Industry Costs and Research Aggregation in Dynamic Competition

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Abstract

We study information acquisition and subsequent price competition in an environment where the cost of each firm is initially unknown and composed of two components, private costs specific to the firm and costs common to all firms in the industry. In this setting, firms choose high initial prices to soften future competition. Moreover, this pricing distortion is exacerbated when firms only possess private information about firm specific costs. This implies that sharing information about industry relevant costs, such as aggregating cost information through a trade association, will lead to higher prices and reduce incentive to acquire information about firm specific costs. As the number of firms in the market increase, the pricing distortion is reduced along with its negative welfare impacts, increasing the likelihood that information sharing improves both consumer and producer surplus.

1 Introduction

Firms within an industry often share information through a trade association. Examples include information about market demand, firm costs, capacity, pricing and sales. Antitrust considerations of collusive potential and the uniformity effect have created clear guidelines for sharing prices and, to a lesser extent, capacity and sales. The impacts, and therefore the guidelines, of sharing cost information through a trade association are not as clear.[1]

In this paper we consider the impact of sharing industry relevant cost information via a trade association in the context of dynamic price competition. Information that is not shared

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[1] See discussions in Kühn and Vives [1994] and Kühn [2001].

through the trade association may otherwise be endogenously revealed through observed prices. The presence of this information exchange agreement can change pricing behavior directly, as there is more information available to the firm, and indirectly as the information content of prices also changes. We additionally analyze the impact of an agreement on incentives to acquire information prior to the exchange. Overall, we show that sharing cost information via a trade association can lead to higher prices and less acquisition of information prior to exchange.

We study a stylized dynamic Bertrand competition model where firms face a commonly known demand curve and have private information about costs. The structure of costs is generalized to allow for both idiosyncratic costs that are firm specific and costs that are common to all firms in the industry, e.g. prices of inputs. In this competitive setting, firms have an incentive to over represent costs by setting a higher price. Understanding that competing firms have other channels to learn about common costs, this signal jamming focuses on inflating beliefs about firm specific costs. The signal is confounded however as a high price could be the result of the firm having information about high costs that are either firm specific or affect all firms in the industry.

Sharing common cost information changes each firm's absolute and relative information structure. Identical information on shared costs implies that unexpectedly high or low prices of another firm is reflective of the specific firm's cost structure. With prices acting as a clearer signal, the incentive to signal jam strengthens, resulting in higher average prices. Additionally, prices are more informative about the firm's idiosyncratic cost structure. This reduces the informational rents of firm-specific cost information lowering firms' willingness to pay to acquire it.

Our analysis begins with two stage price competition between two firms. Each firm's marginal cost has two independent components; one is firm specific and the other, called the common component, is shared with the other firm. The true value of the three total cost components are initially unknown and distributed independently. Each firm has conditionally independent private signals over each of their own components. In the first stage, prices are chosen based on these signals. At the end of this stage, firms obtain full information of their own cost structure, both individual and common components. Additionally, they observe the price chosen by the other firm. Given this information, firms choose prices in the second stage of competition.

For any symmetric level of signal precision we show that the dynamic pricing game has a unique symmetric equilibrium in linear strategies. A complication arises from the fact that price acts as a signal for two sources of private information, only one of which, the firm specific costs, is relevant to the other firm. Prices therefore are informative about private cost information but linear pricing strategies are not separating in the sense of revealing this fully. The optimal first stage pricing strategy depends on the amount of information the price conveys to the other firm about this cost component. In equilibrium, the informativeness of the price necessarily depends on the pricing strategy. Identifying equilibrium requires finding the pricing strategy that leads to a level of price informativeness for which the pricing strategy is optimal.

A linear pricing strategy is characterized by coefficients which determine how much the firm's price changes with the expected cost of each component. In any symmetric linear equilibrium, prices in the first period vary less with firm specific costs than is optimal in the one-shot pricing game. Moreover, the equilibrium value of this coefficient is inversely proportional to the informativeness of the price in equilibrium. This reflects the firm's incentive to signal jam in order to maintain informational rents in the second period.

Equilibrium pricing coefficients also depend on the precision of signals received by firms prior to competition. More precise signals of firm-specific costs increases the incentive to signal jam and leads the firm reduce the pricing coefficient of this information. Conversely, intermediate levels of precision on the common cost component confound the price signal allowing firms to use more information about firm specific costs without losing as much informational rents. Therefore the relationship between the equilibrium pricing coefficient on firm specific cost and common cost signal precision is hump-shaped.

Within this framework, we analyze the impact of firms sharing signals on the common cost component in a verifiable way. The only source of private information each firm still posses is a private cost signal. Therefore, in equilibrium, first stage prices will be a better signal of this information despite a smaller equilibrium pricing coefficient. This increases the incentive to signal jam in the first stage leading to higher prices on average. Signal jamming also distorts the equilibrium pricing strategy away from the optimal one-shot strategy and reduces how much the price adjusts to firm specific expected costs. This lowers the value of this information to the firm, and therefore the willingness to pay for a more precise signal prior to price competition.

Lastly, we characterize how consumer surplus and producer surplus depend on the expectation, variance and covariance of prices in each period of competition. In the setting with two firms, information sharing will reduce consumer surplus and increase producer surplus whenever demand is inelastic enough. We then extend the model by deriving the symmetric linear equilibrium for an arbitrary number of firms, characterizing the consumer and producer surplus as the number of firms become large. In a market with a large number of firms information sharing always increases producer surplus and will also increase consumer surplus when products are not very substitutable or when information prior to sharing is relatively dispersed.

This paper adds to the literature of dynamic oligopoly models with incomplete information and perfect monitoring that examine the information content of prices and the resulting pricing distortions that stem from signaling. Mailath [1989] and Mester [1992] identify the incentive to over represent cost by choosing a high price and more recently Jeitschko et al. [2018] place this incentive in the context of information sharing and information acquisition. Our paper allows for multiple sources of private information allowing for a richer information set and a more complex strategy space. Specifically it allows for firms to posses information that is relevant to their competitors for more than strategic reasons.

As equilibrium strategies do not fully reveal private information this paper also adds to the literature of signal jamming literature in dynamic oligopoly models. In a Cournot setting where market price is observed, Bonatti et al. [2017] characterize with a continuous time model the dynamics of simultaneous signal jamming and learning when firms begin with private cost information. Mirman et al. [1993] look at the case where firms have private information about individual demand curves. In our setting of perfect monitoring, the signal jamming comes from how much weight the firm places on each source of information when choosing price. By reducing the weight on one source of information the firm reduces the informativeness of the price about this source.[2]

Our paper also adds to the literature on optimal use of information from private and public signals. When public signals are perfectly correlated, Angeletos and Pavan [2007] identify both the equilibrium and efficient uses of information in a flexible setting that assumes quadratic payoffs and Gaussian information. They show that when actions are strategic complements agents weight public information relatively more, while the opposite is true under strategic substitutes. While the current paper has this information structure when firms share common cost information, we show these properties hold even when the common costs signals are correlated, but not shared.

When public signals are perfectly correlated, Bergemann and Morris [2013] show firms prefer to share an intermediate level of information about demand uncertainty.[3] When public signals are allowed to be partially correlated Bergemann et al. [2018] show that informational intermediaries may have an incentive to garble private (idiosyncratic) information about consumers before sharing with a firm. This lowers the rents needed to pay the consumer for their information. While intermediate levels of information transmission are not allowed in the decision to share information through the trade association, the pricing strategy in the

[2]Mirman et al. [1994] identify that an increase in the choice of quantity will reduce the informativeness of price in the Cournot setting where incomplete information is symmetric.

[3]Sharing all or no information though a intermediary such as a trade association has been extensively studied in the case of static oligopoly competition. See Raith [1996] and Vives [2001].

first stage of competition determines the amount of information revealed to the other firm in the case where the trade association is not used. This offers an alternative mechanism to partial revelation that emerges through dynamic competition.

Lastly our paper is related to the literature of information acquisition from multiple heterogeneous sources. We show that the promise of a more precise public signal crowds out the willingness to pay for a more precise private signal. This is consistent with the equilibrium result of Colombo et al. [2014]. Additional work on the type of information that is preferred include Morris and Shin [2002] and Myatt and Wallace [2011] in the setting of a generalized beauty contest as well as Myatt and Wallace [2015a] and Myatt and Wallace [2015b] in oligopoly settings with uncertainty on demand.

The rest of the paper is organized as follows. Sections 2 and 3 respectively introduce and analyze the model of two stage price competition with and without information sharing. Section 4 studies the welfare impact of information sharing in the two firm case and in the case when the number of firms is large while extending the equilibrium characterization to n firms. Section 5 examines the value of information with and without information sharing. Section 6 concludes.

2 Model

Two firms, i and j, compete for market share over two periods $t = 1, 2$. Demand is linear in prices, symmetric across firms, and time-independent. Firm i's demand is given by

$$q_{i,t} = a - bp_{i,t} + ep_{j,t}.$$[4]

We assume that demand is weakly more sensitive to a firm's own price than to its opponent's, so that $|e| \leq b$. Each firm faces a constant marginal cost c_i that is same in each period, so profits are

$$\pi_{i,t} = (p_{i,t} - c_i)\, q_{i,t}.$$

Firms are initially uncertain about their marginal costs of production, but know that costs are comprised of an idiosyncratic component θ_i and a common component ρ; their constant marginal cost is the sum of the two components, $c_i = \rho + \theta_i$. We assume that cost

[4] Unless otherwise specified, our equations and inequalities should be taken to be symmetric for agent j.

components are joint-normally distibuted with zero covariance, so that

$$\begin{pmatrix} \theta_i \\ \theta_j \\ \rho \end{pmatrix} \sim N\left(\begin{pmatrix} \mu_\theta \\ \mu_\theta \\ \mu_\rho \end{pmatrix}, \begin{pmatrix} \sigma_\theta^2 & 0 & 0 \\ 0 & \sigma_\theta^2 & 0 \\ 0 & 0 & \sigma_\rho^2 \end{pmatrix}\right).$$

Throughout we will denote the precision of the random variable x by $\tau_x = 1/\sigma_x^2$.

Play proceeds in two stages. In the first stage, each firm receives two noisy signals, $s_{i,\theta}$ and $s_{i,\rho}$, of the values of their idiosyncratic and common costs, respectively. These signals are normally distributed with uncorrelated error terms, and the error terms are uncorrelated between firms. We model these signals as $s_{i,x} = x + \varepsilon_{i,x}$, where $\varepsilon_{i,x}$ is normally distributed with variance $\sigma_{s,x,i}^2$.[5] Upon the realization of their private signals, firms simultaneously select prices $p_{i,1}$ and obtain stage profits $\pi_{i,1}$.

After first-stage profits are obtained, firms become perfectly informed of both the common and their (individual) idiosyncratic cost components. Each also witnesses its opponent's first-stage price, but remains unaware of its opponent's idiosyncratic cost component.[6] Firms then compete again by simultaneously selecting prices and obtain stage profits $\pi_{i,2}$.

The game ends after the second stage, and ex post utility is the (undiscounted) sum of stage profits,

$$u_i\left(p_i, p_j\right) = \pi_{i,1}\left(p_{i,1}, p_{j,1}\right) + \pi_{i,t}\left(p_{i,2}, p_{j,2}\right).$$

We restrict attention to *subgame perfect equilibria in linear strategies.*

Analysis proceeds in three parts. We first determine properties of equilibrium in this base model. Then, we allow firms to share information about their common cost component prior to the two-stage competition. Lastly, we allow firms to acquire more precise information about their costs, and compare the amount of information acquired in the setting where firms share common cost information to the setting where they do not.

3 Equilibrium

We compute the pricing equilibrium in the two stage model by backwards induction. In a subgame-perfect equilibrium second-period prices are best responses to available information.

[5] For the majority of our results we will assume symmetry, so that $\sigma_{s,x,i}^2 = \sigma_{s,x,j}^2$. Allowing for heterogeneity in the variance of the error term is essential when we discuss information acquisition.

[6] Since demand is a deterministic function of firm prices, the assumption that firms witness each others' prices is sufficient to imply that they are perfectly informed of their own private cost c_i; alternatively, if they witness their own sales volume they will be perfectly aware of their opponent's price. That they obtain perfect knowledge of each of the components of $c_i = \rho + \theta_i$ is an additional assumption.

However, even in an equilibrium where first period prices are strictly monotone in each signal, it is impossible for private information to be fully-revealed as in a standard separating equilibrium as private information is two-dimensional while actions are one-dimensional and monotone in information. Residual uncertainty in the second stage is an important feature in our model, affecting firms' first-period pricing through their ability to distort publicly-available information about their costs.

3.1 Second period pricing

In the second period, each firm knows its own marginal costs precisely, but knows only the distribution over its opponent's costs. Letting $F^j(\cdot;\rho,\mathbf{p}_1) \equiv F^j$ be the distribution of firm j's second period price conditional on firm i's available information,[7] the profit maximization problem is

$$\max_p \int (p - c_i)(a - bp + ex)\, dF^j(x).$$

Lemma 1. *Firm i's optimal second period price is*

$$p_{i2}^\star = \frac{1}{2b}\left(a + bc_i + e\mathbb{E}\left[p_{j2}^\star \middle| \rho, \mathbf{p}_1\right]\right).$$

Firm i's maximum second period expected profit is

$$\mathbb{E}\left[\pi_{i2}^\star \middle| \rho, \mathbf{p}_1\right] = \frac{1}{4b}\left(a - bc_i + e\mathbb{E}\left[p_{j2}^\star \middle| \rho, \mathbf{p}_1\right]\right)^2.$$

Thus firm i's second period price is an affine function of the demand intercept, its (known) cost $c_i = \rho + \theta_i$, and its expectation over firm j's second period price. Profits then have a standard quadratic form.

Lemma 2. *In any equilibrium, expected second period prices of a firm given publically available information are*

$$\mathbb{E}\left[p_{j,2}^\star \middle| \rho, \mathbf{p}_1\right] = \frac{1}{4b^2 - e^2}\left((2b+e)a + 2b^2\mathbb{E}\left[c_j \middle| \rho, p_{j,1}\right] + be\mathbb{E}\left[c_i \middle| \rho, p_{i,1}\right]\right),$$

which result in the following expected second period profits:

$$\mathbb{E}\left[\pi_{i2}^\star \middle| \rho, \mathbf{p}_1\right] = \frac{1}{4}\left(\frac{1}{4b^2 - e^2}\right)^2\left((4b+2e)a - 4b^2c_i + \left(\mathbb{E}\left[c_i \middle| \rho, \mathbf{p}_1\right] - c_i\right)e^2 + 2be\mathbb{E}\left[c_j \middle| \rho, \mathbf{p}_1\right]\right)^2.$$

[7] Firm i also knows θ_i, $s_{i,\theta}$, and $s_{i,\rho}$, but these offer no payoff-relevant information in the second stage (beyond θ_i, ρ, and $p_{i,1}$) and may be ignored. Equivalently, $(\rho, \mathbf{p}_1)$ is firm i's knowledge of public information.

Lemma 2 connects firms i's expected second period profits to its first period price. These profits increase in $E[c_i|\rho, \mathbf{p}_1]$, the expectation of firm i's cost given information available to firm j in the second period. Therefore firm i has an incentive to over-represent its cost, leading firm j to increase its second period price, softening competition for firm i.[8]

3.2 First period pricing

First period prices are set to optimize the sum of profits over two periods. Although first period prices have no direct effect on second period profits, firm i's price affects firm j's beliefs regarding firm i's costs. This is shown directly in Lemma 2, where p_{i1} enters only in $\mathbb{E}[c_i|\rho, \mathbf{p}_1]$.

Firm i's first period profit maximization problem is

$$\max_p \mathbb{E}\left[\pi_{i1}|\, s_{i\rho}, s_{i\theta}\right] + \mathbb{E}\left[\pi^\star_{i2}|\, s_{i\rho}, s_{i\theta}\right] = \max_p \mathbb{E}\left[\left(a - bp + ep^\star_{j1}\right)(p - c_i) + \pi^\star_{i2}\middle|\, s_{i\rho}, s_{i\theta}\right].$$

A marginal increase in first period price affects first period profits in a standard way, and has an additional effect on second period profits by manipulation of the opposing firm's second period beliefs which changes second period price choices. This gives Lemma 3.

Lemma 3. *Optimal first period prices are given by*

$$p^\star_{i,1} = \left(\frac{1}{2b}\right)\mathbb{E}\left[bc_i + a + e\hat{p}_{j,1}|\, s_{i,\rho}, s_{i,\theta}\right]$$
$$+ e\left(\frac{1}{2b}\right)^2 \mathbb{E}\left[\left(a - bc_i + e\mathbb{E}\left[p^\star_{j,2}|\,\rho, p^\star_{i,1}, \hat{p}_{j,1}\right]\right)\frac{\partial}{\partial p_{i,1}}\mathbb{E}\left[p^\star_{j,2}|\,\rho, p^\star_{i,1}, \hat{p}_{j,1}\right]\middle|\, s_{i,\rho}, s_{i,\theta}\right].$$

We constrain attention to equilibria in pricing strategies that are linear in the expected value of each cost component.[9] A linear first period price can be expressed as

$$p_{i,1} = p_{i,0} + \mathbb{E}\left[\theta_i|\, s_{i,\theta}\right] p_{i,\theta} + \mathbb{E}\left[\rho|\, s_{i,\rho}\right] p_{i,\rho}.$$

Under linear strategies, each firm's first period price choice is a normally distributed random variable from the perspective of the other firm. Therefore, $(c_i, \rho, p_{i,1})$ are distributed joint-normally, which implies that $\mathbb{E}[c_i|\rho, p_{i,1}]$ is linear in $p_{i,1}$. Moreover, the effect of an

[8]This describes the reaction of firm j when the firms are selling substitutes, $e > 0$. When $e < 0$, a higher value of $E[c_i|\rho, \mathbf{p}_1]$ leads to a lower $p^\star_{j2}$ which still increases $\pi^\star_{i2}$. When $e = 0$, the price and profit equations reduce to the standard monopoly model.

[9]This forces each firm to commit to using information about each cost component at a fixed level for all possible signals, $(s_{i,\theta}, s_{i,\rho})$, it may receive. As we show, these strategies are best responses to the opponent's linear pricing rule, even allowing for nonlinear pricing rules, when the firm has received its private signals. It is possible that there exist equilibria in nonlinear pricing rules.

increase in firm i's first period price on firm j's second period beliefs, and hence second period price, is constant and independent of the level of price. Conditioning beliefs on this relationship gives Lemma 4.

Lemma 4. *The marginal effect of firm i's first period price on firm j's expected second period price is*

$$\frac{\partial}{\partial p_{i,1}} \mathbb{E}\left[p^{\star}_{j,2} \middle| \rho, p_{i,1} \right] = \frac{be}{4b^2 - e^2} \kappa_i,$$

$$\textit{where } \kappa_i \equiv \frac{\partial}{\partial p_{i,1}} \mathbb{E}\left[c_i \middle| \rho, p_{i,1} \right] = \frac{\sigma^2_\theta \bar{\tau}_{s,\theta} p_\theta}{\sigma^2_\rho \left(1 - \bar{\tau}_{s,\rho}\right) \bar{\tau}_{s,\rho} p^2_\rho + \sigma^2_\theta \bar{\tau}_{s,\theta} p^2_\theta} \textit{ and } \bar{\tau}_{s,x,i} = \frac{\tau_{s,x,i}}{\tau_x + \tau_{s,x,i}}.$$

The term κ_i captures the relative informativeness of firm i's first period price regarding the its idiosyncratic cost component θ_i, the remaining source of asymmetric information in the second period when ρ is commonly known. Despite observing ρ, firms do not observe each other's first period signal on the common cost component, $s_{i,\rho}$. Because the first period price depends on the realization of $s_{i,\rho}$, it can be thought of a noisy signal of $s_{i,\theta}$, Therefore the informativeness of the price in determining θ_i depends not only on the variance of the price relative to $s_{i,\theta}$ but also relative to $s_{i,\rho}$.[10]

For $x \in \{\rho, \theta\}$, $\bar{\tau}_{s,x,i}$ is the relative contribution of the normally distributed noise in firm i's signal around the true parameter x, to the precision of the signal $s_{i,x}$. When signals are very noisy, $\bar{\tau}_{s,x,i}$ will be close to zero; when signals give a more precise prediction of the true cost parameter, $\bar{\tau}_{s,x,i}$ will be close to one. When signals are more precise, they have a larger role in the formation of expectations over the cost parameters.

