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Reserve prices eliminate low revenue equilibria in uniform price auctions ☆

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Uniform price auctions frequently admit equilibria which raise zero seller revenue. We show that when demand is sufficiently strong — when market supply is more than covered by any bidder's opponents — the introduction of a reserve price improves revenue not only by directly increasing the market clearing price, but also by eliminating low revenue equilibria in which the market clearing price is almost always equal to the reserve. The condition on demand is sharp, and when it is not satisfied there exist equilibria in which the market clearing price almost always equals the reserve. Our results therefore fully characterize the existence of low revenue equilibria in terms of bidder demand at a given reserve price. This sharp characterization extends directly to the case of stochastic supply, and low revenue equilibria also fail to exist when supply is stochastic and elastic.

1. Introduction

Uniform price auctions for multiple units frequently admit low revenue, collusive-seeming equilibria.[1] In these equilibria bidders implicitly agree on a joint allocation and submit large bids for their individual allocations, independent of their actual values for the items, while bidding zero for all other quantities. These bids ensure that the good is allocated deterministically, and the market clearing price is always zero. Reserve prices are implemented to guard against such outcomes: by accepting only bids above some specified threshold, it is certain that whenever the good is sold per-unit revenue will be weakly above the specified price.[2] We show that reserve prices also affect expected revenues through equilibrium selection: the introduction of a reserve price ensures that there is occasionally competition for aggregate supply, unraveling putative equilibria where bidders appear to coordinate on the reserve price.

We define a low revenue equilibrium as an equilibrium in which the per-unit price is almost always equal to the reserve price, and show that reserve prices eliminate such equilibria. The mechanism by which these equilibria are eliminated is

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[1] Such equilibria are "collusive-seeming" because they implement natural collusive outcomes through standard Nash equilibria, and do not rely on any collusive agreement.

[2] Binding reserve prices introduce inefficiency in auction outcomes, and an injudiciously selected reserve price can lower expected revenues. We ignore the question of selecting a revenue-maximizing reserve price and focus on the ability of reserve prices to eliminate low revenue equilibria, similar to equilibrium selection.

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straightforward. In a low revenue equilibrium without a reserve price, submitted bids are high for all units won and zero for all units lost. When the market price is determined by the first rejected bid, the low bids determine the seller's revenue. With the introduction of a reserve price, winning a unit is unprofitable when the bidder's value for the unit is below the reserve price. Then, in equilibrium, bids must occasionally be below the reserve price, and bidders are noncompetitors with positive probability. If equilibrium per-unit revenues are almost always equal to the reserve price, equilibrium strategies have the same general structure as with no reserve, where allocations are constant among bidders, up to tiebreaking at the reserve. Assuming that any group of $n-1$ bidders has excess demand at the reserve price, and any bidder is noncompetitive with positive probability, there is positive probability of excess demand at prices slightly above the reserve, and tiebreaking at the reserve price is occasionally necessary. As in other auction settings, tiebreaking cannot occur in equilibrium, and low revenue equilibria fail to exist.

Our results are similar to those obtained in models of elastic supply (LiCalzi and Pavan, 2005) and adjustable supply (Back and Zender, 2001; McAdams, 2007).[3] In common-value environments without private information, these papers show that low revenue equilibria can be eliminated with appropriate supply elasticity. McAdams (2007) further shows that ex post adjustable supply can revenue-dominate any reserve price. Aside from the distinction that we work in a more general signal-value framework, key intuitions pass through to our model.[4] Nonetheless our focus is distinct, in that we first focus on the elimination of low revenue equilibria through reserve prices alone, and later give a sufficient condition for an elastic supply curve to eliminate low revenue equilibria, in a general value framework. This contrasts with results for common value auctions, including Back and Zender (1993), LiCalzi and Pavan (2005), and McAdams (2007), which show that reserve prices *cannot* eliminate certain low revenue equilibria. Back and Zender (1993) shows that low revenue equilibria exist in common-value auctions when the reserve price is below the minimum value for a unit and when there are only $n=2$ bidders; our results assume the reserve is above the minimum value, and implicitly require that there are $n \geq 3$ bidders. The chief modeling distinction between LiCalzi and Pavan (2005) and McAdams (2007) and this paper is their use of a full-information common-value context, as opposed to our assumption that bidders retain private information; full information eliminates the possibility that the reserve price causes occasional nonparticipation. In the language of our results, residual competition is eliminated when there are only two bidders, or when bidders have identical information.

More broadly, our work adds to a literature on modifying multi-unit auctions with small adjustments to eliminate undesirable equilibria. As discussed above, elastic supply can be used to eliminate low revenue equilibria in common-value auctions, and we extend this result to a general value framework. These equilibria are also eliminated when the market price is set at the last accepted bid rather than the first rejected bid (Burkett and Woodward, 2020), when the space of acceptable bids is coarse (Kremer and Nyborg, 2004b), when tiebreaking rules are carefully selected (Kremer and Nyborg, 2004a), and when a "buy-it-now" opportunity is available (Tsuchihashi, 2016). Blume and Heidhues (2004) show that in a single-unit second price auction with at least three bidders, reserve prices induce truthful reporting, and therefore eliminate low revenue equilibria; Blume et al. (2009) extend this to the case of the multi-unit Vickrey auction. Since degenerate, single unit first rejected bid uniform price auctions are second price auctions, our results have clear ties to Blume and Heidhues (2004).[5]

The existence of zero revenue equilibria in uniform price auctions without reserve prices means that collusive outcomes may not be distinguishable from equilibrium outcomes — under optimal collusion, bidders obtain all units at zero price. For this reason zero revenue equilibria are considered "collusive-seeming" equilibria. Although we establish that uniform price auctions with reserve prices do not admit low revenue equilibrium with the (external) appearance of collusion, we do not make any claim to eliminate formal collusive agreements.[6] Chassang and Ortner (2015) show that constraining bids from above inhibits the ability of participants to tacitly collude. As in our model, the modification of the mechanism (in their case, constraining bids above) acts as a tool for equilibrium selection, eliminating "bad" equilibria. A useful implication of our results is that reserve prices can potentially aid in the detection of collusion, even with minimal effect on the operation of the mechanism. Even a small reserve price must, in equilibrium, occasionally yield market prices which are strictly above the reserve, thus a concerned regulator observing a sequence of market prices could test whether bidders are actually colluding.

