Auctions and Other Games with Max-Min Players

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Multi-unit auctions

Multi-unit auctions common when principal allocates many homogeneous units.

- Treasury securities
  - 2016: \$8.6tn (U.S.), 526bn € (Fr.), £146bn (U.K.)
- Quantitative easing
- Electricity distribution

Know little about equilibrium in these auctions in presence of private information.

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Appeal to authority

"Unfortunately, computing equilibrium strategies in (asymmetric) discriminatory multi-unit auctions is still an open question [...]."

Hortaçsu and Kastl, 2012

General concept

Max-min utility provides a tractable approach to private information.

- Equilibrium existence
- Strategy selection to combat "anything goes" results
    - Natural limit of risk aversion
    - Limit as ambiguity aversion allows for arbitrary concentration
    - Relation to optimizing "but for"
- Uniqueness of selection

Game theoretic results extend to related settings—oligopoly, cooperation, etc. In the process of formalizing.

Multi-unit auction results

- Equilibrium existence/uniqueness
- In pay-as-bid auctions:
  - Near-efficiency with private values
  - Rent near-extraction with private values
- Revenue and efficiency comparisons across mechanisms
- Clean generalization to interdependent value case

Related literature

Multi-unit auctions (theory)
Maskin and Riley, 1989; Engelbrecht-Wiggans and Kahn, 2002; Ausubel et al., 2015; Burkett and Woodward, 2016

Multi-unit auctions (empirics)
Février et al., 2002; Castellanos and Oviedo, 2004; Armantier and Sbaï, 200x; Kang and Puller, 2008; Hortaçsu and McAdams, 2010

Divisible-good auctions
Wilson, 1979; Klemperer and Meyer, 1989; Back and Zender, 1993; Wang and Zender, 2002; Anderson et al., 2013; Pycia and Woodward, 2016

Max-min mechanism design
Lo, 1998; Bose et al., 2006; Chen et al., 2007; de Castro and Yannelis, 2010; Bodoh-Creed, 2012; Di Tillio et al., 2012; Bose and Renou, 2014; Lopomo et al., 2014; de Castro et al. 2015; Wolitzky, 2016

Model

Presentation model:

- n bidders
- Q indivisible units, $1 \leq Q \leq (n-1)d$
- Value for k^{th} unit is $\theta^i_k \in [0, \bar{\theta}]$; assume full support, $\theta^i \in [0, \bar{\theta}]^d$
- Weakly decreasing bids $b^i_k \in \{0, \varepsilon, \ldots, \bar{m}\varepsilon\}$ (wlog $\bar{m}\varepsilon \leq \bar{\theta}$)
- Q highest bids win; ties broken by random bidder order

Multi-unit auctions

If allocation is q_i, utility is

$$\sum_{k=1}^{q_i} \theta_k^i - t^i\left(b^i, b^{-i}\right)$$

Pay-as-bid: price discrimination against reported demand,

$$t^i\left(b^i, b^{-i}\right) = \sum_{k=1}^{q_i} b_k^i$$

Uniform price: constant per-unit marginal price,

$$t^i\left(b^i, b^{-i}\right) = b^{(Q)} q_i$$

Max-min equilibrium

Definition (Max-min equilibrium)

A strategy profile $(s_i)_{i=1}^n$ is a *max-min equilibrium* if for all agents i, all types θ_i, and all actions $\tilde{a}_i \in A_i$,

$$\inf_{\theta_{-i}} u^i\left(s_i\left(\theta_i\right), s_{-i}\left(\theta_{-i}\right); \theta\right) \geq \inf_{\theta_{-i}} u^i\left(\tilde{a}_i, s_{-i}\left(\theta_{-i}\right); \theta\right).$$

A strategy profile is a max-min equilibrium if for any other action there is a belief over opponent types that generates lower worst-case utility.

Max-min equilibrium: existence

Theorem

There exists a max-min equilibrium.

Max-min equilibrium: IPV first-price auction

Except for very high types ($\bar{\theta} \geq \bar{m}\varepsilon$), anything goes: any bid weakly below value is supportable in equilibrium.