The term $\bar{\tau}_{s,x,i} p_{i,x}$ is the derivative of first period price with respect to $s_{i,x}$, and affects the informativeness of the first period price about the firms cost. Therefore the choice of strategy in the first period for a given level of information precision will directly impact the value of κ_i. Specifically, as either $p_{i,\theta}$ or $p_{i,\rho}$ increases, κ_i decreases. If a firm increases $p_{i,x}$ while precisions remains constant, it is increasing the variance of price and therefore changes in price will be less informative of the underlying primitives of the model. Moreover, the incentive constraints of the equilibrium strategy in the first period depend on the value of κ_i.[11] This fixed point problem is expressed in the single-variable equation in Theorem 1. Importantly, since pricing strategies are not observed, κ_i is not affected by firm i's selection of price; it is determined by the pricing strategy the firm is believed to be following.

[10] Note that σ^2_ρ does not appear in the denominator of κ_i since ρ is commonly observed.

[11] Note that a deviation does not indicate a specific misreport of marginal cost but rather an iso-information curve of feasible $s_{i,\rho}, s_{i,\theta}$. These iso-information curves depend on the value of κ_i.

Theorem 1. *There exists a unique symmetric equilibrium in linear pricing strategies. The equilibrium strategies are determined by the value of κ in equilibrium which satisfies the following single variable equation:*

$$\kappa = \frac{\sigma_\theta^2 \bar{\tau}_{s,\theta} p_\theta}{\sigma_\rho^2 \left(1 - \bar{\tau}_{s,\rho}\right) \bar{\tau}_{s,\rho} p_\rho^2 + \sigma_\theta^2 \bar{\tau}_{s,\theta} p_\theta^2},$$

$$\text{subject to } p_\theta = \frac{1}{2 + \beta\kappa} \text{ and } p_\rho = \frac{1 - \left(\frac{b-e}{2b-e}\right)\beta\kappa}{2 - \frac{e}{b}\bar{\tau}_{s,\rho} - \frac{1}{2}\left(1 - \bar{\tau}_{s,\rho}\right)\beta^2\kappa^2},$$

where $\beta = \frac{e^2}{4b^2 - e^2}$.

There are two strategic effects we can identify in the first period prices. First, due to the correlation of one cost signal and the independence of the other signal, firms may want to act more heavily on one of these signals than the other if they prefer to have their prices correlated in the first period. Additionally, firms benefit from having private information in the second period and therefore prefer to not reveal precise information about their idiosyncratic cost term. The implications of the first effect are in Proposition 1 and those of the second effect are in Proposition 2.

Proposition 1. *In equilibrium, $p_\rho < p_\theta$ when goods are complements ($e < 0$) and $p_\rho > p_\theta$ when goods are substitutes ($e > 0$). $p_\rho = p_\theta$ when markets are independent ($e = 0$).*

When $e > 0$, so that goods are substitutes, firms' first period prices are more sensitive to information on the common cost component than to information on their idiosyncratic cost component. If a firm receives a high signal on the common cost component this often implies the other firm will set a high price, increasing demand and making it optimal to further increase price. When $e < 0$, so that goods are complements, prices are strategic substitutes and will not respond strongly to the common cost signal. When $e = 0$, so that there are no cross-firm demand effects, there is no need to either adjust for the opponent's price and information about each cost component affects first period prices identically. Moreover, in the monopoly case there will be no attempt to conceal information regarding cost. However, in general the information conveyed by first period prices will affect second period profits. Proposition 2 illustrates firms' incentives to not reveal too much information on their idiosyncratic cost component.

Proposition 2. *The equilibrium values of p_θ and κ are inversely related: p_θ increases when κ decreases and vice versa. Additionally, p_θ is decreasing and κ is increasing in $\bar{\tau}_{s,\theta}$, and there is a $\tau^\star$ such that for all $\bar{\tau}_{s,\rho} > \tau^\star$, κ is increasing and p_θ is decreasing in $\bar{\tau}_{s,\rho}$, and for all $\bar{\tau}_{s,\rho} < \tau^\star$, κ is decreasing and p_θ is increasing in $\bar{\tau}_{s,\rho}$. When $e > 0$, $\tau^\star > 1/2$ and when $e < 0$, $\tau^\star < 1/2$.*

When $\bar{\tau}_{s,\theta}$ is close to one, signals relatively precise information about θ_i. To maintain the strategic advantage of private information, the firm will use less of the information from a precise signal when determining first period price. If this signal is not precise, then even if the price fully reflects the information in the signal, it will still maintain private information in the second period from learning the true value of θ_i.

The presence of uncertainty on the common component of cost adds noise to the relationship between first period price and the signal on idiosyncratic cost. When this relationship is more noisy, the price reveals less information about the idiosyncratic signal, allowing the firm to use this information in its pricing decision without revealing too much information. If the signal about the common cost is relatively imprecise, $\bar{\tau}_{s,\rho}$ close to 0, then firms do not learn much information from this signal, and relatively little noise is added to this relationship. Additionally, if the signal is very precise, $\bar{\tau}_{s,\rho}$ close to 1, then when firms learn the true value of ρ in the second round, they will learn, with little error, what signal $s_{i,\rho}$ their opponents received and will be able to tease apart the noise in the pricing strategy. Therefore an intermediate level of precision $\bar{\tau}_{s,\rho}$ on signal $s_{i,\rho}$ will maximize p_θ for a given value of $\bar{\tau}_{s,\theta}$.

In general the incentives to hide idiosyncratic cost information leads firms to be less responsive to their idiosyncratic cost signal than is optimal in a one-stage game (without the informational channels implied by our two-stage model), so relaxing signal jamming incentives leads to an increased sensitivity of price to information on the idiosyncratic cost component.

3.3 Sharing industry relevant information

We now consider the effect of the firms sharing information about costs through a trade association. We assume that signals about the common cost component are shared, while those of firm's idiosyncratic shocks are not. Information shared via a trade association is that which is relevant to the production process of all firms, e.g. input costs, and firms prefer to maintain private information about idiosyncratic costs.

When firms share their signals about their common cost component they will have the same expectation about this parameter. This simplifies the two stage competition model to a variation of the single cost component models in Mailath [1989] and Jeitschko et al. [2018]. While there are still two cost components, the informational structure is simplified so that firms posses private information about their only idiosyncratic cost components; the remaining uncertainty regarding the common cost component is common to both firms. While the optimality conditions look similar in this setting, the equilibrium pricing strategies in the first period fully reveal the private information of each firm. We briefly outline the significant differences from the previous section.

In the second period the information that is available to each firm now includes $s_\rho = (s_{i,\rho}, s_{j,\rho})$. The new first order conditions are given in Lemma 5.

Lemma 5. *Firm i's optimal second period price is*

$$p^c_{i,2} = \frac{1}{2b}\left(a + bc_i + e\mathbb{E}\left[p^c_{j,2}\middle|\, \rho, p_1, s_\rho\right]\right).$$

Firm i's optimal first period price is

$$\begin{aligned} p^c_{i,1} = &\left(\frac{1}{2b}\right)\mathbb{E}\left[bc_i + a + ep^c_{j,1}\middle|\, s_\rho, s_{i,\theta}\right] \\ &+ e\left(\frac{1}{2b}\right)^2 \mathbb{E}\left[\left(a - bc_i + e\mathbb{E}\left[p^c_{j,2}\middle|\, \rho, s_\rho, p_1\right]\right)\frac{\partial}{\partial p_{i,1}}\mathbb{E}\left[p^c_{j,2}\middle|\, \rho, s_\rho, p_1\right]\middle|\, s_\rho, s_{i,\theta}\right]. \end{aligned}$$

In a linear equilibrium, the first period price is $p^c_{i,1} = p_{0,c} + p_{\theta,c}E[\theta_i|s_{i,\theta}] + p_{\rho,c}E[\rho|s_\rho]$. Because s_ρ and $p_{i,1}$ are publicly observable, then in equilibrium, the value of $s_{i,\theta}$ can be inferred by competing firms. Therefore the expectation of each firm's cost in the second period given publicly available information is $\mathbb{E}[c_i|\rho, s_\rho, p_{i,1}] = \rho + \mathbb{E}[\theta_i|s_{i,\theta}]$, where $s_{i,\theta}$ can be determined from the first period price. Moreover, an increase in the first period price will increase this expectation by the inverse of the equilibrium coefficient $p_{\theta,c}$. The effect of firm i's first period price on firm j's second period price takes into account this informational parameter as well as effects on demand,

$$\frac{\partial}{\partial p_{i,1}}\mathbb{E}\left[p^c_{j,2}\middle|\, \rho, p_{i,1}\right] = \frac{be}{4b^2 - e^2}\kappa^c, \text{ where } \kappa^c \equiv \frac{\partial}{\partial p_{i,1}}\mathbb{E}\left[c_i\middle|\, \rho, s_\rho, p_{i,1}\right] = \frac{1}{p_{\theta,c}}$$

In the unique linear equilibrium, $p_{\theta,c}$ is strictly less than p_θ. Therefore firms use less idiosyncratic information in their first period price choice once they have shared common cost information.

Proposition 3. *In the unique equilibrium in linear pricing strategies the coefficient on idiosyncratic information is less than the corresponding coefficient in the equilibrium without information sharing:*

$$p_{\theta,c} = \frac{1-\beta}{2} \leq p_\theta.$$

This inequality is strict when $e \neq 0$.

We can similarly compare the informativeness of first period price about the underlying cost before and after the firms share information on their common cost component. Because

$p_{\theta,c} \leq p_\theta$,

$$\beta\kappa = \frac{1-2p_\theta}{p_\theta} \leq \frac{1-2p_{\theta,c}}{p_{\theta,c}} = \frac{\beta}{p_{\theta,c}} = \beta\kappa^c \quad \Longleftrightarrow \quad \kappa \leq \kappa^c.$$

Following from Proposition A, this inequality is strict when $e \neq 0$.

Once firms have shared common cost information, firms' second period prices are more responsive to the price choices in the first period. In this setting, it is easier to soften future competition and therefore firms have a greater incentive to choose a higher first period price. The increase in expected price imposes a first order negative effect on consumer welfare in the market.

Theorem 2. *Expected first period prices are higher when firms share signals about common cost information,* $\mathbb{E}[p_{i,1}] \leq \mathbb{E}[p^c_{i,1}]$. *Moreover, expected second period prices are the same regardless of firms sharing information or not.*

From the first order conditions in each case, the only differences in determining the optimal price is the expected price by the competing firm in the second period and the rate at which a first period price increase by a firm affects this second period price by the competitor. The rate of increase is higher when firms share common cost information. The expectation of second period prices does not depend on information sharing. The best response function of each firm is linear in the beliefs about the competing firms costs, and on average, the beliefs must be correct in equilibrium.

4 Welfare and profits

Given the equilibrium characterizations of the two information regimes, we turn to the welfare impact of sharing industry relevant cost information via a trade association. Lemmas 6 and 7 give the consumer and producer surplus, derived from the demand structure introduced in Section 2 and the associated utility of the representative consumer[12]

$$\mathbb{E}\left[u\left(\mathbf{q};\mathbf{p}\right)\right] = \frac{a}{b-e}(q_i+q_j) - \frac{1}{2}\left(\frac{b}{b^2-e^2}\right)(q_i^2+q_j^2) - \left(\frac{e}{b^2-e^2}\right)q_iq_j - (p_iq_i+p_jq_j). \quad (1)$$

Lemma 6. *Expected consumer surplus in each period of competition is represented by the following utility function.*

$$\mathbb{E}\left[u\left(\mathbf{p}\right)\right] = \left(-2a+(b-e)\mathbb{E}[p_i]\right)\mathbb{E}[p_i] + b\,\mathrm{Var}(p_i) - e\,\mathrm{Cov}(p_i,p_j).$$

[12]It is assumed that consumer utility is quadratic in consumption and linear in payments.

When expected demand is positive in the symmetric equilibrium, then $a > (b-e)\mathbb{E}[p_i]$. It follows that consumer surplus decreases as $\mathbb{E}[p_i]$ and $\mathrm{Cov}(p_i, p_j)$ increase and increases with $\mathrm{Var}(p_i)$.

Lemma 7. *Expected producer surplus in each period t of competition is given by*

$$\begin{aligned}\mathbb{E}\left[\Pi_t\right] = 2\left[\left(a-(b-e)\,\mathbb{E}\left[p_{it}\right]\right)\left(\mathbb{E}\left[p_{it}\right]-\mathbb{E}\left[c_i\right]\right)+b\left(\mathrm{Cov}\left(c_i,p_{it}\right)-\mathrm{Var}\left(p_{it}\right)\right)\right.\\ \left.-e\left(\mathrm{Cov}\left(c_i,p_{jt}\right)-\mathrm{Cov}\left(p_i,p_{jt}\right)\right)\right].\end{aligned} \tag{2}$$

The producer surplus increases with expected price. From Proposition **??**, when firms share industry cost information expected prices increase in the first period and are the same in the second period. Given that the demand parameter a is large relative to b, and therefore e, i.e. the demand for each product is relatively inelastic, the change in expected price will dominate all other welfare effects from information sharing.

Proposition 4. *For a large relative to b and e, sharing common cost information will increase expected producer surplus and decrease expected consumer surplus.*

4.1 Extension to n firms

Proposition 4 shows that expected producer surplus increases and expected consumer surplus falls when information is shared, provided $b \gg |e|$. A more general comparison is hampered by the fact that price coefficients are not monotone in precision $\overline{\tau}_\rho$. It can be shown that equilibrium inference κ is not monotone in precision, making it difficult to directly apply standard methods from comparative statics.

To obtain sharp predictions for surplus we therefore turn to the n-firm analogue of our basic model. In this model there are n firms, and firm i's demand is

$$q_n^i\left(p_i,p_{-i}\right)=\frac{1}{n-1}\left(a-bp_i+\frac{e}{n-1}\sum_{j\neq i}p_j\right). \tag{3}$$

When $n=2$ the normalization terms $n-1$ are identically 1; this returns our base model, $q_{2i}\equiv q_i$. All other assumptions from the base model — for example, conditionally independent signals of ρ — are maintained. This demand function is generated by consumer utility

$$\begin{aligned}u\left(\mathbf{q};\mathbf{p}\right)=\frac{a}{b-e}\sum_{i=1}^{n}q_i-\frac{n-1}{2}\left(\frac{(n-1)\,b-(n-2)\,e}{\left((n-1)\,b+e\right)(b-e)}\right)\sum_{i=1}^{n}q_i^2\\ -\frac{n-1}{2}\left(\frac{e}{\left((n-1)\,b+e\right)(b-e)}\right)\sum_{i=1}^{n}\sum_{j\neq i}q_iq_j-\sum_{i=1}^{n}p_iq_i.\end{aligned} \tag{4}$$

We maintain our focus on equilibria in linear pricing strategies. Unlike the base model we do not establish uniqueness. The form of equilibrium pricing coefficients depends in a natural way on the number of firms. We therefore view the n-firm extension as providing a natural approximation to results in our base model.

The linear equilibrium analysis of the n-firm extension is not substantially different from that of the base case, and we omit most of the basic calculations.

Theorem 3. *In the linear equilibrium of the n-firm model,*

$$p_{i1}\left(s_{i\theta}, s_{i\rho}\right) = p_{0n} + p_{\theta n}\mathbb{E}\left[\theta_i | \, s_{i\theta}\right] + p_{\rho n}\mathbb{E}\left[\rho | \, s_{i\rho}\right],$$

where

$$p_{\theta n} = \frac{1}{2+\beta_n\kappa}, \quad p_{\rho n} = \frac{b - \left(\frac{b-e}{2b-e}\right)\beta_n\kappa}{2b - e\overline{\tau}_\rho - \frac{1}{2}\left(1-\overline{\tau}_\rho\right)\beta_n^2\kappa^2}, \quad \text{and } \beta_n = \frac{e^2}{\left(2b-e\right)\left(2\left(n-1\right)b+e\right)}.$$

An indirect implication of Theorem 3 is that equilibrium inference from prices does not substantively change in the extension to n firms. That is, what firm $j \neq i$ can learn about θ_i from p_{i1} depends only on κ, which retains the same form as in the base case. It is not the case that κ is identical in the base case and the n-firm extension, but κ depends in the same way on p_θ and p_ρ regardless of the number of firms. Intuitively this is straightforward: once ρ is known, firm j can separate inferences about firm i's initial information from inferences about firm k's initial information. Because signals of ρ are conditionally uncorrelated, nothing learned about firm k can affect what is learned about firm i. If the independence assumption were relaxed, or if ρ were not made public in the second period, this would no longer be the case.

The linear equilibrium of the n-firm extension retains the cross-dependency of price coefficients and inference. We now turn to the limiting case where n becomes large to obtain comparative statics on producer and consumer surplus.

4.2 Welfare for n large

Even when the number of firms is large, inference about any particular firm remains relatively stable. How information affects prices when markets are large depends mostly on the incentive of any one firm to hide information from its opponents. As it turns out, in the limit no firm faces any incentive to obfuscate its private information. This is intuitive, as when the market is large any firm is one of many; since individual complementarities (or substitutabilities) $e/(n-1)$ are going to zero as the market becomes large, exposing private information does not dramatically affect opponent pricing incentives. This is true even as aggregate complementarities $\sum_{j\neq i} e/(n-1) = e$ are held constant.[13]

Recalling that equilibrium inference κ is bounded (between 0 and 3), the effect of firm i's revelation, $\beta_n \kappa$, goes to 0 as n becomes large, since

$$\lim_{n\nearrow\infty} \beta_n = \lim_{n\nearrow\infty} \frac{e^2}{(2b-e)\left(2\left(n-1\right)b+e\right)} = 0.$$

This implies a simple analytic form for equilibrium prices with a large number of firms.

Theorem 4. *In the linear equilibrium of the large-n extension, equilibrium prices are*

$$p_{i1}\left(s_{i\theta}, s_{i\rho}\right) = p_{0\infty} + p_{\theta\infty}\mathbb{E}\left[\theta_i \middle| s_{i\theta}\right] + p_{\rho\infty}\mathbb{E}\left[\rho \middle| s_{i\rho}\right].$$

Letting $r = e/b$,

$$p_{0\infty} = \frac{1}{2-r}\left(\frac{a}{b} + \frac{1}{2} r\mu_\theta + \mu_\rho\right) - \frac{\mu_\rho}{2-r\overline{\tau}_\rho}, \quad p_{\theta\infty} = \frac{1}{2}, \quad \textit{and} \quad p_{\rho\infty} = \frac{1}{2-r\overline{\tau}_\rho}.$$

The lack of incentives to hide information is immediate in Theorem 4. $p_{\theta\infty} = 1/2$ is exactly the dependence of price on private cost information in the equivalent monopoly problem. $p_{\rho\infty}$ is similar to the dependence of price on commonly known costs in a standard duopoly problem. This dependence is adjusted by $\overline{\tau}_\rho$ to account for the fact that firm i's beliefs about firm j's beliefs are a reversion to the mean of firm i's individual beliefs. That is, if firm i believes $\mathbb{E}[\rho|s_{i\rho}] = \rho_i < \mu_\rho$, firm i believes that firm j believes $\mathbb{E}[\rho|s_j] \in (\rho_i, \mu_\rho)$.

Additionally, in the linear equilibrium with a large number of firms expected first-period price is independent of $\overline{\tau}_\rho$. While $p_{\rho\infty}$ depends on $\overline{\tau}_\rho$, in expectation this is exactly offset by the $\mu_\rho p_{\rho\infty}$ term in $p_{0\infty}$. Then to the extent that information sharing (an increase in $\overline{\tau}_\rho$) alters producer or consumer surplus, it is through either $p_{0\infty}$, $p_{\rho\infty}$, variance or covariance.