This paper proceeds in Section 2 by clarifying the breakdown of canonical low revenue equilibria in the presence of a reserve price. Section 3 sets out our general model, and Section 4 gives our main result: low revenue equilibria cannot exist in uniform price auctions under a broad range of reserve prices. The condition for nonexistence is sharp, and low revenue equilibria do exist when it is not satisfied. Section 5 extends our nonexistence result to stochastic and elastic aggregate supply, and extends our existence result to stochastic and inelastic supply.[7]

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[3] In an experimental setting, Sade et al. (2005) show that adjustable supply can improve uniform price auction revenue. However, subjects do not play equilibrium strategies, and the effect is not statistically significant.

[4] The possible elimination of low revenue equilibria via reserve prices is natural in this context, since reserve prices are implementable as infinitely-elastic supply curves.

[5] Blume and Heidhues (2004) show that equilibrium is unique in single-unit second price auctions with a nontrivial reserve price and $n \geq 3$ bidders. We do not obtain any results on equilibrium uniqueness in similar uniform price auctions.

[6] See, e.g., the seminal works of Graham and Marshall (1987) and McAfee and McMillan (1992).

[7] The nonexistence result in Section 5 implies the main nonexistence result in Section 4, at the cost of additional notational overhead. The economic principles underpinning nonexistence are clearer in Section 4, because there is no need to consider random realizations of supply.

2. Example: breakdown of low revenue equilibrium

There are $n \geq 3$ bidders, $i \in \{1, \ldots, n\}$, participating in a first rejected bid uniform price auction for $Q = n$ units. Each bidder i has independent, private value $v_i \sim \mathcal{U}(0, 1)$ for each unit she obtains.[8] The auction has reserve price $r \in [0, 1]$. Bidder i submits a weakly decreasing bid vector $b_i \geq 0$. Units are allocated to the highest n bids above the reserve price r, and for each unit she obtains, bidder i pays the higher of the $n+1$th-highest bid and the reserve price. Where necessary, ties are broken randomly.[9]

When the reserve price is trivial, $r = 0$,[10] there are many zero-revenue equilibria, sharing a common form. Let $b_i = (1, 0, \ldots, 0) \in \mathbb{R}_+^n$ for all bidders i.[11] Then, independent of type realizations, the highest n bids all equal 1 and the $n+1$th-highest bid is 0. All bidders receive one unit at a price of 0, hence each bidder's interim utility is v_i. This cannot be improved by any deviation: to improve upon receiving a single unit at zero cost would require receiving more than one unit, and to receive more than one unit requires increasing the price per unit to at least 1, which is always above the value for the unit. Then all bidders are best-responding. Since the $n+1$th-highest bid is always zero, the seller's revenue is identically zero.

It is clear that each bidder's strategy must be modified with the introduction of a nontrivial reserve price $r > 0$: bidder i never wants to win an item at a price of r when her value is $v_i < r$. A plausible modification of her original bid is

$$b_i^r(v_i) = \begin{cases} (1, 0, \ldots, 0) & \text{if } v \geq r, \\ (0, 0, \ldots, 0) & \text{otherwise.} \end{cases}$$

If bidder i employs bidding strategy b_i^r while her opponents $j \neq i$ retain type-independent strategies b_j, there is excess supply whenever $v_i < r$: only $n-1$ units are demanded above the reserve price, but n units are available. Then any bidder $j \neq i$ can occasionally improve her utility by submitting the bid

$$b_j^r(v) = \begin{cases} (1, r, 0, \ldots, 0) & \text{if } v \geq r, \\ (0, 0, 0, \ldots, 0) & \text{otherwise.} \end{cases}$$

If all bidders j implement this strategy, ties will need to be broken at price r with positive probability. Tiebreaking implies that some bidder $j \neq 1$ prefers to bid slightly higher, and this is sufficient to break equilibrium.

The economic mechanism behind this unraveling is straightforward. Pinning ex post per-unit revenue to the reserve price (or zero, in the case of no sale) requires the use of bids which can dissuade opponents from entering. In the presence of a nontrivial reserve price, this implies a mass point in the stochastic residual demand curve at the reserve price r; in a low revenue equilibrium it is without loss of generality to assume that there is another mass point at the maximum undominated bid $b = 1$. Any bidder with value between two mass points in the allocation function should bid (weakly) between these two points, and tiebreaking implies that no two bidders should share a common mass point in their (individual) bid distribution functions. Then it is not possible for equilibrium to always generate low revenue.

This unraveling need not occur when there are only $n = 2$ bidders. In the event that bidder 1's value $v_1 < r$ and she prefers to stay out of the auction, bidder 2 will receive every unit for which she submits a bid $b_2 \geq r$. Then bidder 2 does not face any meaningful competition at this price, and can submit a bid of r for the second unit and win it whenever bidder 1's value is low. Residual competition is essential to the elimination of low revenue equilibria.