- Very high types can play $b(\bar{\theta}) = \bar{m}\varepsilon < \bar{\theta}$
  - Bidding higher is impossible
  - Bidding lower implies lose to opponent $\bar{\theta}$, utility 0

- Lower types bid anything below value
  - If bid above value, worst case is winning the auction, negative utility
  - If bid below value, worst case is losing (to, e.g., $\bar{\theta}$), indifferent across all losses

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Can we sharpen predictions, respecting analogy to risk aversion?

Upside dominance

Let $\underline{u}^i(a_i, s_{-i}; \theta_i) = \inf_{\tilde{\theta}_{-i}} u^i(a_i, s_{-i}(\tilde{\theta}_{-i}); \tilde{\theta})$.

Definition (Upside dominance)

Action a_i *upside dominates* action a_i' if there is $\bar{\varepsilon} > 0$ such that for all $\varepsilon \in (0, \bar{\varepsilon})$,

$$\left\{\theta_{-i} : \; u^i\left(a_i, s_{-i}\left(\theta_{-i}\right); \theta\right) \geq \underline{u}^i\left(a_i, s_{-i}; \theta_i\right) + \varepsilon\right\}$$
$$\supseteq \left\{\theta_{-i} : \; u^i\left(a_i', s_{-i}\left(\theta_{-i}\right); \theta\right) \geq \underline{u}^i\left(a_i', s_{-i}; \theta_i\right) + \varepsilon\right\}.$$

This is strict for some $\varepsilon' \in (0, \bar{\varepsilon})$.

Two max-min best responses are upside-dominance ordered if one is more likely to guarantee (possibly small) upside.

Limit of risk aversion

Non-formal analogy: for an appropriate strictly concave function f,

$$\lim_{t \nearrow \infty} \left| \underbrace{f \circ \cdots \circ f}_{t \text{ times}} (u') - \underbrace{f \circ \cdots \circ f}_{t \text{ times}} (u) \right| = 0$$

The magnitude of potential gains becomes irrelevant, only the probability of gains matters.

Upside dominance in first-price auctions

Suppose that equilibrium bid distribution has full support (e.g., reports are essentially truthful). Compare $b' < b < \theta^i$.

- Lower bid b' gives higher margins, lower probability
- Higher bid b gives lower margins, higher probability

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$$\exists \theta^{-i},\ b^{-i}\left(\theta^{-i}\right) \in \left(b', b\right] \implies b \succ_{\mathrm{UD}} b'$$

Filtration

Fix a profile of opponent strategies. Idea:

- Start with full set of actions and opponent types
- Find max-min best responses in these sets
- Remove all opponent types against which the agent is indifferent across all max-min best repsonses
- Repeat until no opponent types removed

Filtration

In FPA, suppose opponents submit highest bid strictly below value.

- I am indifferent across all bids weakly below my value
- All opponent type profiles who bid weakly above my value give me max-min outcomes
- Throw away these opponents, everyone who remains bids strictly below my value
- My unique max-min best response is the highest bid strictly below my value

Filtration

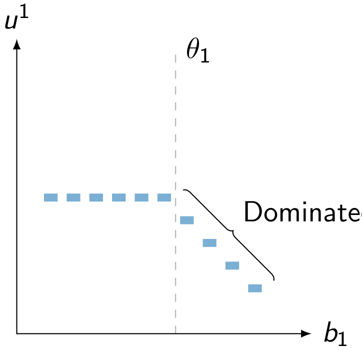

Figure: Worst-case utility, assuming truthful bidding by opponent.

Filtration

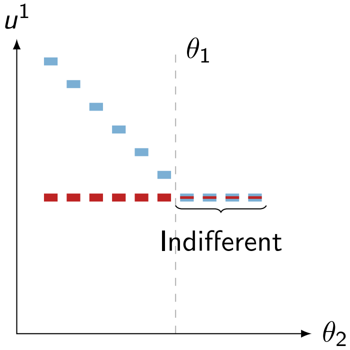

Figure: Maximum and minimum utility from max-min action set.