Proposition 5. *There exist constants $C_u, C_\pi \in \mathbb{R}$ such that for any $\overline{\tau}_\rho$, first-period consumer and producer surplus in a linear equilibrium with a large number firms are given by*

$$\begin{aligned}
\mathbb{E}\left[u_{1\infty}\right] &\propto (1-r)\operatorname{Var}\left(p^\star_{i1}\right) - r\operatorname{Cov}\left(p^\star_{i1}, p^\star_{j1}\right) + C_u, \\
\mathbb{E}\left[\Pi_{1\infty}\right] &\propto \left(\operatorname{Cov}\left(c_i, p^\star_{i1}\right) - \operatorname{Var}\left(p^\star_{i1}\right)\right) - r\left(\operatorname{Cov}\left(c_i, p^\star_{j1}\right) - \operatorname{Cov}\left(p^\star_{i1}, p^\star_{j1}\right)\right) + C_\pi.
\end{aligned}$$

where i, j are any firms such that $i \neq j$.

[13] In the limit there is the additional question of whether the aggregation of these opponent incentives results in a strictly positive effect on the firm's incentives to hide information. Our results answer this in the negative.

The consumer surplus expression in Proposition 5 confirms that expected consumer surplus is increasing in the variance of first-period prices, and decreasing in covariance when goods are substitutes and increasing in covariance when goods are complements. Producer surplus is impacted by variance and covariance in a similar way. Neither consumer surplus nor producer surplus depend directly on expected price simplifying the welfare analysis of information sharing. Therefore, analysis of the effect of information sharing on surplus will depend on the relative change in variance and covariance.

When information is shared in the first period, this is equivalent to firms having common knowledge of ρ prior to setting first period prices. Firms will then choose prices as in Theorem 4 where $\mathbb{E}[\rho|s_\rho] = \rho$ and $\bar{\tau}_\rho = 1$. Comparison of surplus in the first period depends on how the variance and covariance of prices are affected by these changes in the first-period strategy.

Second-period strategies, and therefore second period surplus, are unaffected by information sharing. Common costs, ρ, are public knowledge in the second period, allowing potential inference of opponent information from first-period prices. However pricing strategies depend only on expected opponent prices, and second-period equilibrium is symmetric and linear. This reduces to a linear strategy on expected opponent costs. With a large number of firms the law of large numbers applies, and the sum of expected opponent costs is equivalent to an average opponent cost. This is independent of whether or not information is shared.

Corollaries 1 and 2 summarize the welfare impact of sharing information when the number of firms is large. Information sharing never harms producer surplus and in almost all cases strictly increases it.

Corollary 1. *Producer surplus is increasing in precision $\overline{\tau}_\rho$, and therefore information sharing strictly improves producer surplus whenever $\overline{\tau}_\rho \in (0, 1)$.*

Information sharing improves consumer surplus for a wide range of informational and substitutability parameters, only reducing this surplus when goods are relatively substitutable and individual signal precision prior to sharing is relatively high. Specifically, when $r \lesssim 0.372$ information sharing always improves consumer surplus. For higher values of r surplus may increase when information is relatively dispersed prior to sharing.

Corollary 2. *When goods are complements, consumer surplus is increasing in precision $\overline{\tau}_\rho$. When goods are substitutes, consumer surplus is increasing in precision $\overline{\tau}_\rho$ when $r \lesssim 0.372$ and is single-peaked in precision otherwise.*

5 Value of information

In the previous sections we took the precision of information as exogenous. Treating the precision as a choice variable, we now consider the value of this information to the firms. Specifically, we examine the effect of an (unobserved) marginal increase in precision of cost information on the firm's profits over the two periods of competition.

The precision of cost information allows firms to make better pricing decisions directly affecting expected profits in the first period. However, marginal deviations from equilibrium will not affect the linear pricing coefficients of either firm. By the envelope theorem, marginal changes in precision will not change the optimal pricing strategy of that firm. Additionally, the other firm cannot change their pricing strategy based on an unobserved deviation. Therefore, increases in precision will affect profits only through the variance and covariance terms in equation (2).

Lemma 8. *The marginal changes in first period profits with respect to the information precision on each cost component are*

$$\frac{\partial}{\partial \tau_{i,\theta}} \mathbb{E}\left[\pi_{i,1}\right] = \left(\frac{(1-p_{i,\theta})\, p_{i,\theta}}{\left(\tau_{i,\theta}+\tau_{\theta}\right)^2} \right) b \text{ and } \frac{\partial}{\partial \tau_{i,\rho}} \mathbb{E}\left[\pi_{i,1}\right] = \left(\frac{(1-p_{i,\rho})\, p_{i,\rho}}{\left(\tau_{i,\rho}+\tau_{\rho}\right)^2} \right) (b - \bar{\tau}_{j,\rho} e).$$

First period profits respond to precision of the two components of marginal cost in a similar way with respect to own demand (the terms postmultiplied by b), but the response differs with respect to cross-firm demand (the term postmultiplied by e). To a first approximation, the precision of the informational signals affects the opponent's payoffs only through information on the common cost component; increasing this precision will increase the correlation in first-period prices, and decreasing this precision will reduce the correlation in first-period prices.

A marginal increase in precision does not impact the expected profit in the second period of competition.[14] Because the firm perfectly observes its cost components after the first period, increased precision about these parameters has no impact on the information the firm has in the second period. Since neither firm changes their pricing strategies, inferences about the opposing firms cost structure given the first period price does not change. Lastly, because the increase in precision is not observed by the other firm, the inference its makes will also be unaffected. Therefore neither firm's pricing strategy in the second stage will be affected.

[14] Note that while marginal deviations in investment in precision cannot affect second-period profits, different levels of believed investment will typically generate different second-period profits.

5.1 Sharing industry relevant information

In the case of sharing industry cost information through a trade association, we again find the effect of an increase in information precision on firms' profits. Because all information acquired about the common cost component is shared, the two stages of competition will follow as in 3.3. In this setting, pricing strategies in either stage do not depend on information precision and changes in precision does not change the information available to firms in the second stage. Therefore an increase in precision of either the common or idiosyncratic cost component will only affect expected profits in the first stage.

Lemma 9. *The marginal changes in first period profits with respect to precision on each cost component when firms are sharing information are*

$$\frac{\partial}{\partial \tau_{i,\theta}} \mathbb{E}\left[\pi_{i,1}^{c}\right] = \left(\frac{\left(1-p_{i\theta,c}\right) p_{i\theta,c}}{\left(\tau_{i,\theta}+\tau_{\theta}\right)^{2}}\right) b \text{ and } \frac{\partial}{\partial \tau_{i,\rho}} \mathbb{E}\left[\pi_{i,1}^{c}\right] = \left(\frac{\left(1-p_{i\rho,c}\right) p_{i\rho,c}}{\left(\tau_{i,\rho}+\tau_{j,\rho}+\tau_{\rho}\right)^{2}}\right)(b-e).$$

Comparing the marginal benefit information precision in the private cost component between the two information regimes it is clear that the value of information depends on how much this information is used in the first period of competition. The incentive to signal jam when sharing information via a trade association reduces the extent to which firms use information about private costs which reduces the value of acquiring this information.

Proposition 6. *The value of increased precision in information in private costs is lower when firms share industry relevant information,*

$$\frac{\partial}{\partial \tau_{i,\theta}} \mathbb{E}\left[\pi_{i,1}\right] > \frac{\partial}{\partial \tau_{i,\theta}} \mathbb{E}\left[\pi_{i,1}^{c}\right].$$

Proof. The result follows directly from Lemmas 8 and 9 and Proposition A which implies $p_{i,\theta}^{c} < p_{i,\theta}^{*} < 1/2$. □

In a market with a large number of firms, $p_{n\theta}^{c} = p_{n\theta}^{*} = 1/2$. From Proposition 5, increased precision of private cost information only impacts $\mathrm{Var}(\mathrm{p_{i1}})$ and $\mathrm{Cov}(\mathrm{c_i}, \mathrm{p_{i1}})$, as in the case of two firms. Therefore, the value of increased precision of private cost information does not depend on information sharing on the common cost component in large markets.

6 Conclusion

One of the important roles of an industry association is to aggregate industry relevant information to share with the firms in the industry. To analyze the competitive impact of sharing cost information that is pertinent to all firms in the industry we first need to understand how the information is used and inferred without a sharing agreement. To this end, we study a dynamic pricing competition model that allows for uncertainty in common cost and private cost parameters. We characterize the symmetric linear equilibrium of this model and use it to examine how information sharing affects competition and welfare within the industry.

In the setting with two firms, information sharing incentivizes firms to further soften competition which leads to higher prices on average and reduces the value of acquiring firm specific cost information prior to competition. In settings where demand is relatively inelastic, information sharing can reduce consumer surplus while increasing producer surplus. As the number of firms in the market increases, the effect of competition softening is reduced; in particular, as the number of firms in the market becomes arbitrarily large it vanishes. In a market with a large number of firms, information sharing no longer has an impact on expected prices and under a wide range of market characteristics can lead to both higher producer surplus and consumer surplus.

Information sharing of industry relevant costs leading to higher producer surplus indicates that agreements to share this information may not stem from purely collusive motives. In fact, in cases where there are many firms and industry relevant information is dispersed among the firms sharing is likely to improve welfare. However, these agreements can still be a concern for competition in concentrated markets where the agreements can lead to higher and more coordinated prices even in the absence of an explicit or implicit collusive agreement.

References

George-Marios Angeletos and Alessandro Pavan. Efficient use of information and social value of information. *Econometrica*, 75(4):1103–1142, 2007.

Dirk Bergemann and Stephen Morris. Robust predictions in games with incomplete information. *Econometrica*, 81(4):1251–1308, 2013.

Dirk Bergemann, Alessandro Bonatti, et al. Markets for information: An introduction. *Cowles Foundation for Research in Economics Discussion paper*, 2018.

Alessandro Bonatti, Gonzalo Cisternas, and Juuso Toikka. Dynamic oligopoly with incomplete information. *The Review of Economic Studies*, 84(2):503–546, 2017.

Luca Colombo, Gianluca Femminis, and Alessandro Pavan. Information acquisition and welfare. *The Review of Economic Studies*, 81(4):1438–1483, 2014.

Thomas D Jeitschko, Ting Liu, and Tao Wang. Information acquisition, signaling and learning in duopoly. *International Journal of Industial Organization*, 2018.

Kai-Uwe Kühn. Fighting collusion by regulating communication between firms. *Economic Policy*, 16(32):168–204, 2001.

Kai-Uwe Kühn and Xavier Vives. Information exchanges among firms and their impact on competition. *manuscript, Institut d'Analisi Economica, Barcelona*, 1994.

George J Mailath. Simultaneous signaling in an oligopoly model. *The Quarterly Journal of Economics*, 104(2):417–427, 1989.

Loretta J Mester. Perpetual signalling with imperfectly correlated costs. *The Rand Journal of Economics*, pages 548–563, 1992.

Leonard J Mirman, Larry Samuelson, and Amparo Urbano. Duopoly signal jamming. *Economic Theory*, 3(1):129–149, 1993.

Leonard J Mirman, Larry Samuelson, and Edward E Schlee. Strategic information manipulation in duopolies. *Journal of Economic Theory*, 62(2):363–384, 1994.

Stephen Morris and Hyun Song Shin. Social value of public information. *American Economic Review*, 92(5):1521–1534, 2002.

David P Myatt and Chris Wallace. Endogenous information acquisition in coordination games. *The Review of Economic Studies*, 79(1):340–374, 2011.

David P Myatt and Chris Wallace. Cournot competition and the social value of information. *Journal of Economic Theory*, 158:466–506, 2015a.

David P Myatt and Chris Wallace. Information use and acquisition in price-setting oligopolies. *The Economic Journal*, 2015b.

Michael Raith. A general model of information sharing in oligopoly. *Journal of economic theory*, 71(1):260–288, 1996.

Xavier Vives. *Oligopoly pricing: old ideas and new tools*. MIT press, 2001.

A Proofs for Section 3

Proof of Lemma 1. This follow's directly from firm i's first-order condition with respect to second-period price,

$$(a - bp_{i2}^\star) + \int exdF^j(x)\,dx - (p_{i2}^\star - c_i)\,b = 0 \quad \Longrightarrow \quad p_{i2}^\star = \frac{1}{2b}\left(a + bc_i + e\mathbb{E}\left[p_{j2}^\star \middle| \rho, \mathbf{p}_1\right]\right).$$

Substituting in to the firm's profit function yields the expression in Lemma 1. □

Proof of Lemma 2. This follows from Lemmas 16 and 17, derived for the model with n firms. We give a proof for the two-firm case below.

Because firm i's optimal second-period price, given in Lemma 1, holds given any information set, it also holds in expectation. That is,

$$\mathbb{E}\left[p_{i2}^\star \middle| \rho, \mathbf{p}_1\right] = \frac{1}{2b}\mathbb{E}\left[a + bc_i + ep_{j2}^\star \middle| \rho, \mathbf{p}_1\right].$$

Then

$$2b\mathbb{E}\left[p_{i2}^\star \middle| \rho, \mathbf{p}_1\right] - e\mathbb{E}\left[p_{j2}^\star \middle| \rho, \mathbf{p}_1\right] = a + b\mathbb{E}\left[c_i \middle| \rho, \mathbf{p}_1\right].$$

This gives two equations, one each for firm i and firm j, in two unknowns, $\mathbb{E}[p_{i2}^\star|\rho, \mathbf{p}_1]$ and $\mathbb{E}[p_{j2}^\star|\rho, \mathbf{p}_1]$. Algebraic rearrangement yields the first expression in Lemma 2, and substituting in to the firm's profit function yields the second. □

Proof of Lemma 3. This follows from standard monopoly profit maximization and application Lemma 1 to firm i's second-period profits,

$$\begin{aligned}&\max_p \mathbb{E}\left[\pi_{i1} \middle| s_{i\rho}, s_{i\theta}\right] + \mathbb{E}\left[\pi_{i2}^\star \middle| s_{i\rho}, s_{i\theta}\right] \\ &= \max_p \mathbb{E}\left[\left(a - bp + ep_{j1}^\star\right)(p - c_i) + \frac{1}{4b}\left(a - bc_i + e\mathbb{E}\left[p_{j2}^\star \middle| \rho, \mathbf{p}_1\right]\right)^2 \middle| s_{i\rho}, s_{i\theta}\right].\end{aligned}$$

Note that second-period profits depend on p_{i1} only through $\mathbf{p}_1$'s effect on firm j's beliefs. Without substituting in with the expression in Lemma 1 we could have obtained a similar reduction by applying the envelope theorem (firm i's second-period price is optimal, conditional on its first-period price). Firm i's first order condition is

$$0 = \mathbb{E}\left[\left(a + bc_i + ep_{j1}^\star\right) - 2bp + \frac{e}{2b}\left(a - bc_i + e\mathbb{E}\left[p_{j2}^\star \middle| \rho, \mathbf{p}_1\right]\right)\frac{\partial}{\partial p_{i1}}\mathbb{E}\left[p_{j2}^\star \middle| \rho, \mathbf{p}_1\right] \middle| s_{i\rho}, s_{i\theta}\right].$$

Rearrangement gives the desired result. □

Lemma 10. *Expected costs conditional on second-period information are*

$$\mathbb{E}\left[c_i|\,\rho,\mathbf{p}_1\right]=(\mu_\theta+\mu_\rho)+(1-\kappa_i\overline{\tau}_\rho p_{i\rho})(\rho-\mu_\rho)+\kappa_i\left(p_{i1}-(p_{i0}+p_{i\theta}\mu_\theta+p_{i\rho}\mu_\rho)\right),$$

$$\textit{subject to } \kappa_i=\frac{\overline{\tau}_\theta\sigma_\theta^2 p_{i\theta}}{\overline{\tau}_\theta\sigma_\theta^2 p_{i\theta}^2+(1-\overline{\tau}_\rho)\,\overline{\tau}_\rho\sigma_\rho^2 p_{i\rho}^2}.$$

Proof. Note that, conditional on ρ, p_{j1} conveys no information about c_i. Then consider the joint distribution of c_i, p_{i1}, and ρ. Let $\tau_x=1/\sigma_x^2$ be the precision of the random variable x, and let $\overline{\tau}_x=\tau_{\varepsilon_x}/(\tau_x+\tau_{\varepsilon_x})$ be the relative precision of the signal s_x, $x\in\{\theta_i,\theta_j,\rho\}$. Under a linear pricing strategy,

$$\begin{aligned}p_{i1}&=p_{i0}+p_{i\theta}\mathbb{E}\left[\theta_i|\,s_{i\theta}\right]+p_{i\rho}\mathbb{E}\left[\rho|\,s_{i\rho}\right]\\&=\left(p_{i0}+(1-\overline{\tau}_\theta)\,p_{i\theta}\mu_\theta+(1-\overline{\tau}_\rho)\,p_{i\rho}\mu_\rho\right)+p_{i\theta}\overline{\tau}_\theta s_{i\theta}+p_{i\rho}\overline{\tau}_\rho s_{i\rho}.\end{aligned}$$

Then c_i, ρ, and p_{i1} are jointly normal,

$$(c_i,\rho,p_{i1})^T\sim N\left(\left(\begin{array}{c}\mu_\theta+\mu_\rho\\ \mu_\rho\\ \mathbb{E}\left[p_{i1}\right]\end{array}\right),\left(\begin{array}{ccc}\sigma_\theta^2+\sigma_\rho^2 & \sigma_\rho^2 & \overline{\tau}_\theta\sigma_\theta^2 p_{i\theta}+\overline{\tau}_\rho\sigma_\rho^2 p_{i\rho}\\ \sigma_\rho^2 & \sigma_\rho^2 & \overline{\tau}_\rho\sigma_\rho^2 p_{i\rho}\\ \overline{\tau}_\theta\sigma_\theta^2 p_{i\theta}+\overline{\tau}_\rho\sigma_\rho^2 p_{i\rho} & \overline{\tau}_\rho\sigma_\rho^2 p_{i\rho} & \overline{\tau}_\theta\sigma_\theta^2 p_{i\theta}^2+\overline{\tau}_\rho\sigma_\rho^2 p_{i\rho}^2\end{array}\right)\right).$$

Then the conditional expectation of c_i, given ρ and p_{i1}, is

$$\begin{aligned}&\mathbb{E}\left[c_i|\,\rho,p_{i1}\right]=(\mu_\theta+\mu_\rho)+\Sigma_{12}\Sigma_{22}^{-1}\left((\rho,p_{i1})^T-(\mu_\rho,\mathbb{E}\left[p_{i1}\right])^T\right),\\&\Sigma_{12}=\left(\sigma_\rho^2,\overline{\tau}_\theta\sigma_\theta^2 p_{i\theta}+\overline{\tau}_\rho\sigma_\rho^2 p_{i\rho}\right),\quad \Sigma_{22}=\left(\begin{array}{cc}\sigma_\rho^2 & \overline{\tau}_\rho\sigma_\rho^2 p_{i\rho}\\ \overline{\tau}_\rho\sigma_\rho^2 p_{i\rho} & \overline{\tau}_\theta\sigma_\theta^2 p_{i\theta}^2+\overline{\tau}_\rho\sigma_\rho^2 p_{i\rho}^2\end{array}\right).\end{aligned}$$

Write the matrix product as $\Sigma_{12}\Sigma_{22}^{-1}=(m_{i1},m_{i2})$. Then

$$\begin{aligned}m_{i1}&=\frac{1}{\left(\overline{\tau}_\theta\sigma_\theta^2 p_{i\theta}^2+\overline{\tau}_\rho\sigma_\rho^2 p_{i\rho}^2\right)\sigma_\rho^2-\overline{\tau}_\rho^2\sigma_\rho^4 p_{i\rho}^2}\left(\overline{\tau}_\theta\sigma_\theta^2\sigma_\rho^2 p_{i\theta}^2+\overline{\tau}_\rho\sigma_\rho^4 p_{i\rho}^2-\overline{\tau}_\theta\overline{\tau}_\rho\sigma_\theta^2\sigma_\rho^2 p_{i\theta}p_{i\rho}-\overline{\tau}_\rho^2\sigma_\rho^4 p_{i\rho}^2\right)\\&=\frac{\overline{\tau}_\theta\sigma_\theta^2 p_{i\theta}^2+\overline{\tau}_\rho\sigma_\rho^2 p_{i\rho}^2-\overline{\tau}_\theta\overline{\tau}_\rho\sigma_\theta^2 p_{i\theta}p_{i\rho}-\overline{\tau}_\rho^2\sigma_\rho^2 p_{i\rho}^2}{\left(\overline{\tau}_\theta\sigma_\theta^2 p_{i\theta}^2+\overline{\tau}_\rho\sigma_\rho^2 p_{i\rho}^2\right)-\overline{\tau}_\rho^2\sigma_\rho^2 p_{i\rho}^2}=1-\kappa_i\overline{\tau}_\rho p_{i\rho};\\m_{i2}&=\frac{1}{\left(\overline{\tau}_\theta\sigma_\theta^2 p_{i\theta}^2+\overline{\tau}_\rho\sigma_\rho^2 p_{i\rho}^2\right)\sigma_\rho^2-\overline{\tau}_\rho^2\sigma_\rho^4 p_{i\rho}^2}\left(-\overline{\tau}_\rho\sigma_\rho^4 p_{i\rho}+\overline{\tau}_\theta\sigma_\theta^2\sigma_\rho^2 p_{i\theta}+\overline{\tau}_\rho\sigma_\rho^4 p_{i\rho}\right)\\&=\frac{\overline{\tau}_\theta\sigma_\theta^2 p_{i\theta}}{\overline{\tau}_\theta\sigma_\theta^2 p_{i\theta}^2+(1-\overline{\tau}_\rho)\,\overline{\tau}_\rho\sigma_\rho^2 p_{i\rho}^2}=\kappa_i.\end{aligned}$$

The result is then immediate. □

Proof of Lemma 4. This follows immediately from Lemma 10. □

Lemma 11. *There is an equilibrium in symmetric linear pricing strategies, where*

$$\kappa = \frac{\sigma_\theta^2 \overline{\tau}_\theta p_\theta}{(1-\overline{\tau}_\rho)\,\overline{\tau}_\rho \sigma_\rho^2 p_\rho^2 + \overline{\tau}_\theta \sigma_\theta^2 p_\theta},$$
$$\textit{subject to } p_\theta = \frac{1}{2+\beta\kappa} \textit{ and } p_\rho = \frac{1-\left(\frac{1-r}{2-r}\right)\beta\kappa}{2 - r\overline{\tau}_\rho - \frac{1}{2}(1-\overline{\tau}_\rho)\,\beta^2\kappa^2},$$

where $r = e/b$ and $\beta = r^2/(4-r^2)$.

