{y}^\lambda, b_{-i,t}; z) \to t(q; \overline{y}^\lambda, b_{-i,\star}; z)$. Furthermore, $||\overline{y}^\lambda - \left[(y \wedge v^i(\cdot; s_i)) + \lambda \right] || \leq \overline{Q}\lambda$. Since $y \wedge v^i(\cdot; s_i) \leq y$, monotonicity and uniform continuity of transfers in own bid imply that there is a continuous $h^b : \mathbb{R}_+ \to \mathbb{R}_+$, $h^b(0) = 0$, such that for any q,

$$\lim_{t \nearrow \infty} \tau \left(q; \overline{y}^\lambda, b_{-i,t}, z \right) \leq \tau (q; y, b_{-i,\star}, z) + h^b(\lambda). \tag{5}$$

Inequality (5) transforms the left-hand side of (4) into

$$\lim_{t\nearrow\infty}\left[\tau\left(q_{i,t}^{\lambda};\overline{y}^{\lambda},b_{-i,t},z\right)-\tau\left(q_{i,\star};y,b_{-i,\star},z\right)\right]-\int_{q_{i,\star}}^{q_{i,t}^{\lambda}}v^{i}\left(x;s_{i}\right)dx$$
$$\leq h^{b}\left(\lambda\right)+\lim_{t\nearrow\infty}\tau\left(q_{i,t}^{\lambda};\overline{y}^{\lambda},b_{-i,t},z\right)-\tau\left(q_{i,\star};\overline{y}^{\lambda},b_{-i,t},z\right)-\int_{q_{i,\star}}^{q_{i,t}^{\lambda}}v^{i}\left(x;s_{i}\right)dx$$
$$\leq h^{b}\left(\lambda\right)-\lim_{t\nearrow\infty}\int_{q_{i,\star}}^{q_{i,t}^{\lambda}}v^{i}\left(x;s_{i}\right)-\overline{y}^{\lambda}\left(x\right)dx.$$

Since $\lim_{\lambda'\searrow 0}h^{b}(\lambda')=0$, all that remains to establish the existence of the desired h is to show

$$\lim_{t\nearrow\infty}\int_{q_{i,\star}}^{q_{i,t}^{\lambda}}v^{i}\left(x;s_{i}\right)-\overline{y}^{\lambda}\left(x\right)dx\geq 0. \tag{6}$$

Since $\overline{y}^{\lambda}\leq v^{i}(\cdot;s_{i})$ by construction, if $q_{i,\star}\leq\lim_{t\nearrow\infty}q_{i,t}^{\lambda}$ inequality (6) is trivially satisfied. Therefore assume that $\overline{q}_{i}^{\lambda}\equiv\lim_{t\nearrow\infty}q_{i,t}^{\lambda}<q_{i,\star}$. Recall that $b_{-i,t}\to b_{-i,\star}$. Then for bidder i's quantity to be higher under y against $b_{-i,\star}$ than in the limit under $\overline{y}^{\lambda}$ against $b_{-i,t}$, it must be that $\overline{y}^{\lambda}(q)\leq y(q)$ for all $q\in(\overline{q}_{i}^{\lambda},q_{i,\star})$. By construction this is only possible when $y\geq v^{i}(q;s_{i})$ for all such q, and hence $\overline{y}^{\lambda}(q)=v^{i}(q;s_{i})$ for all such q. Then $\lim_{t\nearrow\infty}\int_{q_{i,\star}}^{q_{i,t}^{\lambda}}v^{i}(x;s_{i})-\overline{y}^{\lambda}(x)dx=0$. In either case inequality (4) is satisfied. □

Lemma 23 (Limit surplus splitting). *The divisible-good auction model satisfies Condition 6.*

Proof. Suppose that there is a sequence of strategies $\langle(\beta^{k,t})_{k=1}^{n}\rangle_{t=1}^{\infty}$ converging to the feasible strategy profile $(\beta^{\star,k})_{k=1}^{n}$ such that there is an agent i with

$$\Pr_{s_{i}}\left(\lim_{t\nearrow\infty}U^{i}\left(\beta^{i,t}\left(s_{i}\right),\beta^{-i,t};s_{i}\right)>U^{i}\left(\beta^{i,\star}\left(s_{i}\right),\beta^{-i,\star};s_{i}\right)\right)>0.$$