Filtration

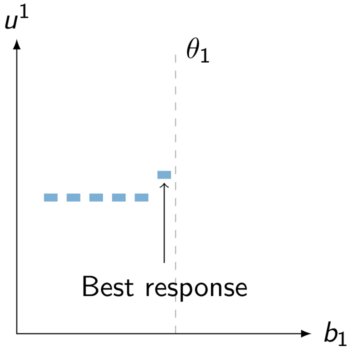

Figure: Worst-case utility in reduced opponent type space.

Upside dominant equilibrium

Definition (Upside-dominant equilibrium)

A strategy profile $(s_i)_{i=1}^n$ is an *upside-dominant equilibrium* if it is a max-min equilibrium, and for each agent i and type θ^i there is no action $\tilde{a}_i$ that upside dominates $s_i(\theta^i)$.

Pay-as-bid: equilibrium existence

Theorem

The pay-as-bid auction admits an upside-dominant equilibrium.

Proof is constructive, but intuition should generalize:

- WLOG actions are monotone in type
- In equilibrium, worst outcome is when opponent has high type
- Start at PSNE in full-information auction with only high types
- Sweep types downward, filling in upside-dominant max-min best response

Pay-as-bid: equilibrium

There is an equilibrium in which

$$b_k^i\left(\theta^i\right) = \begin{cases} \max\left\{\kappa\varepsilon : \ \kappa\varepsilon < \theta_k^i\right\} & \text{if } \theta_k^i > 0, \\ 0 & \text{otherwise.} \end{cases}$$

- Full support of values implies full support of bids, implies all allocations feasible
- Then sum of bids is weakly below sum of values
- If bid for k above value for k, can reduce bid on k without sacrificing net utility
- If bid below prescribed bid, can increase and capture (small) gain against some opponents, keeping all existing positive margins strict

Pay-as-bid: uniqueness

If $n \geq 3$ and the bidding grid is evenly spaced, equilibrium bids are unique for all $\theta \leq (\bar{m} - 1)\varepsilon < \bar{\theta} - \varepsilon$.

Conditions have to do with tiebreaking. Generally:

- For any grid we have (essential) uniqueness for n sufficiently large
- For any n, equilibria (b^i) and $(\hat{b}^i)$ differ by $||b^i - \hat{b}^i|| = O(\text{maximum grid step})$

Properties of equilibrium

Except for highest types, bidders report as truthfully as possible (respecting IR).

If sufficiently high bids are available:

- Ex post allocation is essentially efficient (gap is O(maximum grid step))
- Ex post revenue captures essentially all bidder rents (gap is O(maximum grid step))
- Essentially no role for reserve prices or supply restrictions

Uniform-price: equilibrium

Pay-as-bid logic implies same equilibrium in uniform-price auction.
Except for lowest types, bids are strictly below values in all equilibria.

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- Pay-as-bid bids weakly exceed uniform-price bids
- Pay-as-bid revenue is strictly higher for all strictly-decreasing type realizations
- Uniform-price is (weakly) less efficient

Interdependent values

Consider interdependent single-unit auction model,

$$v^i = \theta^i + \alpha \sum_{j \neq i} \theta^j \quad (\alpha > 0)$$

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Pay-as-bid: equilibrium unchanged,

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Uniform-price: equilibrium still bounded by pay-as-bid equilibrium

Equilibrium properties

In pay-as-bid, bids are unchanged *even though values increase almost surely*.

- Bidders retain rents, give away (most of) minimum possible rents conditional on own type
- Inefficient outcomes arise
  - Suppose $\theta_1^i \gg \theta_Q^i > \theta_1^j \gg \theta_Q^j$; then i gets Q units, j gets 0
  - Then for α sufficiently large,

$$v_1^j = \theta_1^j + \alpha\theta_1^i > \theta_Q^i + \alpha\theta_Q^j = v_Q^i$$

- Holding average ex post values constant, revenues decrease in interdependence

Nonetheless, still no role for reserve price or supply optimization in pay-as-bid.

Conclusion

Considered canonical multi-unit auction formats with max-min bidders.

- Existence of upside-dominant equilibria
- Near-uniqueness of equilibrium in pay-as-bid
  - Near-full rent extration in private values case
  - Near-efficiency in private values case
- Revenue and efficiency dominance of pay-as-bid
- Equilibrium strategies carry over simply to interdependent values model

Working on extending results to more general class of models.