Proof. From Lemmas 3 and 4, first-period prices are given by

$$4bp_{i1}^\star = 2\mathbb{E}\left[bc_i + a + ep_{j1}^\star \middle| s_{i\rho}, s_{i\theta}\right] + \mathbb{E}\left[\left(a - bc_i + e\mathbb{E}\left[p_{j2}^\star \middle| \rho, p_{i1}^\star, p_{j1}^\star\right]\right)\beta\kappa_i \middle| s_{i\rho}, s_{i\theta}\right].$$

Lemma 2 gives second-period expected prices,

$$\mathbb{E}\left[p_{j2}^\star \middle| \rho, \mathbf{p}_1\right] = \frac{1}{4b^2-e^2}\left((2b+e)\,a + 2b^2\mathbb{E}\left[c_j \middle| \rho, p_{j1}\right] + be\mathbb{E}\left[c_i \middle| \rho, p_{i1}\right]\right).$$

Following Lemma 10,

$$\begin{aligned}
\mathbb{E}\left[\mathbb{E}\left[c_j \middle| \rho, \mathbf{p}_1\right] \middle| s_{i\theta}, s_{i\rho}\right] &= \mu_\theta + \mathbb{E}\left[\rho \middle| s_{i\rho}\right], \\
\mathbb{E}\left[\mathbb{E}\left[c_i \middle| \rho, \mathbf{p}_1\right] \middle| s_{i\theta}, s_{i\rho}\right] &= \mu_\theta + \mathbb{E}\left[\rho \middle| s_{i\rho}\right] \\
&\quad + \kappa_i p_{i\theta}\left(\mathbb{E}\left[\theta_i \middle| s_i\right] - \mu_\theta\right) + \kappa_i p_{i\rho}(1-\overline{\tau}_\rho)\left(\mathbb{E}\left[\rho \middle| s_{i\rho}\right] - \mu_\rho\right).
\end{aligned}$$

Substituting in gives

$$\begin{aligned}
4bp_{i1}^\star = {}& 2\mathbb{E}\left[bc_i + a + ep_{j1}^\star \middle| s_{i\rho}, s_{i\theta}\right] + \beta\kappa_i\mathbb{E}\left[\left(a - bc_i + \frac{e}{4b^2-e^2}\left((2b+e)\,a\right)\right) \middle| s_{i\rho}, s_{i\theta}\right] \\
&+ \frac{e\beta\kappa_i}{4b^2-e^2}\mathbb{E}\left[2b^2(\mu_\theta+\rho) + be\left(\mu_\theta + \rho + \kappa_i p_{i\theta}(\theta_i - \mu_\theta) + \kappa_i p_{i\rho}(1-\overline{\tau}_\rho)(\rho-\mu_\rho)\right) \middle| s_{i\rho}, s_{i\theta}\right].
\end{aligned}$$

Recall that $\mathbb{E}[p_{j1}^\star | s_{i\rho}, s_{i\theta}] = p_{j0} + p_{j\theta}\mu_\theta + p_{j\rho}\overline{\tau}_\rho\mathbb{E}[\rho|s_{i\rho}] + p_{j\rho}(1-\overline{\tau}_\rho)\mu_\rho$. Matching coefficients gives

$$4bp_{i\theta} = 2b - b\beta\kappa_i + \frac{be^2\beta\kappa_i^2 p_{i\theta}}{4b^2-e^2}; \tag{5}$$
$$4bp_{i\rho} = 2b + 2e\overline{\tau}_\rho p_{j\rho} - b\beta\kappa_i + \frac{e\beta\kappa_i}{4b^2-e^2}\left(2b^2 + be\left(1 + (1-\overline{\tau}_\rho)\,\kappa_i p_{i\rho}\right)\right). \tag{6}$$

In a symmetric equilibrium, $p_{i\theta} \equiv p_\theta$, $p_{i\rho} \equiv p_\rho$, and $\kappa_i \equiv \kappa$ for both firms $i \in \{1,2\}$. Then the coefficients in equations (5) and (6) can be solved,

$$\begin{aligned}
p_\theta &= \frac{2-\beta\kappa}{4-\beta^2\kappa^2} = \frac{1}{2+\beta\kappa};\\
p_\rho &= \frac{1}{2}\left(1-\frac{1}{2}\beta\kappa+\frac{1}{r}\beta^2\kappa+\frac{1}{2}\beta^2\kappa\right)\left(1-\frac{1}{2}r\overline{\tau}_\rho-\frac{1}{4}\left(1-\overline{\tau}_\rho\right)\beta^2\kappa^2\right)^{-1}\\
&= \frac{1-\left(\frac{1-r}{2-r}\right)\beta\kappa}{2-r\overline{\tau}_\rho-\frac{1}{2}\left(1-\overline{\tau}_\rho\right)\beta^2\kappa^2}.
\end{aligned}$$

The conditional definition of κ follows from Lemma 4. □

Proof of Theorem 1. The expression for linear price coefficients follows from Lemma 11. Substituting price coefficients into κ and letting $\hat{\kappa} \equiv \beta\kappa$ gives

$$\begin{aligned}
&\underbrace{(2+\hat{\kappa})^2\left(1-\left(\frac{1-r}{2-r}\right)\hat{\kappa}\right)^2\left(1-\overline{\tau}_\rho\right)\overline{\tau}_\rho\sigma_\rho^2\hat{\kappa}}_{\text{LHS}(\hat{\kappa})}\\
&\qquad = \underbrace{\left(2-r\overline{\tau}_\rho-\frac{1}{2}\left(1-\overline{\tau}_\rho\right)\hat{\kappa}^2\right)^2\left(\left(2+\hat{\kappa}\right)\beta-\hat{\kappa}\right)\overline{\tau}_\theta\sigma_\theta^2}_{\text{RHS}(\hat{\kappa})}.
\end{aligned}$$

Note that $\text{LHS}(0) = 0 < \text{RHS}(0)$. Furthermore, we show in Appendix D that $\hat{\kappa} \leq r^2/(2-r^2) \equiv \bar{\kappa}$; then we have $\text{LHS}(\bar{\kappa}) > 0 = \text{RHS}(\bar{\kappa})$. Since both LHS and RHS are continuous in $\hat{\kappa}$, it follows that there is a $\hat{\kappa} \in [0, \bar{\kappa}]$ that solves $\text{LHS}(\hat{\kappa}) = \text{RHS}(\hat{\kappa})$.

It is clear that RHS is decreasing in $\hat{\kappa}$, since $(2+\hat{\kappa})\beta + \hat{\kappa} = 2\beta - (1-\beta)\hat{\kappa}$. We now show that LHS is either increasing, or increasing-then-decreasing and concave; in the latter case, we show also that RHS is convex where LHS is decreasing. Since $\text{LHS}(0) < \text{RHS}(0)$ and $\text{LHS}(\bar{\kappa}) > \text{RHS}(\bar{\kappa})$, this is sufficient to show that there is a unique $\hat{\kappa}$ such that $\text{LHS}(\hat{\kappa}) = \text{RHS}(\hat{\kappa})$.

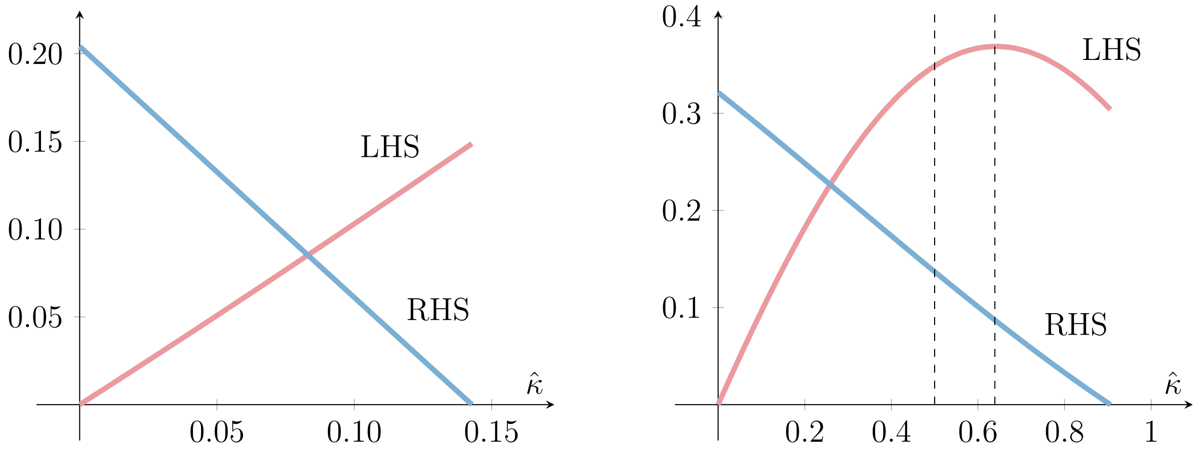

Figure 1: A graphical depiction of the proof of equilibrium existence and uniqueness. The existence of an equilibrium amounts to finding a κ such that $\text{LHS}(\kappa) = \text{RHS}(\kappa)$. Since $\text{LHS}(0) < \text{RHS}(0)$ and $\text{LHS}(\overline{\kappa}) > \text{RHS}(\overline{\kappa})$ and both functions are continuous, such a κ is guaranteed to exist. Additionally, RHS is decreasing. We show that either LHS is increasing (left panel) or increasing and then decreasing (right panel). In the former case, it is clear that there is a unique point of intersection and hence a unique equilibrium. In the latter case, we show that LHS is concave where it is decreasing and RHS is convex anywhere LHS is decreasing. Then $\text{LHS} - \text{RHS}$ is concave, ensuring that equilibrium κ is unique. Dashed lines appear at $\hat{\kappa} = 1/2$ and $\hat{\kappa} = (\sqrt{249} - 3)/20$, the bounds applied in the proof.

To begin, the derivative of LHS is given by

$$
\begin{aligned}
\frac{d\,\text{LHS}}{d\hat{\kappa}} &= 2\left(2+\hat{\kappa}\right)\left(1-\left(\frac{1-r}{2-r}\right)\hat{\kappa}\right)^2\left(1-\overline{\tau}_\rho\right)\overline{\tau}_\rho\sigma_\rho^2\hat{\kappa} \\
&\quad - 2\left(\frac{1-r}{2-r}\right)\left(2+\hat{\kappa}\right)^2\left(1-\left(\frac{1-r}{2-r}\right)\hat{\kappa}\right)\left(1-\overline{\tau}\right)\overline{\tau}\sigma_\rho^2\hat{\kappa} \\
&\quad + \left(2+\hat{\kappa}\right)^2\left(1-\left(\frac{1-r}{2-r}\right)\hat{\kappa}\right)^2\left(1-\overline{\tau}\right)\overline{\tau}\sigma_\rho^2 \\
&\propto \left(2+\hat{\kappa}\right)\left(1-\left(\frac{1-r}{2-r}\right)\hat{\kappa}\right) \\
&\quad \times\left[2\left(1-\left(\frac{1-r}{2-r}\right)\hat{\kappa}\right)\hat{\kappa} - 2\left(\frac{1-r}{2-r}\right)\left(2+\hat{\kappa}\right)\hat{\kappa} + \left(2+\hat{\kappa}\right)\left(1-\left(\frac{1-r}{2-r}\right)\hat{\kappa}\right)\right].
\end{aligned}
$$

The leading terms are positive for $\hat{\kappa} \in [0, \overline{\kappa}]$. The trailing term is

$$
\begin{aligned}
&2\left(1-\left(\frac{1-r}{2-r}\right)\hat{\kappa}\right)\hat{\kappa} - 2\left(\frac{1-r}{2-r}\right)\left(2+\hat{\kappa}\right)\hat{\kappa} + \left(2+\hat{\kappa}\right)\left(1-\left(\frac{1-r}{2-r}\right)\hat{\kappa}\right) \\
&\qquad \propto -5\left(1-r\right)\hat{\kappa}^2 + 3r\hat{\kappa} + 2\left(2-r\right). \qquad (7)
\end{aligned}
$$

This is a negative quadratic in $\hat{\kappa}$, and is strictly positive when $\hat{\kappa} = 0$; thus LHS is either increasing for $\hat{\kappa} \in [0, \bar{\kappa}]$, or it is increasing-then-decreasing on this range. When goods are complements ($r \geq 0$) this is positive for all relevant $\hat{\kappa}$ and the proof is complete. We then focus on the case where goods are substitutes ($r < 0$).

Replacing the leading positive terms in LHS gives

$$\frac{d\,\text{LHS}}{d\hat{\kappa}} \propto \left(2\left(2-r\right) + 3r\hat{\kappa} - 5\left(1-r\right)\hat{\kappa}^2\right)\left(2\left(2-r\right) + r\hat{\kappa} - \left(1-r\right)\hat{\kappa}^2\right).$$

This implies

$$\frac{d^2\,\text{LHS}}{d\hat{\kappa}^2} \propto 8\left(2-r\right)r - 6\left(3r^2 - 12r + 8\right)\hat{\kappa} - 24\left(1-r\right)r\hat{\kappa}^2 + 20\left(1-r\right)^2\hat{\kappa}^3.$$

This is negative at $\hat{\kappa} = 0$ and $\hat{\kappa} = 1 \geq \bar{\kappa}$. Moreover,

$$\frac{d^3\,\text{LHS}}{d\hat{\kappa}^3} \propto -\left(18r^2 - 72r + 48\right) - 48\left(1-r\right)r\hat{\kappa} + 60\left(1-r\right)^2\hat{\kappa}^2.$$

This is a positive quadratic in $\hat{\kappa}$, thus $d^2\,\text{LHS}\,/d\hat{\kappa}^2$ is either decreasing, decreasing-then-increasing, or increasing for $\hat{\kappa} \in [0, \bar{\kappa}]$. Since $d^2\,\text{LHS}\,/d\hat{\kappa}^2 \leq 0$ for $\hat{\kappa} \in \{0, \bar{\kappa}\}$, it follows that $d^2\,\text{LHS}\,/d\hat{\kappa}^2 \leq 0$ for all $\hat{\kappa} \in [0, \bar{\kappa}]$, and LHS is concave.

If LHS is decreasing, it must be that the quadratic in (7) is negative. The zeros of this quadratic are given by

$$\hat{\kappa}_{\pm} \in \frac{3r}{10-10r} \pm \frac{1}{10-10r}\sqrt{9r^2 + 8\left(2-r\right)\left(5-5r\right)} = \frac{1}{10-10r}\left(3r + \sqrt{49r^2 - 120r + 80}\right).$$

LHS is decreasing only if goods are substitutes, $r < 0$, so only the "+" solution is valid. Note that

$$\begin{aligned}\frac{d\hat{\kappa}_+}{dr} &\overset{\text{sign}}{=} \left(3 + \frac{49r - 60}{\sqrt{49r^2 - 120r + 80}}\right)\left(10 - 10r\right) + 10\left(3r + \sqrt{49r^2 - 120r + 80}\right) \\ &\overset{\text{sign}}{=} 20 - 11r + 3\sqrt{49r^2 - 120r + 80} > 0.\end{aligned}$$

Then $\hat{\kappa}_+$ is minimized when $r = -1$ (since $r \in [-1, 1]$ and $d\hat{\kappa}_+/dr < 0$ when $r < 0$). This gives that if LHS is decreasing at $\hat{\kappa}$,

$$\hat{\kappa} \geq \bar{\kappa}_+ = \frac{1}{20}\left(-3 + \sqrt{249}\right) \geq \frac{15-3}{20} > \frac{1}{2}.$$

Finally, we compute the second derivative of RHS with respect to $\hat{\kappa}$ to show that RHS is convex,

$$\begin{aligned}
\frac{d^2\,\mathrm{RHS}}{d\hat{\kappa}^2} &= \frac{d}{d\hat{\kappa}}\Bigg[-2\left(1-\overline{\tau}_\rho\right)\hat{\kappa}\left(2-r\overline{\tau}_\rho-\frac{1}{2}\left(1-\overline{\tau}_\rho\right)\hat{\kappa}^2\right)\left(2\beta-(1-\beta)\,\hat{\kappa}\right) \\
&\qquad\qquad -(1-\beta)\left(2-r\overline{\tau}_\rho-\frac{1}{2}\left(1-\overline{\tau}_\rho\right)\hat{\kappa}^2\right)^2\Bigg] \\
&= 2\left(1-\overline{\tau}_\rho\right)\left(2-r\overline{\tau}_\rho-\frac{1}{2}\left(1-\overline{\tau}_\rho\right)\hat{\kappa}^2\right)\left(4\left(1-\beta\right)\hat{\kappa}-4\beta\right) \\
&\quad + 4\left(1-\overline{\tau}_\rho^2\right)\hat{\kappa}^2\left(2\beta-(1-\beta)\,\hat{\kappa}\right).
\end{aligned}$$

Note that all involved terms are positive for $\hat{\kappa}\in[0,\bar{\kappa}]$, with the potential exception of $4(1-\beta)\hat{\kappa}-\beta$. As shown above, $\hat{\kappa}\geq 1/2$ whenever LHS is decreasing, hence

$$4\left(1-\beta\right)\hat{\kappa}-4\beta\geq 2\left(1-\beta\right)-4\beta=2-6\beta\geq 0. \qquad \text{(since } \beta\leq 1/3\text{)}$$

Then $d^2\,\mathrm{RHS}\,/d\hat{\kappa}^2>0$ when LHS is decreasing. Then where LHS is decreasing it is convex and RHS is concave, implying a unique intersection. □

Proof of Proposition 1. When $e=0$, $\beta=e^2/(4b^2-e^2)=0$. Then $p_\theta=p_\rho=1/2$. Otherwise, we compare

$$\begin{aligned}
p_\theta \gtrless p_\rho \iff&& \frac{1}{2+\beta\kappa} &\gtrless \frac{1-\left(\frac{b-e}{2b-e}\right)\beta\kappa}{2-\frac{e}{b}\overline{\tau}_\rho-\frac{1}{2}\left(1-\overline{\tau}_\rho\right)\beta^2\kappa^2} \\
\iff&& 2-r\overline{\tau}_\rho-\frac{1}{2}\left(1-\overline{\tau}_\rho\right)\beta^2\kappa^2 &\gtrless \left(2+\beta\kappa\right)\left(1-\left(\frac{1-r}{2-r}\right)\beta\kappa\right) \\
\iff&& -r\overline{\tau}_\rho-\frac{1}{2}\left(1-\overline{\tau}_\rho\right)\beta^2\kappa^2 &\gtrless \left(\frac{r}{2-r}\right)\beta\kappa-\left(\frac{1-r}{2-r}\right)\beta^2\kappa^2. \qquad (8)
\end{aligned}$$

When $r>0$ the left-hand side of (8) is maximized when $\overline{\tau}_\rho=0$. This leads to

$$p_\theta<p_\rho \quad\Longleftarrow\quad \left(\frac{1-r}{2-r}-\frac{1}{2}\right)\beta\kappa<\frac{r}{2-r}.$$

The left-hand side is negative and the right-hand side is positive, so $p_\theta<p_\rho$.