Then there is a set S_{i}, $\Pr(s_{i}\in S_{i})>0$, and for each $s_{i}\in S_{i}$ a set $S_{-i}(s_{i})$, $\Pr(s_{-i}\in S_{-i}(s_{i}))>0$, such that for all $s_{i}\in S_{i}$ and $s_{-i}\in S_{-i}(s_{i})$,

$$\lim_{t\nearrow\infty}u^{i}\left(\beta^{i,t}\left(s_{i}\right),\beta^{-i,t}\left(s_{-i}\right);s_{i}\right)>u^{i}\left(\beta^{i,\star}\left(s_{i}\right),\beta^{-i,\star}\left(s_{-i}\right);s_{i}\right).$$

Then for these same $s = (s_i, s_{-i})$,

$$\lim_{t \nearrow \infty} \mathbb{E}_z \left[q^i \left(\beta^t \left(s \right) ; z \right) \right] > \mathbb{E}_z \left[q^i \left(\beta^\star ; z \right) \right].$$

It follows that

$$\Pr \left(\lim_{t \nearrow \infty} q^i \left(\beta^t \left(s \right) ; z \right) > q^i \left(\beta^\star \left(s \right) ; z \right) \right) > 0.$$

Lemma 21 establishes that for any $s_i \in S_i$, $\beta^{i,\star}(s_i)$ is constant on intervals on which quantity does not converge, hence $\overline{\varphi}^{i,\star}(\cdot; s_i)$ is discontinuous at this bid level. Since $\overline{\varphi}^{i,\star}(\cdot; s_i)$ is a monotone function on a compact domain, it has at most countably-many discontinuities, so at least one such quantity interval is realized with positive probability (otherwise bidder i's utility is almost surely converging). Considering such positive-probability intervals, there is a subset of signals $\hat{S}_i \subseteq S_i$ such that these positive-probability intervals intersect, and it is without loss of generality to assume that this subset has positive measure; otherwise, the interval $[0, \bar{Q}]$ can be covered by uncountably-many disjoint sets of positive measure, a contradiction. Lastly, market clearing implies that agent i's quantity loss is some other agent's quantity gain, and since there are only a finite number of agents it is again without loss of generality to assume that in all cases at least some of the discrete gain goes to agent $j \neq i$. Then let $\hat{S}_i \subseteq S_i$ be a positive-probability set such that there are $q_{i,\ell}, q_{i,r} \in [0, \bar{Q}]$ such that for all $s_i \in \hat{S}_i$,

$$\Pr{}_{s_{-i},z} \Bigg(q^i \left(\beta^{i,\star} \left(s_i \right), \beta^{-i,\star} \left(s_{-i} \right) ; z \right) \leq q_{i,\ell} < q_{i,r} \leq \lim_{t \nearrow \infty} q^i \left(\beta^{i,t} \left(s_i \right), \beta^{-i,t} \left(s_{-i} \right) ; z \right) \Bigg) > 0.$$

For $s_i \in \hat{S}_i$, let $\hat{S}_{-i}(s_i)$ be given by

$$\hat{S}_{-i} \left(s_i \right) = \Bigg\{ \left(s_{-i}, z \right) : \; q^i \left(\beta^{i,\star} \left(s_i \right), \beta^{-i,\star} \left(s_{-i} \right) ; z \right) \leq \; q_{i,\ell} < q_{i,r} \leq \lim_{t \nearrow \infty} q^i \left(\beta^{i,t} \left(s_i \right), \beta^{-i,t} \left(s_{-i} \right) ; z \right) \Bigg\}.$$

Lemma 21 implies that $\beta^{i,\star}(s_i)$ is constant on $(q_{i,\ell}, q_{i,r})$ for all $s_i \in \hat{S}_i$. Additionally, this bid must equal to the bid placed by any agent who, at the limit, receives agent i's sacrificed quantity. Then if $s_i, s_i' \in \hat{S}_i$ are such that $\beta^{i,\star}(s_i) \neq \beta^{i,\star}(s_i')$ on $(q_{i,\ell}, q_{i,r})$, it must be that $\hat{S}_{-i}(s_i) \cap \hat{S}_{-i}(s_i') = \emptyset$.