3. Model

A single seller is allocating $Q \in \mathbb{N}$ units in a first rejected bid uniform-price auction with reserve price r.[12] There are $n \geq 3$ bidders, $i \in \{1, \ldots, n\}$, participating in the auction. Bidder i has signal $s_i \sim F^i$ with support $[0, 1]^d$, and value profile $v^i(s_i, s_{-i}) \in \mathbb{R}_+^Q$ which is coordinatewise weakly decreasing, $v_k^i(s_i, s_{-i}) \geq v_{k+1}^i(s_i, s_{-i})$ for all $k \in \{1, \ldots, Q-1\}$.[13] The value profile v^i is monotone increasing in all arguments, and the support of the conditional value profile $v^i(\cdot, s_{-i})$ is a closed subset of $[0, 1]^Q$, contains 0, and is constant in s_{-i}. The conditional support of signal $s_i | s_{-i}$ is also constant in s_{-i}. Subject

[8] It is straightforward to extend the results in this section to the case of asymmetric distributions and correlation, as long as the support of opponent types is constant in each agent's private signal. We explicitly allow for this in our general results. The constraint to $Q = n$ units is for expositional purposes only.

[9] As in many auction models, the tiebreaking rule is irrelevant in equilibrium.

[10] A reserve price of zero is equivalent to the standard assumption that bids must be positive, and hence is redundant. In this example, as in the main results, the support of bidder values contains 0. Then any strictly positive reserve price will encourage bidders to occasionally demand fewer units than they value, affecting the form of equilibrium strategies. In this sense, $r = 0$ is "trivial," while $r > 0$ is "nontrivial." If the lower bound of values is some $\underline{v} > 0$, a nontrivial reserve price is such that $r > \underline{v}$.

[11] In the main results we address the genericity of this form of zero-revenue equilibrium bids.

[12] For expositional simplicity, our main results focus on the case of fixed supply. Section 5 extends our results to stochastic and/or elastic supply. Arguments under elastic stochastic supply follow roughly the same path as under a deterministic reserve price, but technical necessities obscure the intuition.

[13] It is not essential that v^i represents bidder i's ex post value profile. All results apply equally in the case in which v^i is bidder i's expected value profile prior to the realization of some exogenous randomness.

to the assumption of conditionally full support, value profiles may depend arbitrarily on bidder signals, which may in turn have nontrivial correlation.

Value distributions which satisfy these assumptions include:

- Independent private values, $v^i(s) = v^i(s_i)$, generated by independent signals;
- Private values, $v^i(s) = v^i(s_i)$, generated by affiliated signals with conditional full support;
- Semi-interdependent values, where each bidder receives a private value signal s_{i1} and a common value signal s_{i2}, and values have a common value component $\sum_{i=1}^n s_{i2}$ and a private value component s_{i1} that ensures full support;
- Common values, generated by affiliated signals with conditional full support, provided the range of potential values is not restricted by opponent signals.[14]

Because the support of bidder i's value profile $v^i(\cdot, s_{-i})$ contains $0 \in \mathbb{R}_+^Q$ and is independent of s_{-i}, for any bidder i, opponent signal profile s_{-i}, and strictly positive reserve price r, there is strictly positive probability that the bidder's value profile is below the reserve, $\Pr(v^i(s_i, s_{-i}) < r) > 0$. Furthermore, given any set of bidders $\mathcal{I}$, there is strictly positive probabiliy that each bidder in the set simultaneously has a value profile strictly below the reserve.

Bidders submit weakly decreasing bid vectors $b \in \mathbb{R}_+^Q$ to the auctioneer. In the (first rejected bid) uniform price auction with reserve price $r \geq 0$, market prices are determined by[15]

$$p^\star = \max\left\{r, \inf\left\{p : \#\left\{(i,k) : b_k^i \geq p\right\} \leq Q\right\}\right\}.$$

The market clearing price $p^\star$ is the lowest price at which the market quantity Q is (weakly) underdemanded. This is equal to the $Q + 1$th-highest bid. Bidder i's allocation is determined by her submitted bid and the market clearing price, and is comprised of her base demand above the market clearing price plus some additional rationed quantity. Let $\underline{q}_i$ be the base quantity assigned to bidder i and Δ_i be her rationable surplus quantity,

$$\underline{q}_i = \max\left\{k : b_k^i > p^\star\right\}, \quad \Delta_i = \#\left\{k : b_k^i = p^\star\right\}.$$

Denote by $G(\cdot; b)$ the distribution over feasible allocations, conditional on submitted bids. For any $q \in \text{Supp}\, G(\cdot; b)$, $\sum_{i=1}^n q_i = \min\{Q, \sum_{i=1}^n q_i + \Delta_i\}$, and for all i, $\underline{q}_i \leq q_i \leq \underline{q}_i + \Delta_i$. Let G^i be the marginal distribution of bidder i's allocation, and assume that G^i depends only on submitted bids. Then bidder i's utility is given by

$$u^i(b; v) = \mathbb{E}_{q_i \sim G^i(\cdot; b)}\left[\sum_{q=1}^{q_i}\left(v_q^i - p^\star\right)\right].$$

Importantly, tiebreaking cannot occur with positive probability in equilibrium: a small deviation will discretely improve at least one bidder's utility.[16] Because tiebreaking is irrelevant to our arguments, where bidding strategies are clear from context we will write $q^i(s)$ for $q^i(b_i(s_i), b_{-i}(s_{-i}))$, the allocation agent i receives when the aggregate type profile is s.