When $r<0$ the left-hand side of (8) is minimized when $\overline{\tau}_\rho=0$. This leads to

$$p_\theta>p_\rho \quad\Longleftarrow\quad \left(\frac{1-r}{2-r}-\frac{1}{2}\right)\beta\kappa>\frac{r}{2-r}.$$

The left-hand side is positive and the right-hand side is negative, so $p_\theta>p_\rho$. □

Proof of Proposition 2. The inverse relationship of p_θ and κ follows immediately from the definition $p_\theta = 1/(2+\beta\kappa)$.

The remaining relationships follow from the quintic implicit equation for κ,

$$\underbrace{(2+\beta\kappa)^2 \left(1 - \left(\frac{1-r}{2-r}\right)\beta\kappa\right)^2 \sigma_{s,\rho}^2 \overline{\tau}_\rho^2 \kappa}_{\text{LHS}(\kappa)}$$
$$= \underbrace{\left(2 - r\overline{\tau}_\rho - \frac{1}{2}\left(1-\overline{\tau}_\rho\right)\beta^2\kappa^2\right)^2 \left(2-(1-\beta)\kappa\right)\sigma_\theta^2 \overline{\tau}_\theta}_{\text{RHS}(\kappa)}.$$

Note that LHS is constant in $\overline{\tau}_\theta$ and RHS is linearly increasing in $\overline{\tau}_\theta$. All involved functions are continuous and differentiable, hence we check

$$\frac{d}{d\overline{\tau}_\rho}\left[\text{LHS}\left(\kappa\right) - \text{RHS}\left(\kappa\right)\right] = \left(\frac{\partial \text{LHS}}{\partial\kappa} - \frac{\partial \text{RHS}}{\partial\kappa}\right)\frac{\partial\kappa}{\partial\overline{\tau}_\theta} + \left(\frac{\partial \text{LHS}}{\partial\overline{\tau}_\theta} - \frac{\partial \text{RHS}}{\partial\overline{\tau}_\theta}\right).$$

Then

$$\frac{\partial\kappa}{\partial\overline{\tau}_\theta} = \frac{\frac{\partial \text{RHS}}{\partial\overline{\tau}_\theta}}{\frac{\partial \text{LHS}}{\partial\kappa} - \frac{\partial \text{RHS}}{\partial\kappa}}. \tag{9}$$

At the unique κ such that LHS(κ) = RHS(κ) it is the case that $\partial\text{LHS}(\kappa)/\partial\kappa > \partial\text{RHS}(\kappa)/\partial\kappa$, it follows that κ is increasing in $\overline{\tau}_\theta$.

To compute comparative statics with respect to $\overline{\tau}_\rho$, we first check

$$\begin{aligned}
\frac{2 - r\overline{\tau}_\rho - \frac{1}{2}\left(1-\overline{\tau}_\rho\right)\beta^2\kappa^2}{\overline{\tau}_\rho\sqrt{\sigma_{s,\rho}^2}} &= \frac{1}{\sqrt{\sigma_{s,\rho}^2}}\left(\frac{1}{\overline{\tau}_\rho}\left(2 - \frac{1}{2}\beta^2\kappa^2\right) - \left(r - \frac{1}{2}\beta^2\kappa^2\right)\right) \\
&= \left(\left(\frac{\sqrt{\sigma_{s,\rho}^2}}{\sigma_\rho^2} + \frac{1}{\sqrt{\sigma_{s,\rho}^2}}\right)\left(2 - \frac{1}{2}\beta^2\kappa^2\right) - \frac{1}{\sqrt{\sigma_{s,\rho}^2}}\left(r - \frac{1}{2}\beta^2\kappa^2\right)\right) \\
&= \frac{1}{\sqrt{\sigma_\rho^2}}\left(\sqrt{\frac{\sigma_{s,\rho}^2}{\sigma_\rho^2}}\left(2 - \frac{1}{2}\beta^2\kappa^2\right) + \sqrt{\frac{\sigma_\rho^2}{\sigma_{s,\rho}^2}}\left(2-r\right)\right).
\end{aligned}$$

Fixing σ_ρ^2, $\overline{\tau}_\rho$ increases when $\sigma_{s,\rho}^2$ decreases (and vice versa). Letting $R_\rho = \sqrt{\sigma_{s,\rho}^2/\sigma_\rho^2}$, we define LHS^R and RHS^R as

$$\underbrace{(2+\beta\kappa)^2\left(1-\left(\frac{1-r}{2-r}\right)\beta\kappa\right)^2\kappa}_{\text{LHS}^R(\kappa)}$$
$$=\underbrace{\frac{1}{\sqrt{\sigma_\rho^2}}\left(\left(2-\frac{1}{2}\beta^2\kappa^2\right)R_\rho+(2-r)\frac{1}{R_\rho}\right)(2-(1-\beta)\kappa)\,\sigma_\theta^2\overline{\tau}_\theta}_{\text{RHS}^R(\kappa)}.$$

Note that LHS^R is constant in R_ρ. Holding κ fixed, the extent to which RHS^R is affected by R_ρ is given by

$$\frac{\partial}{\partial R_\rho}\left[\left(2-\frac{1}{2}\beta^2\kappa^2\right)R_\rho+(2-r)\frac{1}{R_\rho}\right]=\left(2-\frac{1}{2}\beta^2\kappa^2\right)-\frac{2-r}{R_\rho^2}. \tag{10}$$

Note that $\beta\kappa\leq 1$ and $r\leq 1$, so the above is negative when R_ρ^2 is small and positive when R_ρ^2 is large. Since $R_\rho^2=(1-\overline{\tau}_\rho)/\overline{\tau}_\rho$, the above is negative when $\overline{\tau}_\rho$ is large and positive when $\overline{\tau}_\rho$ is small. An analysis similar to equation (9) implies that κ is decreasing in R_ρ (increasing in $\overline{\tau}_\rho$) when $\overline{\tau}_\rho$ is large and increasing in R_ρ (decreasing in $\overline{\tau}_\rho$) when $\overline{\tau}_\rho$ is small. To see single-peakedness, note that as R_ρ increases, (10) also increases. Starting from a point at which κ is locally constant, a slight increase in R_ρ from a point at which κ is locally constant must cause κ to rise; otherwise, κ is falling, implying that (10) is even more positive, a contradiction.

Finally, note that

$$\left(2-\frac{1}{2}\beta^2\kappa^2\right)-\frac{2-r}{R_\rho^2}=2\left(\frac{R_\rho^2-1}{R_\rho^2}\right)+\left(\frac{r-\frac{1}{2}\beta^2\kappa^2R_\rho^2}{R_\rho^2}\right).$$

When $R_\rho^2=1$, this is simply $r-\beta^2\kappa^2/2\overset{\text{sign}}{=}r$. Then (10) is positive at $R_\rho^2=1$ ($\overline{\tau}_\rho=1/2$) when $r>0$, and negative at $R_\rho^2=1$ when $r<0$. From single-peakedness, it follows that κ is minimized at $\overline{\tau}^\star>1/2$ when $r>0$ and at $\overline{\tau}^\star<1/2$ when $r<0$. □

Proof of Lemma 5. Second-period prices p_{i2}^c follow from the same methodology applied in the proof of Lemma 1. First-period prices p_{i1}^c follow from the same methodology applied in the proof of Lemma 3. □

Lemma 12. *When common cost information is shared, expected second period prices are*

$$\mathbb{E}\left[p_{i2}^c\middle|\,\rho,s_\rho,\mathbf{p}_1\right]=\frac{1}{4b^2-e^2}\left((2b+e)\,a+2b^2\mathbb{E}\left[c_i\middle|\,\rho,s_\rho,\mathbf{p}_1\right]+be\mathbb{E}\left[c_j\middle|\,\rho,s_\rho,\mathbf{p}_1\right]\right).$$

Proof. Following Lemma 5, we have

$$\mathbb{E}\left[p_{i2}^{c}\middle|\,\rho,s_{\rho},\mathbf{p}_{1}\right]=\frac{1}{2b}\left(a+b\mathbb{E}\left[c_{i}\middle|\,\rho,s_{\rho},\mathbf{p}_{1}\right]+e\mathbb{E}\left[p_{j2}^{c}\middle|\,\rho,s_{\rho},\mathbf{p}_{1}\right]\right).$$

This yields two linear equations in two unknowns. Solving this linear system gives the desired equation. □

Proof of Proposition A. We begin by computing $p_{\theta c}$, then address comparisons to the no-information-sharing regime.

Lemma 5 and the statement that $\partial\mathbb{E}[p_{j2}^{c}|\rho,s_{\rho},\mathbf{p}_{1}]/dp_{i1}=b\beta/ep_{i\theta c}$ give that first-period prices are

$$p_{i1}^{c}=\frac{1}{2b}\mathbb{E}\left[a+bc_{i}+ep_{j1}^{c}\middle|\,s_{i}\right]+\frac{1}{4b}\mathbb{E}\left[\left(a-bc_{i}+e\mathbb{E}\left[p_{j2}^{c}\middle|\,\rho,s_{\rho},\mathbf{p}_{1}\right]\right)\frac{\beta}{p_{i\theta c}}\middle|\,s_{i}\right].\qquad(11)$$

Following Lemma 12 we have

$$\begin{aligned}&\mathbb{E}\left[-bc_{i}+e\mathbb{E}\left[p_{j2}^{c}\middle|\,\rho,s_{\rho},\mathbf{p}_{1}\right]\middle|\,s_{i}\right]\\&=\mathbb{E}\left[\mathbb{E}\left[-bc_{i}+ep_{j2}^{c}\middle|\,\rho,s_{\rho},\mathbf{p}_{1}\right]\middle|\,s_{i}\right]\\&=\mathbb{E}\left[\mathbb{E}\left[-bc_{i}+\frac{e}{4b^{2}-e^{2}}\left(\left(2b+e\right)a+2b^{2}c_{j}+bec_{i}\right)\middle|\,\rho,s_{\rho},\mathbf{p}_{1}\right]\middle|\,s_{i}\right]\\&=\frac{1}{4b^{2}-e^{2}}\mathbb{E}\left[\mathbb{E}\left[\left(2b+e\right)ea+2b^{2}ec_{j}-2\left(2b^{2}-e^{2}\right)bc_{i}\middle|\,\rho,s_{\rho},\mathbf{p}_{1}\right]\middle|\,s_{i}\right].\end{aligned}$$

In a linear equilibrium, $s_{i\theta}$ is perfectly revealed by p_{i1}. Then the above is

$$\begin{aligned}&\mathbb{E}\left[-bc_{i}+e\mathbb{E}\left[p_{j2}^{c}\middle|\,\rho,s_{\rho},\mathbf{p}_{1}\right]\middle|\,s_{i}\right]\\&=\frac{1}{4b^{2}-e^{2}}\mathbb{E}\left[\left(2b+e\right)ea-2\left(2b^{2}-e^{2}\right)bc_{i}+2b^{2}e\mathbb{E}\left[c_{j}\middle|\,\rho,s_{\rho},\mathbf{p}_{1}\right]\middle|\,s_{i}\right].\end{aligned}$$

In the linear equilibrium, $p_{i1}=p_{i0c}+p_{i\theta c}\mathbb{E}\left[\theta_{i}\middle|\,s_{i}\right]+p_{i\rho c}\mathbb{E}\left[\rho\middle|\,s_{i}\right]$. Restricting equation (11) to terms which depend on $\mathbb{E}[\theta_{i}|s_{i}]$ gives

$$p_{i\theta c}=\frac{1}{2}-\frac{1}{2}\left(\frac{2b^{2}-e^{2}}{4b^{2}-e^{2}}\right)\frac{\beta}{p_{i\theta c}}=\frac{1}{2}+\frac{1}{4}\left(\frac{\left(\beta-1\right)\beta}{p_{i\theta c}}\right)\quad\Longrightarrow\quad 4p_{i\theta c}^{2}-2p_{i\theta c}-\left(\beta-1\right)\beta=0.$$

The solutions of this quadratic are

$$\begin{aligned}p_{i\theta c}&=\frac{1}{8}\left(2\pm\sqrt{4+16\left(\beta-1\right)\beta}\right)\\&=\frac{1}{4}\pm\frac{1}{4}\sqrt{4\beta^{2}-4\beta+1}=\frac{1}{4}\left(1\pm\left(2\beta-1\right)\right)\in\left\{-\frac{1}{2}\beta,\frac{1}{2}\left(1-\beta\right)\right\}.\end{aligned}$$

Since $\beta = e^2/(4b^2 - e^2) \geq 0$, one solution is positive and the other is negative.[15] Then $p_{i\theta c} = p_{\theta c} = (1-\beta)/2$ for both firms.

Recall that $p_\theta = 1/(2+\beta\kappa)$. By its definition in Lemma 4, $\kappa \leq 1/p_\theta = 2 + \beta\kappa$; then $\kappa \leq 2/(1-\beta)$. It follows that

$$p_\theta \geq \frac{1}{2 + \frac{2\beta}{1-\beta}} = \frac{1-\beta}{2} = p_{\theta c}.$$

The κ inequality is strict whenever $\beta > 0$ and $\sigma_\rho^2(1-\overline{\tau}_\rho)\tau_\rho p_\rho^2 > 0$; since we have assumed signals are informative, this is true whenever $e > 0$.

□

Proof of Theorem 2. Following from Lemma 2 ex-ante expected second period prices are

$$\begin{aligned}
\mathbb{E}[p_{j2}^*] &= \frac{1}{4b^2 - e^2}\big((2b+e)\,a + 2b^2\mathbb{E}\left[\mathbb{E}\left[c_j|\,\rho, p_{j1}\right]\right] + be\mathbb{E}\left[\mathbb{E}\left[c_i|\,\rho, p_{i1}\right]\right]\big) \\
&= \frac{1}{4b^2 - e^2}\left(\mathbb{E}\left[(2b+e)a + (2b^2 + be)\left(\mu_\theta + \mathbb{E}\left[\rho|s_{i,\rho}\right]\right)\right]\right. \\
&\quad + be\mathbb{E}\left[\kappa\left(\mathbb{E}[p_{i,1}|s] - p_0 + p_\theta\mu_\theta + p_\rho\mathbb{E}[\rho|s]\right)\right]\big)
\end{aligned}$$

The later term equals zero $\mathbb{E}[\mathbb{E}[p_{i,1}|s] - p_0 + p_\theta\mu_\theta + p_\rho E[\rho|s])] = 0$. It follows that in equilibrium

$$\mathbb{E}[p_{j2}^*] = \frac{a + b(\mu_\rho + \mu_\theta)}{2b - e}.$$

Similarly, with information sharing

$$\begin{aligned}
\mathbb{E}[p_{j2}^c] &= \frac{1}{4b^2 - e^2}\big((2b+e)\,a + 2b^2\mathbb{E}\left[\mathbb{E}\left[c_j|\,\rho, s_\rho, p_{j1}\right]\right] + be\mathbb{E}\left[\mathbb{E}\left[c_i|\,\rho, s_\rho, p_{i1}\right]\right]\big) \\
&= \frac{1}{4b^2 - e^2}\big((2b+e)\,a + 2b^2\mathbb{E}\left[\rho + \mathbb{E}[\theta_j|s_{j,\theta}]\right] + be\mathbb{E}\left[\rho + \mathbb{E}[\theta_i|s_{i,\theta}]\right]\big) \\
&= \frac{a + b(\mu_\rho + \mu_\theta)}{2b - e}.
\end{aligned}$$

In equilibrium the ex-ante expected price for each firm in the second period are the same with and without information sharing.

[15] When $e = 0$, $-\beta/2 = 0$. This solution can be ruled out by second order conditions, but we omit this exercise: if $p_{i\theta c} = 0$, prices do not depend on private cost information, which is not possible in our equilibrium.

From Lemma 3 the first period price in the symmetric equilibrium is

$$\mathbb{E}[p_{i,1}^*] = \frac{1}{2b-e}\mathbb{E}\left[\mathbb{E}\left[bc_i + a + \frac{e}{2b}\left(a - bc_i + e\mathbb{E}\left[p_{j,2}^*\middle|\, \rho, s_{i,\rho}, p_1\right]\right)\frac{be}{4b^2-e^2}\kappa \middle|\, s_{i,\rho}, s_{i,\theta}\right]\right]$$
$$= \frac{1}{2b-e}\left(a + b(\mu_\rho + \mu_\theta) + \frac{e^2\kappa}{2(4b^2-e^2)}\mathbb{E}\left[\mathbb{E}\left[\left(a - bc_i + e\mathbb{E}\left[p_{j,2}^*\middle|\, \rho, s_{i,\rho}, p_1\right]\right)\middle|\, s_{i,\rho}, s_{i,\theta}\right]\right]\right).$$

Similarly, from Lemma 5 when firms are sharing common cost information expected first period prices are

$$\mathbb{E}[p_{i,1}^c] = \frac{1}{2b-e}\mathbb{E}\left[bc_i + a + \frac{e}{2b}\left(a - bc_i + e\mathbb{E}\left[p_{j,2}^c\middle|\, \rho, s_\rho, p_1\right]\right)\frac{be}{4b^2-e^2}\kappa^c \middle|\, s_\rho, s_{i,\theta}\right]$$
$$= \frac{1}{2b-e}\left(a + b(\mu_\rho + \mu_\theta) + \frac{e^2\kappa^c}{2(4b^2-e^2)}\mathbb{E}\left[\mathbb{E}\left[\left(a - bc_i + e\mathbb{E}\left[p_{j,2}^c\middle|\, \rho, s_\rho, p_1\right]\right)\middle|\, s_\rho, s_{i,\theta}\right]\right]\right).$$

All terms are identical except κ and κ^c. From Proposition $\kappa^c \geq \kappa$ where the inequality is strict when $e \neq 0$. Therefore $\mathbb{E}[p_{i,1}^c] \geq \mathbb{E}[p_{i,1}]$ where the inequality is strict when $e \neq 0$. □

B Proofs for Section 4

Lemma 13. *The utility specification in* (4) *generates demand specification* (3)*. In particular, when $n = 2$ the utility specification in* (B) *generates demand specification in Section 2.*

Proof. Fix a price vector $\mathbf{p}$. Given utility as in (4), the consumer's first-order condition with respect to quantity q_i is

$$\frac{a}{b-e} - (n-1)\left(\frac{(n-1)\,b - (n-2)\,e}{((n-1)\,b + e)\,(b-e)}\right)q_i - (n-1)\left(\frac{e}{((n-1)\,b + e)\,(b-e)}\right)\sum_{j\neq i} q_j = p_i.$$

Let $Q = \sum_{i=1}^n q_i$. Summing up both sides of the equation over all firms i gives

$$\frac{na}{b-e} - (n-1)\left(\frac{(n-1)\,b - (n-2)\,e}{((n-1)\,b + e)\,(b-e)}\right)Q - (n-1)^2\left(\frac{e}{((n-1)\,b + e)\,(b-e)}\right)Q = \sum_{i=1}^n p_i.$$

Algebraic rearrangement yields

$$(n-1)\,Q = na - (b-e)\sum_{i=1}^n p_i.$$

Note that $\sum_{j\neq i} q_j = Q - q_i$. Then the consumer's first-order condition with respect to q_i

can be written

$$\frac{a}{b-e} - (n-1)^2 \left(\frac{1}{(n-1)\,b+e} \right) q_i - (n-1) \left(\frac{e}{((n-1)\,b+e)\,(b-e)} \right) Q = p_i.$$