From this and the fact that $\Pr(s_{-i} \in \hat{S}_{-i}(s_i)) > 0$, it follows that there is a bid level p and a positive-probability set $\tilde{S}_i \subseteq \hat{S}_i$ of agent i's signal realizations such that for all $s_i, s_i' \in \tilde{S}_i$ and $q, q' \in (q_{i,\ell}, q_{i,r})$,

$$\left[\beta^{i,\star}(s_i)\right](q) = p = \left[\beta^{i,\star}(s_i')\right](q').$$

Let S be defined as

$$\begin{aligned} S = \Bigg\{ (s,z) : \; & q^i\left(\beta^{i,\star}(s_i), \beta^{-i,\star}(s_{-i}); z\right) \leq \; q_{i,\ell} \\ & < q_{i,r} \leq \lim_{t \nearrow \infty} q^i\left(\beta^{i,t}(s_i), \beta^{-i,t}(s_{-i}); z\right), \\ & \text{and } \left[\beta^{i,\star}(s_i)\right](q) = \left[\beta^{i,\star}(s_i)\right](q') \; \forall q, q' \in (q_{i,\ell}, q_{i,r}), \\ & \text{and } \lim_{t \nearrow \infty} q^j\left(\beta^{j,t}(s_j), \beta^{-j,t}(s_{-j}); z\right) < q^j\left(\beta^{j,\star}(s_j), \beta^{-j,\star}(s_{-j}); z\right) \Bigg\}. \end{aligned}$$

Then there is an agent j, quantities $q_{j,\ell}, q_{j,r} \in [0, \bar{Q}]$ with $q_{j,\ell} < q_{j,r}$, and a set $\hat{S} \subseteq S$ with $\Pr((s,z) \in \hat{S}) > 0$ such that for all $(s,z) \in \hat{S}$,

$$\begin{aligned} \lim_{t \nearrow \infty} q^j\left(\beta^{j,t}(s_j), \beta^{-j,t}(s_{-j}); z\right) \leq q_{j,\ell} < q_{j,r} \leq q^j\left(\beta^{j,\star}(s_j), \beta^{-j,\star}(s_{-j}); z\right) \\ \text{and } \left[\beta^{j,\star}(s_j)\right](q) = p = \left[\beta^{j,\star}(s_j')\right](q') \; \forall q, q' \in (q_{j,\ell}, q_{j,r}). \end{aligned}$$

Fix s_j and define an alternative bid $\bar{b}_\lambda^{j,t}$ by

$$\bar{b}_{j,t}^\lambda = \left[\left[\beta^{j,\star}(s_j) \vee \beta^{j,t}(s_j)\right] + \lambda\right] \wedge v^j(\cdot; s_j).$$

Define $d_t = ||\beta^{j,t}(s_j) - \beta^{j,\star}(s_j)||$. Since the divisible-good model with standard transfers satisfies Condition 3, for all opponent signal realizations s_{-j} and random realizations z,

$$u_z^j\left(\bar{b}_{j,t}^\lambda, \beta^{-j,t}(s_{-j}); s_j, z\right) \geq u_z^j\left(\beta^{j,t}(s_j), \beta^{-j,t}(s_{-j}); s_j, z\right) - g\left(d_t + \lambda\bar{Q}; s_j\right).$$

Since the right-hand residual can be made arbitrarily small by letting λ be small and t be large, it will suffice to show that with positive probability the above inequality is strict, even without the residual term.