We analyze Bayesian Nash equilibria: a strategy profile $(b^i)_{i=1}^n$ is a Bayesian Nash equilibrium if $b^i(s_i)$ is s_i-almost surely a best response to $b^{-i} = (b^j)_{j \neq i}$.

Remark 1. In the case of (atomless) independent private values, the existence of a pure strategy Bayesian Nash equilibrium in multi-unit uniform price auctions is guaranteed by McAdams (2003) and Reny (2011), but with general interdependent values equilibrium existence remains an open question. In our results ruling out low revenue equilibria, our focus is on the nonexistence of a particular type of equilibrum, and we do not address whether an equilibrium exists in this setting. In our results demonstrating the existence of low revenue equilibria, we prove constructively that an equilibrium exists.

4. Results

Our main results establish that low revenue equilibria cannot exist in the presence of a nontrivial reserve price. Because sufficiently small reserve prices $r > 0$ trivially rule out zero-revenue equilibria,[17] we define a low revenue equilibrium as

[14] For this value specification to be reasonable, it needs to be the case that the support of $v^i(\cdot; s_{-i})$ is independent of s_{-i} for almost all s_{-i}. Since our arguments focus on positive probability events, this assumption on values is sufficient for our results.

[15] In a procurement context, the market clearing price is the maximum that successfully clears the market, and the definition of $p^\star$ must be adjusted accordingly. Subject to this adjustment, all conclusions continue to go through as expected.

[16] In a standard symmetric tiebreaking model, a small upward deviation will discretely improve any tie-broken bidder's utility, provided bids are below values. With general tiebreaking we can only claim that there exists a bidder with a profitable deviation, but this is sufficient. This claim is formalized in the arguments presented in Section 4.

[17] The assumption that r is "sufficiently small" is made only to guarantee that bidders sometimes enter the auction. A too-high reserve price trivially generates a zero-revenue equilibrium by causing all bidders to almost surely remain out of the auction. This notion is formalized in the definition of *excess residual demand*.

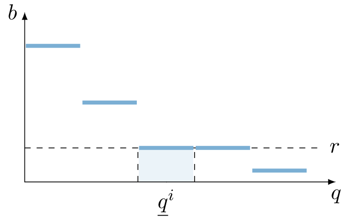

Fig. 1. A bid b has (potentially degenerate) regions strictly above and weakly below the reserve price r.

one in which the market clearing price, equivalent to the per-unit payment to the seller, is almost always weakly below the reserve price. Thus an equilibrium can provide the seller strictly positive expected revenues while still being low revenue.[18]

Definition 1 *(Low revenue).* A strategy profile $(b^i)_{i=1}^n$ is *low revenue* if for s-almost all type profiles $(s_i)_{i=1}^n$,

$$p^\star\left(b^1(s_1), \ldots, b^n(s_n)\right) = r.$$

Our necessary and sufficient condition for the existence of an equilibrium in low revenue strategies (a low revenue equilibrium) is expressed in terms of aggregate demand by all-but-one bidder, at the reserve price r. Define bidder i's *maximum value* for unit k to be $\overline{v}_k^i = \sup\{v_k\colon \Pr(v_k^i(s) \le v_k) < 1\}$, and her *maximum strict demand* at price p to be $\overline{m}^i(p) = \max\{k\colon \overline{v}_k^i > p\}$.

Definition 2 *(Excess residual demand).* A reserve price r *generates excess residual demand* if for all agents i,

$$\sum_{j \neq i} \overline{m}^j(r) > Q.$$

A reserve price r generates excess residual demand if, given a posted price r, there is positive probability that the true aggregate demand of any set of all-but-one bidders exceeds the available supply Q.

Any bid b has (potentially degenerate) ranges on which it is strictly above the reserve price r, on which it equals the reserve price r, and on which it is strictly below the reserve price r. These ranges define a cutoff quantity $\underline{q}^i$, illustrated in Fig. 1, that determines where bids are strictly above the reserve,

$$\underline{q}^i(s_i) = \min\left\{k\colon b_k^i(s_i) \le r\right\} \cup \{0\}.$$

Additionally, there is an (endogenous) minimum possible quantity achievable for any bidder. In a monotone equilibrium this will be the quantity obtained when all opponents receive high signals, however even in nonmonotone equilibria (McAdams, 2007) such a quantity, $q_{\min}^i$, will still exist,

$$q_{\min}^i(s_i) = \min\left\{q\colon \Pr_{s_{-i}}\left(q^i(s_i, s_{-i}) \le q\right) > 0\right\}.$$

Under low revenue strategies there is a natural relationship between $q_{\min}^i$ and $\underline{q}^i$: since the market clearing price is almost always below r and is set by the highest non-accepted bid, it must be that $\underline{q}^i \le q_{\min}^i$.

Lemma 1 *(Low revenue allocations). Under low revenue strategies, $\underline{q}^i(s_i) \le q_{min}^i(s_i)$ for almost all s_i.*

Proof. Suppose otherwise. Then $\Pr_{s_{-i}}(q^i(s_i, s_{-i}) < \underline{q}^i(s_i)) > 0$ for a positive-probability set of s_i. For any such (s_i, s_{-i}), it must be that $p^\star \ge b_{\underline{q}^i(s_i)}^i(s_i) > r$, implying that $p^\star > r$ with strictly positive probability, contradicting low-revenue equilibrium. □

Corollary 1 *(Low-revenue cutoffs). Under low revenue strategies,* $\Pr\left(\sum_{i=1}^n \underline{q}^i(s_i) \le Q\right) = 1$.