This implies

$$\begin{aligned} q_i &= \left(\frac{1}{n-1} \right)^2 \left(\left(\frac{(n-1)\,b+e}{b-e} \right) a - \frac{e}{b-e} \left(na - (b-e) \sum_{j=1}^{n} p_j \right) - ((n-1)\,b+e)\,p_i \right) \\ &= \left(\frac{1}{n-1} \right)^2 \left((n-1)\,a - (n-1)\,bp_i + e \sum_{j \neq i} p_j \right). \end{aligned}$$

This is the demand form given in (3). □

Proof of Lemma 6. From Lemma 13, utility is given by

$$\mathbb{E}\left[u\left(\mathbf{q};\mathbf{p}\right)\right] = \frac{a}{b-e}(q_i+q_j) - \frac{1}{2}\left(\frac{b}{b^2-e^2}\right)(q_i^2+q_j^2) - \left(\frac{e}{b^2-e^2}\right) q_i q_j - (p_i q_i + p_j q_j).$$

Substituting in demand and applying equilibrium symmetry

$$\begin{aligned} \mathbb{E}\left[u\left(\mathbf{q}_t;\mathbf{p}_t\right)\right] &= \frac{2a}{b-e}\mathbb{E}\left[q_{it}\right] - \left(\frac{b}{b^2-e^2}\right)\mathbb{E}\left[q_{it}^2\right] - \left(\frac{e}{b^2-e^2}\right)\mathbb{E}\left[q_{it}q_{jt}\right] - 2\mathbb{E}\left[p_{it}q_{it}\right] \\ &= -2a\mathbb{E}\left[p_{it}\right] + b\mathbb{E}\left[p_{it}^2\right] - e\mathbb{E}\left[p_{it}p_{jt}\right]. \end{aligned}$$

Expressing in terms of the expectation, covariance and variance of prices

$$\mathbb{E}\left[u\left(\mathbf{p}_t\right)\right] = \left(-2a + (b-e)\,\mathbb{E}\left[p_{it}\right]\right)\mathbb{E}\left[p_{it}\right] + b\,\mathrm{Var}\left(p_{it}\right) - e\,\mathrm{Cov}\left(p_{it},p_{jt}\right).$$

□

Proof of Lemma 7. In period t firm i's expected profits are

$$\mathbb{E}\left[\pi_{it}\right] = \mathbb{E}\left[\left(a - bp_{it} + ep_{jt}\right)\left(p_{it} - c_i\right)\right].$$

Note that, when considering ex ante expected profits, it is not necessary to condition on learned information, which disappears by the law of iterated expectations. Then we see

$$\begin{aligned}\mathbb{E}\left[\pi_{it}\right] &= a\mathbb{E}\left[p_{it}-c_i\right] - b\mathbb{E}\left[p_{it}^2 - p_{it}c_i\right] + e\mathbb{E}\left[p_{jt}p_{it} - p_{jt}c_i\right] \\ &= a\mathbb{E}\left[p_{it}-c_i\right] - b\operatorname{Var}\left(p_{it}\right) - b\mathbb{E}\left[p_{it}\right]^2 + b\operatorname{Cov}\left(p_{it},c_i\right) + b\mathbb{E}\left[p_{it}\right]\mathbb{E}\left[c_i\right] \\ &\quad + e\operatorname{Cov}\left(p_{it},p_{jt}\right) + e\mathbb{E}\left[p_{it}\right]\mathbb{E}\left[p_{jt}\right] - e\operatorname{Cov}\left(p_{jt},c_i\right) - e\mathbb{E}\left[p_{jt}\right]\mathbb{E}\left[c_i\right] \\ &= \left(a - b\mathbb{E}\left[p_{it}\right] + e\mathbb{E}\left[p_{jt}\right]\right)\left(\mathbb{E}\left[p_{it}\right] - \mathbb{E}\left[c_i\right]\right) \\ &\quad - b\left(\operatorname{Var}\left(p_{it}\right) - \operatorname{Cov}\left(p_{it},c_i\right)\right) + e\left(\operatorname{Cov}\left(p_{it},p_{jt}\right) - \operatorname{Cov}\left(p_{jt},c_i\right)\right) \\ &= \left(a - (b-e)\,\mathbb{E}\left[p_{it}\right]\right)\left(\mathbb{E}\left[p_{it}\right] - \mathbb{E}\left[c_i\right]\right) \\ &\quad - b\left(\operatorname{Var}\left(p_{it}\right) - \operatorname{Cov}\left(p_{it},c_i\right)\right) + e\left(\operatorname{Cov}\left(p_{it},p_{jt}\right) - \operatorname{Cov}\left(p_{jt},c_i\right)\right).\end{aligned}$$

The final line follows by equilibrium symmetry. Since there are two firms, symmetry implies that expected producer surplus is twice this quantity. □

Lemma 14. *For given values of b and e, consumer surplus decreases with information sharing when a is large enough.*

Proof. From Lemma 6 consumer surplus is

$$\mathbb{E}\left[u\left(\mathbf{p}\right)\right] = \left(-2a + (b-e)\mathbb{E}[p_i]\right)\mathbb{E}[p_i] + b\operatorname{Var}(p_i) - e\operatorname{Cov}(p_i,p_j)$$

Without information sharing, the expected price, variance of price, and covariance of prices in the first period are given by:

$$\begin{aligned}\mathbb{E}\left[p_{i,1}^*\right] &= \frac{1}{2b-e}\left(a + b(\mu_\rho + \mu_\theta) + \frac{e^2\kappa}{2(4b^2-e^2)}\mathbb{E}\left[\left(a - bc_i + e\mathbb{E}\left[p_{j,2}^*\middle|\,\rho, s_{i,\rho}, p_1\right]\right)\right]\right) \\ \operatorname{Var}(p_{i,1}^*) &= p_\theta^2\operatorname{Var}\left(\mathbb{E}[\theta_i|s_{i,\theta}]\right) + p_\rho^2\operatorname{Var}\left(\mathbb{E}[\rho|s_{i,\rho}]\right) = p_\theta^2\frac{\bar{\tau}_{i,\theta}}{\tau_\theta} + p_\rho^2\frac{\bar{\tau}_{i,\rho}}{\tau_\rho} \\ \operatorname{Cov}(p_{i,1}^*, p_{j,1}^*) &= p_\rho^2\operatorname{Cov}\left(\mathbb{E}[\rho|s_{i,\rho}], \mathbb{E}[\rho|s_{j,\rho}]\right) = p_\rho^2\frac{\bar{\tau}_{i,\rho}^2}{\tau_\rho}\end{aligned}$$

With information sharing these become

$$\begin{aligned}\mathbb{E}\left[p_{i,1}^c\right] &= \frac{1}{2b-e}\left(a + b(\mu_\rho + \mu_\theta) + \frac{e^2\kappa^c}{2(4b^2-e^2)}\mathbb{E}\left[\left(a - bc_i + e\mathbb{E}\left[p_{j,2}^c\middle|\,\rho, s_\rho, p_1\right]\right)\right]\right) \\ \operatorname{Var}(p_{i,1}^c) &= p_{\theta,c}^2\operatorname{Var}\left(\mathbb{E}[\theta_i|s_{i,\theta}]\right) + p_{\rho,c}^2\operatorname{Var}\left(\mathbb{E}[\rho|s_{i,\rho}, s_{j,\rho}]\right) = p_{\theta,c}^2\frac{\bar{\tau}_{i,\theta}}{\tau_\theta} + p_{\rho,c}^2\frac{2\tau_{i,\rho}}{\tau_\rho(\tau_\rho + 2\tau_{i,\rho})} \\ \operatorname{Cov}(p_{i,1}^c, p_{j,1}^c) &= p_{\rho,c}^2\operatorname{Var}\left(\mathbb{E}[\rho|s_{i,\rho}, s_{j,\rho}]\right) = p_{\rho,c}^2\frac{2\tau_{s,\rho,i}}{\tau_\rho(\tau_\rho + 2\tau_{i,\rho})}\end{aligned}$$

Given an information sharing agreement the difference in these values compared to the baseline of no information sharing are as follows

$$\Delta\mathbb{E}\left[p_{i,1}\right] = \left(\frac{1}{2b-e}\right)^2 \left(\frac{e^2}{2(4b^2-e^2)}\right)(\kappa^c-\kappa)\left(2ab-2(b-e)(\mu_\rho+\mu_\theta)\right)$$

$$\Delta\mathrm{Var}(p_{i,1}) = \frac{\bar{\tau}_{i,\theta}}{\tau_\theta}(p_{\theta,c}^2-p_\theta^2) + p_{\rho,c}^2\frac{2\tau_{i,\rho}}{\tau_\rho(\tau_\rho+2\tau_{s,\rho,i})} - p_\rho^2\frac{\bar{\tau}_{i,\rho}}{\tau_\rho}$$

$$\Delta\mathrm{Cov}(p_{i,1},p_{j,1}) = p_{\rho,c}^2\frac{2\tau_{i,\rho}}{\tau_\rho(\tau_\rho+2\tau_{i,\rho})} - p_\rho^2\frac{\bar{\tau}_{i,\rho}^2}{\tau_\rho}$$

From Proposition A we have $\kappa^c > \kappa$ for $e \neq 0$, so $\Delta\mathbb{E}\left[p_{i,1}\right] > 0$ and is increasing in a. Equilibrium pricing strategies and price informativeness do not depend on the value of a and therefore for given $b > |e|$ and $e \neq 0$, $\Delta\mathrm{Var}(p_{i,1})$ and $\Delta\mathrm{Cov}(p_{i,1},p_{j,1})$ are constant for all a. Moreover, the difference in the first term of the utility between sharing information and not is decreasing in a:

$$\begin{aligned}
&\frac{\partial}{\partial a}\{\left(-2a+(b-e)\mathbb{E}[p_{i,1}^c]\right)\mathbb{E}[p_{i,1}^c] - \left(-2a+(b-e)\mathbb{E}[p_{i,1}^*]\right)\mathbb{E}[p_{i,1}^*]\} \\
&\quad < -2\left(\mathbb{E}[p_{i,1}^c]-\mathbb{E}[p_{i,1}^*]\right) - 2a\left(\frac{\partial\mathbb{E}[p_{i,1}^c]}{\partial a} - \frac{\partial\mathbb{E}[p_{i,1}^*]}{\partial a}\right) + 2(b-e)\left(\mathbb{E}[p_{i,1}^c]-\mathbb{E}[p_{i,1}^*]\right)\frac{\partial\mathbb{E}[p_{i,1}^c]}{\partial a} \\
&\quad < \left(-2+\frac{2(b-e)}{2b-e}\left(1+\frac{e}{2b-e}\right)\right)\left(\mathbb{E}[p_{i,1}^c]-\mathbb{E}[p_{i,1}^*]\right) < -\varepsilon
\end{aligned}$$

for some $\varepsilon > 0$. To get these inequalities we use that fact that

$$\begin{aligned}
\frac{\partial\mathbb{E}[p_{i,1}^c]}{\partial a} &= \left(\frac{1}{2b-e}\right)\left(1+\left(\frac{e}{2b-e}\right)\frac{be}{4b^2-e^2}\kappa^c\right) \\
&> \left(\frac{1}{2b-e}\right)\left(1+\left(\frac{e}{2b-e}\right)\frac{be}{4b^2-e^2}\kappa\right) + \varepsilon = \frac{\partial\mathbb{E}[p_{i,1}^*]}{\partial a} + \varepsilon
\end{aligned}$$

Therefore we can choose an a such that

$$\left(-2a+(b-e)\mathbb{E}[p_i^c]\right)\mathbb{E}[p_i^c] - \left(-2a+(b-e)\mathbb{E}[p_i]\right)\mathbb{E}[p_i] < b\Delta\mathrm{Var}(p_{i,1}) - \Delta\mathrm{Cov}(p_{i,1},p_{j,1}). \tag{12}$$

For all such a, the consumer surplus in the first period decreases under information sharing.

In the second period, the expectation, covariance, and variance of prices do not change under information sharing. Therefore two-period consumer surplus decreases for all a that satisfy equation (12). □

Lemma 15. *For given values of* b *and* e*, producer surplus increases with information sharing when* a *is large enough.*

Proof. From Lemma 7 producer surplus in each period is

$$\begin{aligned}\mathbb{E}\left[\pi_{it}\right] &= \left(a-(b-e)\,\mathbb{E}\left[p_{it}\right]\right)\left(\mathbb{E}\left[p_{it}\right]-\mathbb{E}\left[c_i\right]\right)\\ &\quad + b\left(\operatorname{Cov}\left(p_{it},c_i\right)-\operatorname{Var}\left(p_{it}\right)\right)-e\left(\operatorname{Cov}\left(p_{jt},c_i\right)-\operatorname{Cov}\left(p_{it},p_{jt}\right)\right).\end{aligned}$$

Similar to the case with consumer surplus, none of the variance or covariance terms depend on the value of a both with and without information sharing. Therefore the following term does not depend on a

$$b\left(\Delta\operatorname{Cov}\left(p_{it},c_i\right)-\Delta\operatorname{Var}\left(p_{it}\right)\right)-e\left(\Delta\operatorname{Cov}\left(p_{jt},c_i\right)-\Delta\operatorname{Cov}\left(p_{it},p_{jt}\right)\right)$$

$$\begin{aligned}\mathbb{E}[\Pi_1] &= 2\left(\left(a-(b-e)\mathbb{E}[p_{i,1,2}]\right)\left(\mathbb{E}[p_{i,1}]-\mathbb{E}[c_i]\right)-b\operatorname{Var}(p_{i,1})+e\operatorname{Cov}\left(p_{i,1},\mathbb{E}[p_{j,1}]\right)\right.\\ &\quad \left.+b\operatorname{Cov}\left(c_i,p_{i,1}\right)-e\operatorname{Cov}\left(c_i,\mathbb{E}[p_{j,1}]\right)\right)\\ &= 2\left(a-(b-e)\mathbb{E}[p_{i,1}]\right)\left(\mathbb{E}[p_{i,1}]-(\mu_\rho+\mu_\theta)\right)+C\end{aligned}$$

As shown in the consumer surplus, the expected value of prices along with the variance and covariance of prices in the second period do not change with information sharing. However there is a change associated with the covariance of prices with costs. This change, which also does not depend on the value of a, is

$$\Delta\mathbb{E}[\Pi_2]=\mathbb{E}[\Pi_2^c]-\mathbb{E}[\Pi_2]=-\frac{be^2}{2(4b^2-e^2)}\frac{\bar{\tau}_{i,\theta}}{\tau_\theta}(1-\kappa p_\theta).$$

The total change in profits with information sharing is then

$$\Delta\mathbb{E}[\Pi_1+\Pi_2]=2\left(a-(b-e)\mathbb{E}[p_{i,1}^c]\right)\left(\mathbb{E}[p_{i,1}^c]-\mathbb{E}[c_i]\right)-\left(2\left(a-(b-e)\mathbb{E}[p_{i,1}^*]\right)\left(\mathbb{E}[p_{i,1}^*]-\mathbb{E}[c_i]\right)\right)+C_2$$

where C_2 does not depends on a. Taking the derivative with respect to a

$$\begin{aligned}
\frac{\partial \Delta \mathbb{E}[\Pi_1 + \Pi_2]}{\partial a} &= 2\left(\mathbb{E}[p^c_{i,1}] - \mathbb{E}[p^*_{i,1}]\right)\left(1 - 2(b-e)\frac{\partial \mathbb{E}[p^c_{i,1}]}{\partial a}\right) \\
&+ 2(a + (b-e)\mathbb{E}[c_i])\left(\frac{\partial \mathbb{E}[p^c_{i,1}]}{\partial a} - \frac{\partial \mathbb{E}[p^*_{i,1}]}{\partial a}\right) \\
&\geq 2(a + (b-e)\mathbb{E}[c_i]\left(\frac{1}{2b-e}\right)^2\left(\frac{be^2}{4b^2-e^2}\right)(\kappa^c - \kappa) \\
&- 2\left(\frac{1}{2b-e}\right)^2\left(\frac{e^2}{2(4b^2-e^2)}\right)(\kappa^c - \kappa)(2ab - 2(b-e)\mathbb{E}[c_i]) \\
&= 2\left(\frac{1}{2b-e}\right)^2\left(\frac{e^2}{2(4b^2-e^2)}\right)(\kappa^c - \kappa)\left(ab + b(b-e)\mathbb{E}[c_i] - ab + (b-e)\mathbb{E}[c_i]\right) > \varepsilon
\end{aligned}$$

The first inequality comes from

$$\left(1 - \frac{2(b-e)}{2b-e}\left(\frac{1}{2b-e}\left(1 + \frac{e^2 b\kappa^c}{4b^2-e^2}\right)\right)\right) \geq -1$$

Therefore we can choose a large enough so that change in total profits is positive. □

Proof of Proposition 4. The follows immediately from Lemmas 14 and 15. □

Lemma 16. *In the n-firm extension expected second-period prices are given by*

$$\begin{aligned}
\mathbb{E}\left[p^\star_{i2} \middle| \rho, \mathbf{p}_1\right] = \frac{a}{2b-e} &+ (n-1)\left(\frac{1}{2(n-1)b+e}\right)\mathbb{E}\left[bc_i \middle| \rho, \mathbf{p}_1\right] \\
&+ \frac{e}{(2(n-1)b+e)(2b-e)}\sum_{j=1}^{n}\mathbb{E}\left[bc_j \middle| \rho, \mathbf{p}_1\right].
\end{aligned}$$

Proof. Firm i's second-period objective is

$$\max_p \mathbb{E}\left[\frac{1}{n-1}\left(a - bp + \frac{e}{n-1}\sum_{j\neq i} p^\star_{j2}\right)(p - c_i)\middle| c_i, \mathbf{p}_1\right].$$

At the optimal price $p^\star_{i2}$, firm i's second-period first-order condition is

$$0 = a - 2bp^\star_{i2} + bc_i + \frac{e}{n-1}\sum_{j\neq i}\mathbb{E}\left[p^\star_{j2}\middle| \rho, \mathbf{p}_1\right].$$

This equation holds given firm i's second-period information; therefore it holds in expectation, conditional on ρ and $\mathbf{p}_1$. This gives

$$0 = a - 2b\mathbb{E}\left[p^{\star}_{i2}\right|\rho, \mathbf{p}_1] + b\mathbb{E}\left[c_i\right|\rho, \mathbf{p}_1] + \frac{e}{n-1}\sum_{j\neq i}\mathbb{E}\left[p^{\star}_{j2}\right|\rho, \mathbf{p}_1\right].$$

Taken over all firms i this is a linear system, $A\mathbb{E}[\mathbf{p}^{\star}_2|\rho, \mathbf{p}_1] = a + b\mathbb{E}[\mathbf{c}|\rho, \mathbf{p}_1]$, where

$$A_{ii} = 2b, \quad A_{ij} = -\frac{e}{n-1} \quad (j \neq i).$$

The matrix A is invertible, with

$$A^{-1}_{ii} = \frac{2(n-1)b - (n-2)e}{(2b-e)(2(n-1)b+e)}, \quad A^{-1}_{ij} = \frac{e}{(2b-e)(2(n-1)b+e)} \quad (j \neq i).$$

This implies the stated result. □

Corollary 3. *In the n-firm extension second-period prices are given by*

$$p^{\star}_{i2} = \frac{a}{2b-e} + \frac{1}{2}\left(c_i + \frac{e^2}{(2(n-1)b+e)(2b-e)}\mathbb{E}\left[c_i\right|\rho, \mathbf{p}_1]\right) + \sum_{j\neq i}\frac{be}{(2(n-1)b+e)(2b-e)}\mathbb{E}\left[c_j\right|\rho, \mathbf{p}_1].$$

Proof. This follows immediately from the proof of Lemma 16. □

Lemma 17. *In the n-firm extension expected second-period profits are given by*

$$\mathbb{E}\left[\pi^{\star}_{i2}\right|c_i, \mathbf{p}_1] = \frac{1}{4b(n-1)}\left(a - bc_i + \frac{e}{n-1}\sum_{j\neq i}\mathbb{E}\left[p^{\star}_{j2}\right|\rho, \mathbf{p}_1\right]\right)^2$$