Let $(s_j, s_{-j}, z) \in \hat{S}$. Since $v^j(q; \cdot)$ is strictly increasing for all q, it is without loss to assume that

$[\beta^{j,\star}(s_j)](q_{j,r}) < v^j(q_{j,r}; s_j)$. Since standard transfers are bounded above by bids, it follows that there is $\delta > 0$ such that

$$\lim_{t \nearrow \infty} u_z^j\left(\beta^{j,t}(s_j), \beta^{-j,t}(s_{-j}); s_j, z\right) < u_z^j\left(\beta^{j,\star}(s_j), \beta^{-j,t}(s_{-j}); s_j, z\right) - 2\delta. \tag{7}$$

Furthermore, for λ sufficiently small and t sufficiently large, $[\beta^{j,\star}(s_j)](q_{j,r}) < \overline{b}_{j,t}^{\lambda}(q_{j,r}) < v^j(q_{j,r}; s_j)$, hence

$$\lim_{t \nearrow \infty} q^j\left(\overline{b}_{j,t}^{\lambda}, \beta^{-j,t}(s_{-j}); z\right) \geq q^j\left(\beta^{j,\star}(s_j), \beta^{-j,\star}(s_{-j}); z\right) \equiv q_{j,\star} \geq q_{j,r}.$$

Uniform continuity of standard transfers in own and opponents' bids implies that for λ sufficiently small,

$$\lim_{t \nearrow \infty} \tau\left(q; \overline{b}_{j,t}^{\lambda}, \beta^{-j,t}(s_{-j}), z\right) - \tau\left(q; \beta^{j,\star}(s_j), \beta^{-j,\star}(s_{-j}), z\right) < \delta.$$

Since $\overline{b}_{j,t}^{\lambda} \geq \beta^{j,t}(s_j)$ is bounded above by marginal value $v^j(\cdot; s_j)$,

$$\begin{aligned}
&\lim_{t \nearrow \infty} u_z^j\left(\overline{b}_{j,t}^{\lambda}, \beta^{-j,t}(s_{-j}); s_j, z\right) \\
&\geq \lim_{t \nearrow \infty} u_z^j\left(\beta^{j,\star}(s_j), \beta^{-j,\star}(s_{-j}); s_j, z\right) \\
&\quad - \left[\tau\left(q_{j,\star}; \overline{b}_{j,t}^{\lambda}, \beta^{-j,t}(s_{-j}), z\right) - \tau\left(q_{j,\star}; \beta^{j,\star}(s_j), \beta^{-j,t}(s_{-j}), z\right)\right] \\
&\quad + \int_{q_{j,\star}}^{q^j\left(\overline{b}_{j,t}^{\lambda}, \beta^{-j,t}(s_{-j}); z\right)} v^j(x; s_j) - \overline{b}_{j,t}^{\lambda}(x)\, dx \geq u_z^j\left(\beta^{j,\star}(s_j), \beta^{-j,\star}(s_{-j}); s_j, z\right) - \delta.
\end{aligned} \tag{8}$$

Putting together (7) and (8) gives

$$\lim_{t \nearrow \infty} u_z^j\left(\overline{b}_{j,t}^{\lambda}, \beta^{-j,t}(s_{-j}); s_j, z\right) > \lim_{t \nearrow \infty} u_z^j\left(\beta^{j,t}(s_j), \beta^{-j,t}(s_{-j}); s_j, z\right) + \delta.$$

Since this δ improvement can be realized with positive probability while costs can be made arbitrarily small, it follows that

$$\lim_{t \nearrow \infty} u^j\left(\overline{b}_{j,t}^{\lambda}, \beta^{-j,t}(s_{-j}); s_j\right) > \lim_{t \nearrow \infty} u^j\left(\beta^{j,t}(s_j), \beta^{-j,t}(s_{-j}); s_j\right).$$

□

Lemma 24 (Divisible-good type insensitivity)**.** *Divisible-good auctions with standard transfers satisfy Condition 9.*

Proof. Let $b_i \in A^i(s_i)$, and for $\lambda > 0$ consider an alternative bid function $\underline{b}_i^\lambda = [v^i(\cdot; s_i) - \lambda]_+ \wedge b_i$. Then $\underline{b}_i^\lambda \in \underline{A}^i(s_i)$, and $\underline{b}_i^\lambda \leq b_i$. Consider any opponent bid profile $b_{-i} = (b_j)_{j \neq i}$. Since standard transfers are monotone in own bid, for any q and z

$$\tau\left(q; \underline{b}_i^\lambda, b_{-i}, z\right) \leq \tau\left(q; b_i, b_{-i}, z\right).$$