Since $\sum_{i=1}^n q^i(s) \le Q$ and $q_{\min}^i(s_i) \le q^i(s)$ for all s, Corollary 1 follows immediately from Lemma 1.

[18] For example, in a two-bidder second-price auction for a single unit, one bidder can submit a strong bid whenever her value is above the reserve while the other bids the reserve when her value is above the reserve. In this equilibrium, the seller's revenue is always weakly less than the reserve price.

The ability of reserve prices to ensure that low revenue strategy profiles are not equilibria hinges on the fact that bidders want to submit low bids when they know their value is almost certainly below the reserve.[19] To support a canonical *zero*-revenue equilibrium, some bidders must commit to bids which are higher than almost all values. Without a reserve price, winning the auction with an unreasonably high bid yields weakly positive utility regardless of the bidder's own type. In the presence of a reserve price, when the bidder's value is extremely low she is better off remaining out of the auction altogether. Thus if a low revenue strategy profile constitutes an equilibrium, each bidder is a nonparticipant with strictly positive probability.

Lemma 2 *(Non-participation). In any equilibrium with a nontrivial reserve $r > 0$, for any bidder i, $b^i_k(s_i) < r$ for all units k with s_i-positive probability.*

Proof. Define s_i to be *low value* if $\Pr_{s_{-i}}(v^i(s_i, s_{-i}) < r) = 1$. By assumption of conditional full support of values, s_i is low value with strictly positive probability. A bidder with a low value signal is willing to bid above the reserve only if she wins zero units with probability one, meaning her opponents are bidding above the reserve with probability one. Because her opponents have low values with positive probability, this implies that some low value opponent wins a strictly positive expected quantity and would be better off bidding below the reserve. It follows that all bidders remain out of the auction when they receive low value signals, and there is strictly positive probability that all bidders do not participate in the auction. □

Because values have conditional full support, if r generates excess residual demand there is positive probability that all bidders have values strictly above the reserve price r. And because signals and values have conditional full support, some bidder who receives a strictly positive allocation when her value is high will, with positive probability, have a low value even when her opponents' values remain high. Then the quantity she would have received if her value were high must be reallocated among her competitors, who will compete to win this residual quantity. This implies that no low revenue strategy profile can be an equilibrium.

Theorem 1 *(Nonexistence of low-revenue equilibrium). If the reserve price $r > 0$ generates excess residual demand, there is no low revenue equilibrium in the first rejected bid uniform price auction.*

Proof. Define s_i to be *high value* if $\Pr_{s_{-i}}(v^i_k(s_i, s_{-i}) > r,\ \forall k \le \overline{m}^j(r)) = 1$. Each bidder i is either high value with positive probability, in which case we let $\overline{S}_i$ be the set of high value types for agent i, or she is almost never high value, in which case we let $\overline{S}_i = \operatorname{Supp} s_i$. Define $\overline{S} = \times_{i=1}^n \overline{S}_i$; by construction, $\Pr(s \in \overline{S}) > 0$. Consider a low revenue strategy profile $(b^i)_{i=1}^n$, and let $q(\overline{S})$ be the set of equilibrium allocations achievable when signals are in $\overline{S}$. Because signals are in $\overline{S}$ with positive probability and allocations are discrete, there is some allocation $q^\star \in q(\overline{S})$ that occurs with strictly positive probability, both conditional on $s \in \overline{S}$ and unconditionally.

Let P be the set of agents receiving a strictly positive allocation at $q^\star$, $P = \{i : q^\star_i > 0\}$. Market supply is positive, $Q > 0$, so $P \neq \emptyset$. Let $x \in \arg\max_{x' \in P} q^\star_{x'}$, and consider the set of signal profiles $\underline{S}$,

$$\underline{S} = \left\{\left(\tilde{s}_x, s_{-x}\right) : s \in \overline{S},\ q\left(s\right) = q^\star,\ \text{and}\ b^x\left(\tilde{s}_x\right) < r\right\}.$$

That is, $\underline{S}$ is the set of signal profiles that, but for the fact that bidder x is low value, are high value and would generate the allocation $q^\star$. Lemma 2 implies that $\Pr(s \in \underline{S}) > 0$. Because the reserve price r generates excess residual demand, there is an agent i such that $q^i(s) < \overline{m}^i(r)$ with positive probability for $s \in \underline{S}$; let U be the set of such agents. Let U' be the set of agents j such that $q^\star_j < q^j(s)$ with positive probability on $\underline{S}$; by Corollary 1, $b^j_{q^\star_j+1}(s_j) \le r$ for all $j \in U'$. Excess residual demand and the fact that x maximizes $q^\star_{x'}$ imply that it is possible to select $i \in U$ and $j \in U'$ with $i \neq j$. Then $b^j_{q^j(s)}(s_j) = r = b^i_{q^i(s)+1}(s_i) < v^i(s_i, s_{-i})$ with positive probability on $\underline{S}$. Then for any $\varepsilon > 0$ bidder i can increase her bid to $b^i(s_i) + \varepsilon$ and ensure that she wins at least one additional unit with strictly positive probability. This deviation will increase the market clearing price by at most ε, and since i is high value when $s \in \underline{S}$, when $\varepsilon > 0$ is sufficiently small this deviation is profitable. It follows that bidder j is not best responding, and $(b^i)_{i=1}^n$ is not an equilibrium. Then a low revenue equilibrium cannot exist. □

The principles voiding low revenue equilibria in multi-unit auctions are essentially the same as those applied to the single-unit second price auction.[20] If a strategy profile induces low revenue, bids must be sufficiently high to dissuade

[19] A similar argument can be made with respect to equilibrium beliefs about values conditional on winning. Arguments are simplified by restricting attention to agents with almost-surely low values.