Proof. This is a standard profit-maximization problem, and follows immediately from the optimization in Lemma 16. □

Proof of Theorem 3. Firm i's first period objective is

$$\max_p \mathbb{E}\left[\frac{1}{n-1}\left(a - bp + \frac{e}{n-1}\sum_{j\neq i}p^{\star}_{j1}\right)(p - c_i) + \pi^{\star}_{i2}(p)\right| s_i\right].$$

Following Lemma 17, the firm's first-order condition is

$$\begin{aligned}2bp_{i1}^{\star} = {} & \mathbb{E}\left[a + bc_i + \frac{e}{n-1}\sum_{j\neq i} p_{j1}^{\star} \middle| s_i \right] \\ & + \frac{e}{2b}\mathbb{E}\left[\left(a - bc_i + \frac{e}{n-1}\sum_{j\neq i} p_{j2}^{\star} \right) \left(\frac{1}{n-1}\sum_{j\neq i} \frac{dp_{j2}^{\star}}{dp_{i1}} \right) \middle| s_i \right].\end{aligned} \tag{13}$$

Conditional on ρ, firm i's price does not affect firm j's beliefs about firm k's price. Then following Corollary 3,

$$\frac{d}{dp_{i1}}\mathbb{E}\left[p_{j2}^{\star} \middle| \rho, \mathbf{p}_1 \right] = \frac{be}{\left(2\left(n-1\right)b + e\right)\left(2b - e\right)}\left(\frac{d}{dp_{i1}}\mathbb{E}\left[c_i \middle| \rho, \mathbf{p}_1 \right] \right).$$

Conditional on ρ, the effect of firm i's first period price on firm j's beliefs about i's costs is completely determined by the relative importance of private and public costs in setting first period prices. That is,

$$\frac{d}{dp_{i1}}\mathbb{E}\left[c_i \middle| \rho, \mathbf{p}_1 \right] = \kappa \equiv \frac{\sigma_\theta^2 \overline{\tau}_\theta p_\theta}{\sigma_\rho^2\left(1 - \overline{\tau}_\rho\right)\overline{\tau}_\rho p_\rho^2 + \sigma_\theta^2 \overline{\tau}_\theta p_\theta^2}.$$

Define $\beta_n = e^2/(2(n-1)b+e)(2b-e)$; note that $\beta_2 = \beta$ as defined in the base two-firm case. Then equation (**??**) becomes, for any $j \neq i$,

$$2bp_{i1}^{\star} = \left(1 + \frac{e}{2b}\right)a + \left(1 - \frac{1}{2}\beta_n\kappa\right)\mathbb{E}\left[bc_i \middle| s_i \right] + e\mathbb{E}\left[p_{j1}^{\star} \middle| s_i \right] + \frac{1}{2}e\beta_n\kappa\mathbb{E}\left[p_{j2}^{\star} \middle| s_i \right].$$

Corollary 3 implies[16]

$$\begin{aligned}\mathbb{E}\left[p_{j2}^{\star} \middle| s_i \right] &= \frac{a}{2b-e} + \frac{1}{2}\left(1 + \beta_n\right)\mathbb{E}\left[c_j \middle| s_i \right] + \sum_{k\neq i,j}\frac{b}{e}\beta_n\mathbb{E}\left[c_k \middle| s_i \right] + \frac{b}{e}\beta_n\mathbb{E}\left[\mathbb{E}\left[c_i \middle| \rho, \mathbf{p}_1 \right] \middle| s_i \right] \\ &= \frac{a}{2b-e} + \left(\frac{1}{2}\left(1 + \beta_n\right) + \left(n-2\right)\frac{b}{e}\beta_n\right)\mathbb{E}\left[c_j \middle| s_i \right] + \frac{b}{e}\beta_n\mathbb{E}\left[\mathbb{E}\left[c_i \middle| \rho, \mathbf{p}_1 \right] \middle| s_i \right].\end{aligned}$$

In the linear pricing equilibrium,

$$p_{i1}^{\star} = p_{0n} + p_{\theta n}\mathbb{E}\left[\theta_i \middle| s_i \right] + p_{\rho n}\mathbb{E}\left[\rho \middle| s_i \right].$$

[16]Note that $b\beta_n/e = be/(2(n-1)b+e)(2b-e)$, so divding by zero is not a relevant concern.

Matching coefficients gives

$$
\begin{aligned}
2bp_{\theta n} &= b + \frac{e}{2b}\left(-b + b\beta_n\kappa p_{\theta n}\right)\frac{b}{e}\beta_n\kappa \\
&= \left(b - \frac{1}{2}b\beta_n\kappa\right) + \frac{1}{2}b\beta_n^2\kappa^2 p_{\theta n}; \\
2bp_{\rho n} &= b + e\overline{\tau}_\rho p_{\rho n} \\
&\quad + \left(\frac{e}{2b}\right)\left(-b + \left(\frac{1}{2}\left(1+\beta_n\right) + \left(n-2\right)\frac{b}{e}\beta_n + \frac{b}{e}\beta_n + \frac{b}{e}\beta_n\kappa p_{\rho n}\left(1-\overline{\tau}_\rho\right)\right)e\right)\frac{b}{e}\beta_n\kappa \\
&= b + e\overline{\tau}_\rho p_{\rho n} + \frac{1}{2}\left(-b + \frac{1}{2}\left(1+\beta_n\right)e + \left(n-1\right)b\beta_n + b\beta_n\kappa p_{\rho n}\left(1-\overline{\tau}_\rho\right)\right).
\end{aligned}
$$

The stated equalities are immediate. □

Proof of Theorem 4. The expressions for $p_{\theta\infty}$ and $p_{\rho\infty}$ follow immediately from Theorem 3.

As mentioned in the main text, $\lim_{n\nearrow\infty}\beta_n = 0$. In the n-large limit information about firm i does not affect any firm j's second-period pricing strategy. Then firm i's first-period first order conditions (equation (**??**) in the proof of Theorem 3 above) reduce to[17]

$$
2bp_{i1}^{\star} = a + b\mathbb{E}\left[c_i \middle| s_i\right] + e\mathbb{E}\left[p_{j1}^{\star}\middle| s_i\right] \quad \text{for any } j \neq i.
$$

In expectation this is

$$
2b\mathbb{E}\left[p_{i1}^{\star}\right] = a + b\mathbb{E}\left[\mathbb{E}\left[c_i\middle| s_i\right]\right] + e\mathbb{E}\left[\mathbb{E}\left[p_{j1}^{\star}\middle| s_i\right]\right] = a + b\mathbb{E}\left[c_i\right] + e\mathbb{E}\left[p_{j1}^{\star}\right].
$$

In the linear equilibrium this implies

$$
2b\left(p_{0\infty} + p_{\theta\infty}\mu_\theta + p_{\rho\infty}\mu_\rho\right) = a + b\left(\mu_\theta + \mu_\rho\right) + e\left(p_{0\infty} + p_{\theta\infty}\mu_\theta + p_{\rho\infty}\mu_\rho\right).
$$

Algebraic rearrangement gives

$$
\left(2b-e\right)p_{0\infty} = a + b\left(\mu_\theta + \mu_\rho\right) - \left(2b-e\right)p_{\theta\infty}\mu_\theta - \left(2b-e\right)p_{\rho\infty}\mu_\rho.
$$

Substituting in for $p_{\theta\infty}$ and $p_{\rho\infty}$ yields the stated equation for $p_{0\infty}$. □

Lemma 18. *There exists a constant $C_u \in \mathbb{R}$ such that for any $\overline{\tau}_\rho$, the linear equilibrium with a large number of firms yields expected first-period consumer surplus*

$$
\mathbb{E}\left[u_{1\infty}\right] \propto \left(1-r\right)\operatorname{Var}\left(p_{i1}^{\star}\right) - r\operatorname{Cov}\left(p_{i1}^{\star}, p_{j1}^{\star}\right) + C_u.
$$

[17] Recall that $\mathbb{E}[\sum_{k\neq i} p_{k1}^{\star}|s_i]/(n-1) = \mathbb{E}[p_{j1}^{\star}|s_i]$ for any $j \neq i$.

Proof. For any finite number of firms n, the linear pricing equilibrium yields the first-stage expected consumer surplus given in equation (4). Let d_{in} be scaled demand with n firms,

$$d_{in} = (n-1)\, q_i = a - bp^\star_{i1} + \frac{e}{n-1} \sum_{j \neq i} p^\star_{j1}.$$

Applying symmetry of the linear pricing equilibrium and the linearity of expectation gives

$$\mathbb{E}\left[u_{1\infty}\right] = \lim_{n \nearrow \infty} \mathbb{E}\left[u_{1n}\right] \propto ab\mathbb{E}\left[d_{i\infty}\right] - \frac{1}{2}(b-e)\,\mathbb{E}\left[d^2_{i\infty}\right] - \frac{1}{2} e\mathbb{E}\left[d_{i\infty} d_{j\infty}\right] - (b-e)\, b\mathbb{E}\left[p^\star_{i1} d_{i\infty}\right].$$

In the limit with a large number of firms, $\mathbb{E}[p^\star_{i1}]$ does not depend on $\overline{\tau}_\rho$. Then $\mathbb{E}[d_{i\infty}]$ does not depend on $\overline{\tau}_\rho$. Define $d^a_{in} = d_{in} - a$. Then there is a constant $C_{\pi 1}$ such that the above equation can be written

$$\mathbb{E}\left[u_{1\infty}\right] \propto -(b-e)\,\mathbb{E}\left[\left(d^a_{i\infty} + 2bp^\star_{i1}\right) d^a_{i\infty}\right] - e\mathbb{E}\left[d^a_{i\infty} d^a_{j\infty}\right] + C_{\pi 1}.$$

We compute each piece in turn.

$$\begin{aligned}
\mathbb{E}\left[\left(d^a_{i\infty} + 2bp^\star_{i1}\right) d^a_{i\infty}\right] &= \lim_{n \nearrow \infty} \mathbb{E}\left[\left(d^a_{in} + 2bp^\star_{i1,n}\right) d^a_{in}\right] \\
&= \lim_{n \nearrow \infty} \mathbb{E}\left[-b^2 p^{\star 2}_{i1,n} + \left(\frac{e}{n-1}\right)^2 \left(\sum_{j \neq i} p^\star_{j1,n}\right)^2\right] \\
&= -b^2 \mathbb{E}\left[p^{\star 2}_{i1}\right] + \lim_{n \nearrow \infty} \left(\frac{e}{n-1}\right)^2 \mathbb{E}\left[\sum_{j \neq i} p^{\star 2}_{j1,n} + 2 \sum_{j \neq i} \sum_{k \neq i,j} p^\star_{j1,n} p^\star_{k1,n}\right] \\
&= -b^2 \mathbb{E}\left[p^{\star 2}_{i1}\right] + 2e^2 \mathbb{E}\left[p^\star_{i1} p^\star_{j1}\right] = -b^2 \operatorname{Var}\left(p^\star_{i1}\right) + 2e^2 \operatorname{Cov}\left(p^\star_{i1}, p^\star_{j1}\right) + C_{\pi 2}; \\
\mathbb{E}\left[d^a_{i\infty} d^a_{j\infty}\right] &= \lim_{n \nearrow \infty} \mathbb{E}\left[d^a_{in} d^a_{jn}\right] \\
&= \lim_{n \nearrow \infty} \mathbb{E}\left[\left(-bp^\star_{i1,n} + \frac{e}{n-1} \sum_{k \neq i} p^\star_{k1,n}\right)\left(-bp^\star_{j1,n} + \frac{e}{n-1} \sum_{k \neq j} p^\star_{k1,n}\right)\right] \\
&= b^2 \mathbb{E}\left[p^\star_{i1} p^\star_{j1}\right] - 2be\mathbb{E}\left[p^\star_{i1} p^\star_{j1}\right] + \lim_{n \nearrow \infty} \left(\frac{e}{n-1}\right)^2 \mathbb{E}\left[\left(\sum_{k \neq i,j} p^\star_{k1,n}\right)^2\right] \\
&= \left(b^2 - 2be\right) \mathbb{E}\left[p^\star_{i1} p^\star_{j1}\right] + 2e^2 \mathbb{E}\left[p^\star_{i1} p^\star_{j1}\right] = \left((b-e)^2 + e^2\right) \operatorname{Cov}\left(p^\star_{i1}, p^\star_{j1}\right) + C_{\pi 3}.
\end{aligned}$$

Putting these pieces together leaves

$$\begin{aligned}\mathbb{E}\left[u_{1\infty}\right] &\propto -\left(b-e\right)\left(-b^2\operatorname{Var}\left(p^\star_{i1}\right)+2e^2\operatorname{Cov}\left(p^\star_{i1},p^\star_{j1}\right)\right)-e\left(\left(b-e\right)^2+e^2\right)\operatorname{Cov}\left(p^\star_{i1},p^\star_{j1}\right)+C_{\pi 4}\\ &=\left(b-e\right)b^2\operatorname{Var}\left(p^\star_{i1}\right)-\left(2e^2\left(b-e\right)+e\left(b-e\right)^2+e^3\right)\operatorname{Cov}\left(p^\star_{i1},p^\star_{j1}\right)+C_{\pi 4}\\ &\propto\left(1-r\right)\operatorname{Var}\left(p^\star_{i1}\right)-r\operatorname{Cov}\left(p^\star_{i1},p^\star_{j1}\right)+C_{\pi 5}.\end{aligned}$$

□

Lemma 19. *There exists a constant $C_\pi\in\mathbb{R}$ such that for any $\overline{\tau}_\rho$, the linear equilibrium with a large number of firms yields expected first-period producer surplus*

$$\mathbb{E}\left[\Pi_{1\infty}\right]\propto\left(\operatorname{Cov}\left(c_i,p^\star_{i1}\right)-\operatorname{Var}\left(p^\star_{i1}\right)\right)+\left(\operatorname{Cov}\left(p^\star_{i1},p^\star_{j1}\right)-\operatorname{Cov}\left(c_i,p^\star_{j1}\right)\right)r+C_\pi.$$

Proof. For any finite number of firms n, the linear pricing equilibrium yields first-stage expected producer surplus of

$$\begin{aligned}\mathbb{E}\left[\Pi_{1n}\right] &= \mathbb{E}\left[\sum_{i=1}^{n}\frac{1}{n-1}\left(a-bp^\star_{i1}+\frac{e}{n-1}\sum_{j\neq i}p^\star_{j1}\right)\left(p^\star_{i1}-c_i\right)\right]\\ &=\mathbb{E}\left[\left(a-bp^\star_{i1}+\frac{e}{n-1}\sum_{j\neq i}p^\star_{j1}\right)\left(p^\star_{i1}-c_i\right)\right].\end{aligned}$$

Symmetry in the linear pricing equilibrium implies the second equality; this expression holds for any firm i. With the exception of sensitivity to opponent prices all terms are (first-order) independent of the number of firms n; prices themselves will depend on the number of the firms in the market. Linearity of expectations and symmetry of pricing strategies imply

$$\mathbb{E}\left[\Pi_{1\infty}\right]=\lim_{n\nearrow\infty}\mathbb{E}\left[\Pi_{1n}\right]=\mathbb{E}\left[\left(a-bp^\star_{i1}\right)\left(p^\star_{i1}-c_i\right)\right]+e\mathbb{E}\left[\left(p^\star_{i1}-c_i\right)p^\star_{j1}\right]\qquad(j\neq i).$$

Recall that when n is large, expected prices do not depend on $\overline{\tau}_\rho$. Then there are constants $C_{\pi k}$ such that

$$\begin{aligned}\mathbb{E}\left[\Pi_{1\infty}\right] &= e\mathbb{E}\left[\left(p^\star_{i1}-c_i\right)p^\star_{j1}\right]-b\mathbb{E}\left[\left(p^\star_{i1}-c_i\right)p^\star_{i1}\right]+C_{\pi 1}\\ &= e\mathbb{E}\left[p^\star_{i1}p^\star_{j1}\right]-e\mathbb{E}\left[c_ip^\star_{j1}\right]-b\mathbb{E}\left[p^{\star 2}_{i1}\right]+b\mathbb{E}\left[c_ip^\star_{i1}\right]+C_{\pi 1}\\ &= e\operatorname{Cov}\left(p^\star_{i1},p^\star_{j1}\right)-e\operatorname{Cov}\left(c_i,p^\star_{j1}\right)-b\operatorname{Var}\left(p^\star_{i1}\right)+b\operatorname{Cov}\left(c_i,p^\star_{i1}\right)+C_{\pi 2}\\ &\propto\left(\operatorname{Cov}\left(c_i,p^\star_{i1}\right)-\operatorname{Var}\left(p^\star_{i1}\right)\right)+\left(\operatorname{Cov}\left(p^\star_{i1},p^\star_{j1}\right)-\operatorname{Cov}\left(c_i,p^\star_{i1}\right)\right)r+C_{\pi 3}.\end{aligned}$$

This establishes the stated result. □

Proof of Proposition 5. This follows immediately from Lemmas 18 and 19. □

Proof of Corollary 1. Following Proposition 5, the extent to which producer surplus depends on $\overline{\tau}_\rho$ is given by

$$\mathbb{E}\left[\Pi_{1\infty}\right] \simeq \left(\operatorname{Cov}\left(c_i, p^\star_{i1}\right) - \operatorname{Var}\left(p^\star_{i1}\right)\right) + \left(\operatorname{Cov}\left(p^\star_{i1}, p^\star_{j1}\right) - \operatorname{Cov}\left(c_i, p^\star_{j1}\right)\right) r.$$

To simplify notation we use $\simeq$ to denote an equivalence of all terms that depend directly on $\overline{\tau}_\rho$; that is, $f(\cdot) \simeq g(\cdot)$ if there is $C \in \mathbb{R}$ such that for any $\overline{\tau}_\rho$, $g(\overline{\tau}_\rho) - f(\overline{\tau}_\rho) = C$. We compute in turn:

$$\begin{aligned}
\operatorname{Cov}\left(c_i, p^\star_{i1}\right) &\simeq p_{\rho\infty}\overline{\tau}_\rho\sigma^2_\rho; \\
\operatorname{Var}\left(p^\star_{i1}\right) &\simeq p^2_{\rho\infty}\overline{\tau}^2_\rho\left(\sigma^2_\rho + \sigma^2_{\varepsilon\rho}\right) = p^2_{\rho\infty}\overline{\tau}_\rho\sigma^2_\rho; \\
\operatorname{Cov}\left(p^\star_{i1}, p^\star_{j1}\right) &\simeq p^2_{\rho\infty}\overline{\tau}^2_\rho\sigma^2_\rho; \\
\operatorname{Cov}\left(c_i, p^\star_{j1}\right) &\simeq p_{\rho\infty}\overline{\tau}_\rho\sigma^2_\rho.
\end{aligned}$$

Then

$$\begin{aligned}
\mathbb{E}\left[\Pi_{1\infty}\right] &\simeq \left(p_{\rho\infty}\overline{\tau}_\rho\sigma^2_\rho - p^2_{\rho\infty}\overline{\tau}_\rho\sigma^2_\rho\right) + \left(p^2_{\rho\infty}\overline{\tau}^2_\rho\sigma^2_\rho - p_{\rho\infty}\overline{\tau}_\rho\sigma^2_\rho\right) r \\
&\propto (1-r)\, p_{\rho\infty}\overline{\tau}_\rho - (1 - r\overline{\tau}_\rho)\, p^2_{\rho\infty}\overline{\tau}_\rho \\
&= \frac{(1-r)\,\overline{\tau}_\rho}{2 - r\overline{\tau}_\rho} - \frac{(1 - r\overline{\tau}_\rho)\,\overline{\tau}_\rho}{\left(2 - r\overline{\tau}_\rho\right)^2} = \frac{(1-2r)\,\overline{\tau}_\rho + r^2\overline{\tau}^2_\rho}{\left(2 - r\overline{\tau}_\rho\right)^2}.
\end{aligned}$$

The derivative of this expression with respect to $\overline{\tau}_\rho$ is[18]

$$\frac{d}{d\overline{\tau}_\rho}\mathbb{E}\left[\Pi_{1\infty}\right] = \frac{2 + (2r^2 - 4r)\,\overline{\tau}_\rho}{\left(2 - r\overline{\tau}_\rho\right)^3}.$$