Then if $u_z^i(\underline{b}_i^\lambda, b_{-i}; s_i, z) < u_z^i(b_i, b_{-i}; s_i, z)$, it must be that $\underline{q}_i^\lambda \equiv q^i(\underline{b}_i^\lambda, b_{-i}; z) < q^i(b_i, b_{-i}; z) \equiv q_i$. Write

$$\begin{aligned} u_z^i\left(b_i, b_{-i}; s_i, z\right) &= u_z^i\left(\underline{b}_i^\lambda, b_{-i}; s_i, z\right) + \left[\tau\left(\underline{q}^\lambda; \underline{b}_i^\lambda, b_{-i}, z\right) - \tau\left(q_i; b_i, b_{-i}, z\right)\right] + \int_{\underline{q}_i^\lambda}^{q_i} v^i\left(x; s_i\right) dx \\ &\leq u_z^i\left(\underline{b}^\lambda, b_{-i}; s_i, z\right) + \int_{\underline{q}_i^\lambda}^{q_i} v^i\left(x; s_i\right) - p\left(b_i, b_{-i}; z\right) dx. \end{aligned} \tag{9}$$

By market clearing, it must be that $\underline{b}_i^\lambda(\underline{q}_i^\lambda) \leq p(b_i, b_{-i}; z)$. Bid montonicity and inequality (9) imply

$$u_z^i\left(b_i, b_{-i}; s_i, z\right) \leq u_z^i\left(\underline{b}_i^\lambda, b_{-i}; s_i, z\right) + \int_{\underline{q}_i^\lambda}^{q_i} v^i\left(x; s_i\right) - \underline{b}_i^\lambda\left(x\right) dx. \tag{10}$$

Furthermore, market clearing implies that for $q \in (\underline{q}_i^\lambda, q_i)$, $\underline{b}_i^\lambda(q) < b_i(q)$. Then for all such q, $\underline{b}_i^\lambda(q) = v^i(q; s_i)$. Then (9) becomes

$$u_z^i\left(b_i, b_{-i}; s_i, z\right) \leq u_z^i\left(\underline{b}_i^\lambda, b_{-i}; s_i, z\right) + \left(q_i - \underline{q}_i^\lambda\right)\lambda \leq u_z^i\left(\underline{b}_i^\lambda, b_{-i}; s_i, z\right) + \overline{Q}\lambda.$$

It follows that against any opponent strategy profile $\beta^{-i} = (\beta^j)_{j \neq i}$,

$$U^i\left(\underline{b}_i^\lambda, \beta^{-i}; s_i\right) \geq U^i\left(b_i, \beta^{-i}; s_i\right) - \overline{Q}\lambda.$$

□

Proof of Corollary 4. That quantity is utility-relevant follows from the logic employed in the proof of Lemma 23, which establishes Condition 5. In particular, if quantity is not converging for a positive-measure set of signal realizations, it is without loss to assume that agent i loses quantity in the limit. Then for some realizations of agent signals, the lost quantity intervals overlap, and Lemma 21 then implies that for some of these type realizations the lost quantity intervals overlap at exactly the same bid level. Then because marginal values are strictly monotone in signal, a positive-measure subset of these signal realizations is such that agent i's utility is not converging, implying that quantity is utility-relevant.

With regard to seller revenue, that bidding strategies are converging implies that if $\pi(y^\star(s), z) \neq \lim_{t\nearrow\infty} \pi(y^t(s); z)$, then $q(y^\star(s); z) \neq \lim_{t\nearrow\infty} q(y^t(s); z)$. Then since q is utility-relevant, π is utility-relevant. □

B.3 Discretized model

Let $\langle \varepsilon_t \rangle_{t=0}^{\infty}$ be a decreasing sequence converging to zero, and for any $\varepsilon > 0$ let X_D^ε and X_R^ε be given by

$$X_D^\varepsilon = \mathbb{Z}\varepsilon \cap X_D, \quad X_R^\varepsilon = \mathbb{Z}\varepsilon^2 \cap X_R.$$