[20] In the case of multi-unit auctions, truthful reporting typically fails because an agent's supramarginal bids occasionally determine the market price. Then the nonexistence of low revenue equilibrium cannot be demonstrated as a consequence of truthful reporting, but follows instead from arguments similar to those showing that truthful reporting is a dominant strategy in single-unit second price auctions.

competition for units which are obtained, and bids for any marginal units must equal the reserve price. Because bidders occasionally have value profiles which fall below the reserve price, other bidders are occasionally in competition for any given bidder's demand. This competition ensures that bidders will never co-locate mass points at the reserve price, and hence there is a positive incentive to bid slightly above the reserve price when values are sufficiently large. It follows that with a nontrivial reserve price, per-unit revenues must be strictly above the reserve price with positive probability.

Although the intuition driving the result may be expressed in terms of standard competitive pressures, we detail three aspects of the result. First, the role of the nontrivial reserve price $r > 0$ is to ensure that bidders occasionally want to remain out of the auction, because they can never win units at a profitable price (Lemma 2). If the distribution of values had lower bound $\underline{v} > 0$, Theorem 1 would continue to hold for all reserve prices $r > \underline{v}$ which generate excess residual demand. Second, conditional full support allows us to hold constant the set of positive-probability value profiles. While bidders may use their signals to draw inferences about their ex post values, these inferences cannot change the support of feasible values; see Remark 2 below. Finally, the "drop out" bidder k is selected as one who receives the largest allocation at $q^\star$. This ensures that when this bidder drops out, there will be competition for the newly-available units. It is straightforward to construct examples where bidders with lower allocations drop out and the remaining bidders face no new competitive pressures.

Remark 2. The argument in the proof of Theorem 1 does not depend on the distributions of signals and values, except through the assumption that the support of any agent's signal and value profile is independent of her opponents' signals. To see that some form of this assumption is crucial to the argument, consider a type distribution supported by unit vectors where any realized signal profile is such that one agent has a signal $s_i = 1$ and all other agents $j \neq i$ have $s_j = 0$. Bidder i's signal determines her value for quantity q, $v_k^i(s) = s_i$. When bidder i receives signal $s_i = 1$, she knows that she faces no competition in the auction, and can bid the reserve price r for all units. Conditional full support ensures that such inference is not possible.

Theorem 2 *(Existence of low revenue equilibrium). If the reserve price $r > 0$ does not generate excess residual demand, there is a low revenue equilibrium in the first rejected bid uniform price auction.*

Proof. Suppose that r does not generate excess residual demand. Then there is an agent $i^\star$ such that $\sum_{j \neq i^\star} \overline{m}^j(r) \leq Q$. Consider the strategy profile $(b^i)_{i=1}^n$, where $b_k^{i^\star}(s_{i^\star}) \in \{0, r\}$ for all units k and all signals $s_{i^\star}$, and

$$i \neq i^\star \implies b_k^i(s_i) = \begin{cases} 1 & \text{if } \mathbb{E}_{s_{-i}}\left[v_k^i(s_i, s_{-i}) \middle| s_i \right] \geq r, \\ 0 & \text{otherwise.} \end{cases}$$

Since $\sum_{j \neq i^\star} \overline{m}^j(r) \leq Q$ and $b_k^{i^\star} \in \{0, r\}$, the bids of agents $j \neq i^\star$ never set the market clearing price; then since bidder $i^\star$'s bid is either r or zero, the market clearing price is $p^\star = r$. Then any bidder $j \neq i^\star$ receives her full stated demand (all units for which she bids 1) regardless of opponent signal realizations, and there is no information content in the number of units won. Bidder $j \neq i^\star$ receives each unit she unconditionally values above r at a price of r, and her strategy is a best response.

Constructing bidder $i^\star$'s strategy is more delicate, since there is potentially information content in her allocation. Because bidders $j \neq i^\star$ are submitting bids that are either 0 or 1, given any bid vector $b \in \{0, r\}^Q$ bidder $i^\star$ cannot affect her resulting allocation unless she bids 1 (weakly above her value) or 0 (sacrificing the unit). Then it is a best response for her to submit a bid vector such that her bid for any unit is r if her expected value, conditional on winning this unit, is weakly above r, and zero otherwise. □

Corollary 2 *(Low revenue equilibrium with two bidders). The first rejected bid uniform price auction with $n = 2$ bidders admits a low-revenue equilibrium.*

Putting together Theorem 1 and Theorem 2 gives a complete characterization of when low-revenue equilibria exist.

Proposition 1 *(Characterization of low revenue equilibrium). The first rejected bid uniform price auction with reserve price $r > 0$ admits a low revenue equilibrium if and only if r does not generate excess residual demand.*

Given the relative generality of our model of the first rejected bid uniform price auction, Proposition 1 suggests that bootstrapping competition by implementing a reserve price can be helpful in a wide range of circumstances.