Since $r\overline{\tau}_\rho \leq 1$, the denominator is always positive. The numerator is linear in $\overline{\tau}_\rho$ and thus will obtain extrema at $\overline{\tau}_\rho = 0$ and $\overline{\tau}_\rho = 1$. At $\overline{\tau}_\rho = 0$ the numerator is $2 > 0$; at $\overline{\tau}_\rho = 1$ the numerator is $2(r-1)^2 \geq 0$. Then producer surplus is strictly increasing in $\overline{\tau}_\rho$ for all $\overline{\tau}_\rho < 1$. It follows that producer surplus is strictly greater when information is shared, provided $\overline{\tau}_\rho \in (0, 1)$. □

Proof of Corollary 2. Following Proposition 5, the extent to which consumer surplus depends

[18] Recall that $\simeq$ is a relationship removing terms constant in $\overline{\tau}_\rho$. Then the derivative of $\mathbb{E}[\Pi_{1\infty}]$ depends only on the terms on the right-hand side of $\simeq$ whether or not the constant terms are ignored, and can be written as $=$.

on $\overline{\tau}_\rho$ is given by

$$\mathbb{E}\left[u_{1\infty}|\,\overline{\tau}_\rho\right] \simeq (1-r)\operatorname{Var}\left(p^\star_{i1}\right) - r\operatorname{Cov}\left(p^\star_{i1}, p^\star_{j1}\right).$$

We compute in turn:

$$\begin{aligned}\operatorname{Var}\left(p^\star_{i1}\right) &\simeq p^2_{\rho\infty}\overline{\tau}_\rho\sigma^2_\rho;\\ \operatorname{Cov}\left(p^\star_{i1}, p^\star_{j1}\right) &\simeq p^2_{\rho\infty}\overline{\tau}^2_\rho\sigma^2_\rho.\end{aligned}$$

Then

$$\mathbb{E}\left[u_{1\infty}|\,\overline{\tau}_\rho\right] \simeq (1-r)\,p^2_{\rho\infty}\overline{\tau}_\rho\sigma^2_\rho - rp^2_{\rho\infty}\overline{\tau}^2_\rho\sigma^2_\rho \propto \frac{(1-r)\,\overline{\tau}_\rho - r\overline{\tau}^2_\rho}{\left(2-r\overline{\tau}_\rho\right)^2}.$$

To determine the effects of information sharing, we compare expected consumer surplus with precision $\overline{\tau}_\rho$ against precision $\overline{\tau}'_\rho = 1$,

$$\begin{aligned} & & \mathbb{E}\left[u_{1\infty}|\,\overline{\tau}_\rho\right] \gtrless \mathbb{E}\left[u_{1\infty}|\,\overline{\tau}'_\rho = 1\right] \\ \iff & & \frac{(1-r)\,\overline{\tau}_\rho - r\overline{\tau}^2_\rho}{\left(2-r\overline{\tau}_\rho\right)^2} \gtrless \frac{1-2r}{\left(2-r\right)^2} \\ \iff & & \left(r^3+3r^2-4r\right)\overline{\tau}^2_\rho + \left(4-4r-r^3-3r^2\right)\overline{\tau}_\rho + (8r-4) \gtrless 0. \end{aligned} \tag{14}$$

When $\overline{\tau}_\rho = 0$ the left-hand side is $8r-4$, and the value of information sharing will depend on whether $r \gtrless 1/2$. When $\overline{\tau}_\rho = 0$, the left-hand side is 0. The left-hand side of the above inequality is a quadratic in $\overline{\tau}_\rho$, so properties of the parabola will determine the effect of information sharing on consumer surplus. In particular, it is sufficient to analyze the slope of the parabola at $\overline{\tau}_\rho = 1$. If this slope is positive the left-hand side of (14) is negative and information sharing improves expected consumer welfare; if this slope is negative information sharing may lower expected consumer welfare, depending on initial informational precision.

The derivative of the left-hand side of (14) with respect to $\overline{\tau}_\rho$, evaluated at $\overline{\tau}_\rho = 1$, is

$$2\left(r^3+3r^2-4r\right) + \left(4-4r-r^3-3r^2\right) = r^3+3r^2-12r+4 = -\left(2-r\right)\left(r^2+5r-2\right).$$

Then the slope of the left-hand term will depend on $-(r^2+5r-2) \gtrless 0$. Solving the quadratic gives

$$r^\perp = -\frac{5}{2}+\frac{1}{2}\sqrt{33} \approx 0.372.$$

When $r \lesssim 0.372$ the slope of the left-hand term is positive at $\overline{\tau}_\rho = 1$, and when $r \gtrsim 0.372$ the slope of the left-hand term is negative at $\overline{\tau}_\rho = 1$. Then when $r \lesssim 0.372$ information sharing strictly improves expected consumer surplus, and when $r \gtrsim 0.372$ information sharing may harm expected consumer surplus, depending on the initial level of precision $\overline{\tau}_\rho$.

□

C Proofs for Section 5

Proof of Lemma 8. From Lemma 7 producer surplus in the first period is

$$\begin{aligned}\mathbb{E}\left[\pi_{i,1}\right] &= \left(a-(b-e)\,\mathbb{E}\left[p_{i,1}\right]\right)\left(\mathbb{E}\left[p_{i,1}\right]-\mathbb{E}\left[c_i\right]\right)\\ &\quad - b\left(\mathrm{Var}\left(p_{i,1}\right)-\mathrm{Cov}\left(p_{i,1},c_i\right)\right)+e\left(\mathrm{Cov}\left(p_{i,1},p_{j,1}\right)-\mathrm{Cov}\left(p_{j,1},c_i\right)\right).\end{aligned}$$

Expected cost and first period pricing strategies are not effected by marginal increases in precision so the first term can be treated as a constant, C_i.

Writing the variance and covariance terms in terms of the precision parameters we first note that for $x=\theta,\rho$

$$\mathrm{Var}\left(\mathbb{E}[x|s_{i,x}]\right)=\frac{\tau_{i,x}}{\left(\tau_{i,x}+\tau_x\right)\tau_x},$$

and for ρ,

$$\mathrm{Cov}\left(\mathbb{E}[\rho|s_{i,\rho}],\mathbb{E}[\rho|s_{j,\rho}]\right)=\frac{\tau_{i,\rho}\tau_{j,\rho}}{\left(\tau_{i,\rho}+\tau_\rho\right)\left(\tau_{j,\rho}+\tau_\rho\right)\tau_\rho}.$$

Then the terms become

$$\begin{aligned}\mathrm{Var}(p_{i,1}) &= \frac{\tau_{i,\theta}p_{i\theta}^2}{\left(\tau_{i,\theta}+\tau_\theta\right)\tau_\theta}+\frac{\tau_{i,\rho}p_{i\rho}^2}{\left(\tau_{i,\rho}+\tau_\rho\right)\tau_\rho},\\ \mathrm{Cov}\left(p_{i,1},c_i\right) &= \frac{\tau_{i,\theta}p_{i\theta}}{\left(\tau_{i,\theta}+\tau_\theta\right)\tau_\theta}+\frac{\tau_{i,\rho}p_{i\rho}}{\left(\tau_{i,\rho}+\tau_\rho\right)\tau_\rho},\\ \mathrm{Cov}(p_{i,1},p_{j,1}) &= \frac{\tau_{i,\rho}\tau_{j,\rho}p_{i\rho}p_{j\rho}}{\left(\tau_{i,\rho}+\tau_\rho\right)\left(\tau_{j,\rho}+\tau_\rho\right)\tau_\rho},\\ \mathrm{Cov}\left(p_{j,1},c_i\right) &= \frac{\tau_{i,\rho}\tau_{j,\rho}p_{j\rho}}{\left(\tau_{i,\rho}+\tau_\rho\right)\left(\tau_{j,\rho}+\tau_\rho\right)\tau_\rho}.\end{aligned}$$

Putting everything together,

$$\mathbb{E}\left[\pi_{i,1}\right]=\left(\frac{\left(1-p_{i\theta}\right)p_{i\theta}\tau_{i,\theta}}{\left(\tau_{i,\theta}+\tau_\theta\right)\tau_\theta}\right)b+\left(\frac{\left(1-p_{i\rho}\right)p_{i\rho}\tau_{i,\rho}}{\left(\tau_{i,\rho}+\tau_\rho\right)\tau_\rho}\right)b-\left(\frac{\left(1-p_{i\rho}\right)p_{j\rho}\tau_{i,\rho}\tau_{j,\rho}}{\left(\tau_{i,\rho}+\tau_\rho\right)\left(\tau_{j,\rho}+\tau_\rho\right)\tau_\rho}\right)e+C_i.$$

The result follows from taking the derivative with respect to each precision parameter. □

Proof of Lemma 9. Producer surplus in the first period becomes

$$\begin{aligned}\mathbb{E}\left[\pi_{i,1}^{c}\right] &= \left(a-(b-e)\,\mathbb{E}\left[p_{i,1}^{c}\right]\right)\left(\mathbb{E}\left[p_{i,1}^{c}\right]-\mathbb{E}\left[c_i\right]\right) \\ &\quad - b\left(\operatorname{Var}\left(p_{i,1}^{c}\right)-\operatorname{Cov}\left(p_{i,1}^{c},c_i\right)\right)+e\left(\operatorname{Cov}\left(p_{i,1}^{c},p_{j,1}^{c}\right)-\operatorname{Cov}\left(p_{j,1}^{c},c_i\right)\right).\end{aligned}$$

For ρ the pertinent variance term is

$$\operatorname{Var}\left(\mathbb{E}[\rho|s_{i,\rho},s_{j,\rho}]\right)=\frac{\tau_{i,\rho}+\tau_{j,\rho}}{\left(\tau_{i,\rho}+\tau_{j,\rho}+\tau_{\rho}\right)\tau_{\rho}},$$

Then the variance and covariance terms are

$$\begin{aligned}\operatorname{Var}(p_{i,1}^{c}) &= \frac{\tau_{i,\theta}p_{i\theta}^{2}}{\left(\tau_{i,\theta}+\tau_{\theta}\right)\tau_{\theta}}+\frac{(\tau_{i,\rho}+\tau_{j,\rho})p_{i\rho}^{2}}{\left(\tau_{i,\rho}+\tau_{j,\rho}+\tau_{\rho}\right)\tau_{\rho}}, \\ \operatorname{Cov}\left(p_{i,1}^{c},c_i\right) &= \frac{\tau_{i,\theta}p_{i\theta}}{\left(\tau_{i,\theta}+\tau_{\theta}\right)\tau_{\theta}}+\frac{(\tau_{i,\rho}+\tau_{j,\rho})p_{i\rho}}{\left(\tau_{i,\rho}+\tau_{j,\rho}+\tau_{\rho}\right)\tau_{\rho}}, \\ \operatorname{Cov}(p_{i,1}^{c},p_{j,1}^{c}) &= \frac{(\tau_{i,\rho}+\tau_{j,\rho})p_{i\rho}p_{j\rho}}{\left(\tau_{i,\rho}+\tau_{j,\rho}+\tau_{\rho}\right)\tau_{\rho}}, \\ \operatorname{Cov}\left(p_{j,1}^{c},c_i\right) &= \frac{(\tau_{i,\rho}+\tau_{j,\rho})p_{j\rho}}{\left(\tau_{i,\rho}+\tau_{j,\rho}+\tau_{\rho}\right)\tau_{\rho}}.\end{aligned}$$

Putting everything together,

$$\mathbb{E}\left[\pi_{i,1}^{c}\right]=\left(\frac{\left(1-p_{i\theta}\right)p_{i\theta}\tau_{i,\theta}}{\left(\tau_{i,\theta}+\tau_{\theta}\right)\tau_{\theta}}\right)b+\left(\frac{\left(1-p_{i\rho}\right)p_{i\rho}(\tau_{i,\rho}+\tau_{j,\rho})}{\left(\tau_{i,\rho}+\tau_{j,\rho}+\tau_{\rho}\right)\tau_{\rho}}\right)b-\left(\frac{\left(1-p_{i\rho}\right)p_{j\rho}(\tau_{i,\rho}+\tau_{j,\rho})}{\left(\tau_{i,\rho}+\tau_{j,\rho}+\tau_{\rho}\right)\tau_{\rho}}\right)e+C_i.$$

The result follows from taking the derivative with respect to each precision parameter. □

Proof of Proposition 6. This proof is included in the main text. □

D Bounds on values

The following inequalities are used throughout the paper.

$$\beta \in \left[0, \frac{1}{3}\right] \tag{15}$$

$$\frac{b-e}{2b-e} \in \left[0, \frac{2}{3}\right] \tag{16}$$

$$\left(\frac{b-e}{2b-e}\right)\beta \in \left[0, \frac{2}{9}\right] \tag{17}$$

$$p_\theta \in \left[\frac{1}{3}, \frac{1}{2}\right] \tag{18}$$

$$\kappa \in \left[0, \frac{2}{1-\beta}\right] \subseteq [0,3] \tag{19}$$

$$\left|\frac{be}{4b^2-e^2}\right|\kappa \in [0,1] \tag{20}$$

$$\beta\kappa \in \left[0, \frac{r^2}{2-r^2}\right] \subseteq \left[0, r^2\right] \subseteq [0,1] \tag{21}$$

$$p_\rho \in \left[\frac{1}{9}, \frac{1}{2}\right] \text{ (when } e < 0) \tag{22}$$

$$p_\rho \in [0.46, 1] \text{ (when } e > 0) \tag{23}$$

D.1 Proofs of bounds

Proof of inequality (15). Since $|e| \leq b$, $\beta = e^2/(4b^2 - e^2) \geq 0$. To establish the upper bound, note that the numerator is increasing in e^2 and the denominator is decreasing in e^2, so the maximum value of β will be attained when e^2 is at its maximum. Since $e^2 \leq b^2$, it follows that $\beta \leq 3$. □

Proof of inequality (16). Since $|e| \leq b$, it is immediate that $(b-e)/(2b-e) \geq 0$. To establish the upper bound we examine the first derivative of the expression,[19]

$$\frac{-(2b-e)+(b-e)}{(2b-e)^2} = -\frac{b}{(2b-e)^2} < 0.$$

Then the derivative is negative everywhere, and the expression is maximized when e is at its minimum, $e = -b$. This gives

$$\frac{b-(-b)}{2b-(-b)} = \frac{2}{3}.$$

[19]Basic intuition about fractions is sufficient for this maximization. We find that straightforward calculus is simpler to analyze.

□

Proof of inequality (17). This follows directly from inequalities (15) and (16). □

Proof of inequalities (18) *and* (19). Since $\beta\kappa \geq 0$ and $p_\theta = 1/(2+\beta\kappa)$, it must be that $p_\theta \leq 1/2$. Further, p_θ will be minimized when $\beta\kappa$ is maximized. Looking at κ in isolation,

$$\kappa = \frac{\sigma_\theta^2 \bar{\tau}_{s,\theta} p_\theta}{\sigma_{s,\rho}^2 \bar{\tau}_{s,\rho}^2 p_\rho^2 + \left(\sigma_\theta^2 + \sigma_{s,\theta}^2\right) \bar{\tau}_{s,\theta}^2 p_\theta^2}.$$

All involved terms are positive, so κ can be bounded above by assuming that $\bar{\tau}_{s,\rho} = 0$. This gives

$$\kappa \leq \frac{\sigma_\theta^2 \bar{\tau}_{s,\theta} p_\theta}{\left(\sigma_\theta^2 + \sigma_{s,\theta}^2\right) \bar{\tau}_{s,\theta}^2 p_\theta^2} = \frac{\sigma_\theta^2 \bar{\tau}_{s,\theta} p_\theta}{\sigma_\theta^2 \bar{\tau}_{s,\theta} p_\theta^2} = \frac{1}{p_\theta}.$$

Let $\underline{p}_\theta$ be the minimum feasible value of p_θ and $\overline{\beta} = 1/3$ be the maximum feasible value of β; then $\kappa \leq 1/\underline{p}_\theta$. It follows that

$$p_\theta \geq \frac{1}{2 + \frac{\beta}{\underline{p}_\theta}} \quad \Longrightarrow \quad \underline{p}_\theta \geq \frac{1}{2 + \frac{\overline{\beta}}{\underline{p}_\theta}}.$$

This gives

$$2\underline{p}_\theta + \overline{\beta} \geq 1 \quad \Longrightarrow \quad \underline{p}_\theta \geq \frac{1}{3}.$$

Then $p_\theta \geq 1/3$. It follows that $\kappa \leq 3$. Since $|e| \leq b$, $be/(4b^2 - e^2) \leq 1/3$, hence

$$\left(\frac{be}{4b^2 - e^2}\right) \kappa \leq \left(\frac{1}{3}\right) 3 = 1.$$

From $\kappa \leq \frac{1}{p_\theta} = 2 + \beta\kappa$ we can bound

$$\kappa \leq \frac{2}{1-\beta} = \frac{4 - r^2}{2 - r^2} \text{ and } \beta\kappa \leq \frac{r^2}{2 - r^2}$$

□

Proof of inequality (20). Note that

$$\left|\frac{be}{4b^2 - e^2}\right| = \frac{b\,|e|}{4b^2 - |e|^2}.$$

The numerator is increasing in $|e|$ while the denominator is decreasing in $|e|$, thus this ratio will be maximized when $|e|$ is maximized, or when $|e| = b$. Then

$$\left|\frac{be}{4b^2 - e^2}\right| \leq \frac{b^2}{4b^2 - b^2} = \frac{1}{3}.$$

Since inequality (19) gives $\kappa \leq 3$, inequality (20) follows. $\square$

Proof of inequality (21). Note that

$$\beta = \frac{e}{b}\left(\frac{be}{4b^2 - e^2}\right).$$

Then inequality (21) follows immediately from inequality (20). $\square$

Proof of inequalities (22) *and* (23). Recall the equilibrium equation for p_ρ,

$$p_\rho = \frac{1 - \left(\frac{1-r}{2-r}\right)\beta\kappa}{(2 - r\bar{\tau}_{s,\rho}) - \frac{1}{2}\left(1 - \bar{\tau}_{s,\rho}\right)\beta^2\kappa^2}, \text{ where } r = \frac{e}{b}.$$

By inequality (21), $\beta\kappa \leq |r|$, so the bound on the denominator will depend on the sign of r.

When $r < 0$, the denominator is bounded below by $2 - \beta^2\kappa^2/2$ and above by $2 - r$. The numerator is bounded above by $1 - \beta\kappa/2$. This gives

$$p_\rho \leq \frac{1 - \frac{1}{2}\beta\kappa}{2 - \frac{1}{2}\beta^2\kappa^2} = \frac{2 - \beta\kappa}{4 - \beta^2\kappa^2} = \frac{1}{2 + \beta\kappa} \leq \frac{1}{2};$$

$$p_\rho \geq \frac{1 - \left(\frac{1-r}{2-r}\right)\beta\kappa}{2 - r} \geq \frac{(2 - r) - (1 - r)}{(2 - r)^2} = \frac{1}{(2 - r)^2} \geq \frac{1}{9}.$$

When $r > 0$, the denominator is bounded below by $2 - r$ and above by $2 - \beta^2\kappa^2/2$. The numerator is bounded below by $1 - \beta\kappa/2$. This gives

$$p_\rho \leq \frac{1 - \left(\frac{1-r}{2-r}\right)\beta\kappa}{2 - r} \leq \frac{(2 - r) - (1 - r)\, r}{(2 - r)^2} = \frac{2 - 2r + r^2}{(2 - r)^2} \leq 1;$$

$$p_\rho \geq \frac{1 - \left(\frac{1-r}{2-r}\right)\left(\frac{r^2}{4-r^2}\right)\kappa}{2 - \frac{1}{2}\left(\frac{r^2}{4-r^2}\right)^2\kappa^2} \geq \frac{1 - \left(\frac{1-r}{2-r}\right)\left(\frac{r^2}{4-r^2}\right)\left(\frac{2}{1-r^2/(4-r^2)}\right)}{2} \geq \frac{1}{2} - \left(\frac{1-r}{2-r}\right)\left(\frac{r^2}{4 - 2r^2}\right) \geq\approx 0.46.$$

$\square$