Let Y^ε be the set of decreasing functions from X_D^ε to X_R^ε. For any s_i, let $A^{i,\varepsilon}(s_i)$ be given by

$$A^{i,\varepsilon}(s_i) = \left\{ y \in Y^\varepsilon : \ y(x) - y(x+\varepsilon) \in (0, \gamma\varepsilon], \text{ and } v^i(k\varepsilon; s_i) > 0 \implies y(k\varepsilon) \leq v^i(k\varepsilon; s_i) + \varepsilon^2 \right\}.$$

That is, the set of feasible bid functions is the set of strictly decreasing and Lipschitz γ-continuous functions on X_D^ε that are weakly below values (on X_D^ε). Reny (2011) establishes existence in discrete discriminatory and uniform-price auctions without the Lipschitz and strict monotonicity constraint. Nonetheless these constraints greatly simplify the proof of existence, and do not affect the proof of existence in the divisible-good model in the main text.

Lemma 25 (Satisfaction of Condition 8)**.** *The ε-discretized model $\mathcal{M}^\varepsilon = (n, u, X^\varepsilon, A^\varepsilon, F)$ satisfies Condition 8.*

Proof. Closure and the lattice structure of $A^{i,\varepsilon}(s_i)$ follow immediately from its definition.

Let $b \in A^i(s_i)$, and let $b_\varepsilon : X_D^\varepsilon \to X_R^\varepsilon$ be given by

$$b_\varepsilon(q) = \left\lfloor \frac{b(q) + \varepsilon^2}{\varepsilon^2} \right\rfloor \varepsilon^2.$$

Since $b \leq v^i(\cdot; s_i)$, $b_\varepsilon \leq v^i(\cdot; s_i) + \varepsilon^2$ and $b_\varepsilon \in A^{i,\varepsilon}(s_i)$. Then $b_{\varepsilon_t} \searrow b$ with $b_{\varepsilon_t} \geq a$ for all t.

Lastly, let $\langle b_t \rangle_{t=1}^{\infty}$ be a sequence of functions, $b_t \in A^{i,\varepsilon_t}(s_i)$ for all t, and assume that $b_t \to b^\star$. Since each b_t is monotone, $b^\star$ is monotone and $b^\star \leq v^i(\cdot; s_i)$ almost everywhere. Then $b^\star$ is L_1 equivalent to a function $\hat{b}^\star \in A^i(s_i)$, and $\langle b_t \rangle_{t=0}^{\infty}$ converges in $A^i(s_i)$. □

Lemma 26 (Satisfaction of Condition 7)**.** *The ε-discretized model $\mathcal{M}^\varepsilon$ admits a monotone pure strategy Bayesian Nash equilibrium.*

Proof. By Proposition 4.4 of Reny (2011) it is sufficient to show that bidder i's utility function is weakly quasisupermodular and satisfies weak single crossing.

Weak single crossing is straightforward. Let $b_i' \geq b_i$; then $q^i(b_i', \cdot; \cdot) \geq q^i(b_i, \cdot; \cdot)$, and

$$\begin{aligned} & U^i\left(b_i', \beta^{-i}; s_i\right) \geq U^i\left(b_i, \beta^{-i}; s_i\right) \\ \iff \quad & \mathbb{E}\left[\int_{q^i(b_i, b_{-i}; z)}^{q^i\left(b_i', b_{-i}; z\right)} v^i\left(x; s_i\right) dx\right] \geq \mathbb{E}\left[\tau\left(q^i\left(b_i', b_{-i}; z\right); b_i', b_{-i}, z\right) - \tau\left(q^i\left(b_i, b_{-i}; z\right); b_i, b_{-i}, z\right)\right]. \end{aligned}$$

The left-hand side is strictly increasing in s_i while the right-hand side is constant in s_i, and weak single crossing is established.

Weak quasisupermodularity derives from the observation that, given any bids b_i and b_i' and any realization z, $\{q^i(b_i, b_{-i}; z), q^i(b_i', b_{-i}; z)\} = \{q^i(b_i \vee b_i', b_{-i}; z), q^i(b_I \wedge b_i', b_{-i}; z\}$, and the presumed submodularity of the standard transfer rule. □