5. Stochastic and elastic supply

Adapting the results to elastic and stochastic supply requires a modification of the definition of low revenue equilibrium. Let $Q : \mathbb{R}_+ \times \Theta \to \mathbb{N}$ be an elastic supply function, so that market supply at price p is $\mathbf{Q} = Q(p; \theta)$, where $\theta \sim G$ has support Θ and is independent of s_i for all bidders i. We assume that Range Q is a finite subset of $\mathbb{N}$, and that Q is

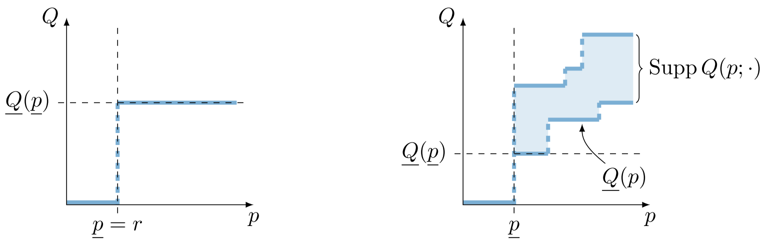

Fig. 2. The value $\underline{p}$ and function $\underline{Q}$ for a reserve price r (left panel), and for elastic stochastic supply Q (right panel).

monotonically increasing in p. Because the set of available quantities is finite and Q is monotone, Q may be interpreted as a right-continuous step function in p. We then redefine the market clearing price,

$$p^\star(b;\theta) = \inf\left\{p : \#\left\{(i,k) : b_k^i \geq p\right\} \leq Q(p;\theta)\right\}.$$

Since Q is increasing in p, an appropriate $p^\star$ always exists.

Defining a low revenue strategy profile when supply is elastic requires defining the minimum feasible market clearing price. For any price p, let $\underline{Q}(p)$ be the minimum feasible supply at p,

$$\underline{Q}(p) = \inf\left\{\tilde{Q} : \Pr\left(Q(p;\theta) > \tilde{Q}\right) > 0\right\}.$$

Because the distribution of aggregate quantity is discrete, $\Pr_\theta(\mathbf{Q} = \underline{Q}(p)) > 0$. The minimum feasible market clearing price is defined where the minimum feasible supply is strictly positive. Let $\underline{p} = \min\{p : \underline{Q}(p) > 0\}$; because Q is right continuous, $\underline{p}$ is well defined. These values are illustrated in Fig. 2.

Definition 3 *(Sometimes excess residual demand).* The elastic supply curve Q *sometimes generates excess residual demand* if for all agents i,

$$\sum_{j\neq i}\overline{m}^j\left(\underline{p}\right) > \underline{Q}\left(\underline{p}\right).$$

If Q does not sometimes generate excess residual demand, it *never generates excess residual demand*.

Some examples clarify this definition:

- When stochastic elastic supply Q is equivalent to deterministic supply $\tilde{Q}$ at reserve price r, $Q(p;\theta) = \tilde{Q}1[p \geq r]$, Q sometimes generates excess residual demand if and only if the reserve price r generates excess residual demand.
- When stochastic elastic supply Q is equivalent to stochastic supply $\tilde{Q}(\theta)$ at reserve price r, $Q(p;\theta) = \tilde{Q}(\theta)1[p \geq r]$, Q sometimes generates excess residual demand if and only if the reserve price r generates excess residual demand at the minimum (nonzero) feasible supply, $\min\{\tilde{Q}(\theta) : \tilde{Q}(\theta) > 0\}$.
- When stochastic elastic supply Q is equivalent to deterministic supply $\tilde{Q}$ under a nondegenerate supply curve, $Q(p;\theta) = \tilde{Q}(p)$, Q sometimes generates excess residual demand if and only if the minimum positive-supply price $\underline{p}$ generates excess residual demand at $\tilde{Q}(\underline{p})$.

Definition 4 *(Essentially low revenue).* A strategy profile $(b^i)_{i=1}^n$ is *essentially low revenue* if with (s,θ)-probability one,

$$p^\star\left(b^1(s_1),\ldots,b^n(s_n);\theta\right) = \underline{p}.$$

When the elastic supply curve Q is equivalent to a reserve price, the definition of an essentially low revenue strategy profile is equivalent to the definition of a low revenue strategy profile. Showing that there is no essentially low revenue equilibrium when Q sometimes generates excess residual demand is substantially similar to the proof of the nonexistence of low revenue equilibrium when a reserve price generates excess residual demand. A technical complication regarding the distribution of supply randomness θ must be dealt with, but does not qualitatively affect the results: because the distribution of supply is discrete, the allocation $q^\star$ from the proof of Theorem 1 may still be assumed to arise with positive probability.

Theorem 3 *(Nonexistence of essentially low revenue equilibrium). If the elastic supply curve Q sometimes generates excess residual demand, the first rejected bid uniform price auction does not admit an essentially low revenue equilibrium.*

Proof. Define s_i to be *high value* if $\Pr_{s_{-i}}(v_k^i(s_i, s_{-i}) > \underline{p},\ \forall k \le \overline{m}^j(\underline{p})) = 1$. Each agent i is either high value with positive probability, in which case we let $\overline{S}_i$ be the set of high value types for agent i, or she is almost never high value, in which case we let $\overline{S}_i = \operatorname{Supp} s_i$. Define $\overline{S} = \times_{i=1}^n \overline{S}_i$; by construction, $\Pr(s \in \overline{S}) > 0$. Define $\underline{\Theta} = \{\theta \in \operatorname{Supp} G : Q(\underline{p}, \theta) = \underline{Q}(\underline{p})\}$ to be the set of aggregate supply-minimizing realizations of θ, conditional on $\underline{p}$; by definition of $\underline{Q}$, $\Pr(\theta \in \underline{Q}(\underline{p})) > 0$. Consider an essentially low revenue strategy profile $(b^i)_{i=1}^n$, and let $q(\overline{S}; \underline{\Theta})$ be the set of allocations achievable when signals are in $\overline{S}$ and aggregate supply is minimized, $\theta \in \underline{\Theta}$. Because signals are in $\overline{S}$ with positive probability and allocations are discrete, there is some allocation $q^\star \in q(\overline{S}; \underline{\Theta})$ that occurs with strictly positive probability, both conditional on $(s, \theta) \in \overline{S} \times \underline{\Theta}$ and unconditionally.

Let P be the set of agents receiving a strictly positive allocation at $q^\star$, $P = \{i : q_i^\star > 0\}$. Market supply is positive at $\underline{p}$, $Q(\underline{p}; \theta) > 0$ for all $\theta \in \underline{\Theta}$, so $P \neq \emptyset$. Let $x \in \arg\max_{x' \in P} q_{x'}^\star$, and consider the set of random realizations $\underline{T}$,

$$\underline{T} = \left\{ \left(\tilde{s}_x, s_{-x}, \theta\right) : s \in \overline{S},\ \theta \in \underline{\theta},\ q(s; \theta) = q^\star,\ \text{and } b^x\left(\tilde{s}_x\right) < \underline{p} \right\}.$$

That is, $\underline{T}$ is the set of random realizations which, but for the fact that bidder k is low value, would generate the allocation $q^\star$ when bidders are high value. An adaptation of Lemma 2 implies that $\Pr((s, \theta) \in \underline{T}) > 0$. Because market price $\underline{p}$ generates excess residual demand when supply is minimized, there is an agent i such that $q^i(s; \theta) < \overline{m}^i(\underline{p})$ with positive probability for $(s, \theta) \in \underline{T}$; let U be the set of such agents. Let U' be the set of agents j such that $q_j^\star < q^j(s; \theta)$ with positive probability on $\underline{S}$; an adaptation of Corollary 1 implies that $b^j_{q_j^\star + 1}(s_j) \le \underline{p}$ for all $j \in U'$. Sometimes excess residual demand and the fact that x maximizes $q_{x'}^\star$ imply that it is possible to select $i \in U$ and $j \in U'$ with $i \neq j$. Then $b^j_{q^j(s;\theta)}(s_j) = \underline{p} = b^i_{q^i(s;\theta)+1}(s_i) < v^i(s_i, s_{-i})$ with positive probability on $\underline{T}$. Then for any $\varepsilon > 0$ bidder i can increase her bid to $b^i(s_i) + \varepsilon$ and ensure that she wins at least one additional unit with strictly positive probability. This deviation will increase the market clearing price by at most ε, and since i is high value when $(s, \theta) \in \underline{T}$, for $\varepsilon > 0$ sufficiently small this deviation is profitable. It follows that an essentially low revenue equilibrium cannot exist. □

The construction used to prove the existence of low revenue equilibrium when a reserve price does not satisfy excess residual demand (Theorem 2) cannot be applied to the case of elastic supply. For intuition, it is sufficient to consider an essentially low revenue strategy profile in which $n - 1$ bidders submit high bids for each unit they value above $\underline{p}$, and the remaining *residual* bidder bids $\underline{p}$ for each unit they value above $\underline{p}$. With a simple reserve price, the residual bidder's decision is straightforward: she can increase her allocation only by bidding above her highest possible value, simultaneously increasing the market clearing price. This is never profitable, hence she is best responding by bidding the reserve price. Under elastic supply, it may be possible for a slight increase in her bid to increase *aggregate* supply, and for this deviation to be profitable even net of the associated increase in market clearing price. Whether or not this is profitable will depend on the correlation of types, the relationship of values to types, the elasticity of supply, and its conditional distribution. A clean characterization of when an essentially low revenue equilibrium does or does not exist is infeasible.

Nonetheless, a partial converse to Theorem 3 is possible when attention is constrained to stochastic supply.

Definition 5 *(Reserve price implementation).* The elastic supply curve Q *implements random supply at reserve price* r if for all (p, θ),

$$Q(p; \theta) = \begin{cases} \tilde{Q}(\theta) & \text{if } p \ge r, \\ 0 & \text{otherwise.} \end{cases}$$

When elastic supply implements a reserve price, an essentially low revenue equilibrium will exist whenever the supply curve never generates excess residual demand. This result is a generalization of Theorem 2, which is constrained to deterministic supply.

Theorem 4 *(Existence of essentially low revenue equilibria under stochastic supply). Suppose that the elastic supply curve Q implements random supply at reserve price r. If Q never generates excess residual supply, the first rejected bid uniform price auction admits an essentially low revenue equilibrium.*

Proof. The proof is identical to that of Theorem 2, replacing Q with $\underline{Q}(r)$. □

Theorems 3 and 4 together imply a full characterization of essentially low revenue equilibria with random supply and a reserve price.

Proposition 2 *(Characterization of essentially low revenue equilibria under stochastic supply). Suppose that the elastic supply curve Q implements random supply at reserve price r. The first rejected bid auction admits an essentially low revenue equilibrium if and only if Q never generates excess residual supply.*

6. Conclusion

We have defined a low revenue equilibrium in a multi-unit auction to be such that the market clearing price almost always equals the reserve price, and have shown that a simple condition on demand is necessary and sufficient for the existence of low revenue equilibria in first rejected bid uniform price auctions. Under a general family of value specifications, low revenue equilibria exist if and only if, with strictly positive probability, any bidder's opponents' demand at the reservation price exceeds aggregate supply. This result extends to the case of stochastic supply, but only the necessary condition extends to general elastic supply curves.

The implementation of a reserve price may then have two effects which are beneficial to the auctioneer. An appropriate reserve price exogenously improves revenue by setting a lower bound for payment. We show that reserve prices may also have an equilibrium selection effect, and imply that collusive bidding strategies are not sustainable in Bayesian Nash equilibrium. Although previous results (Back and Zender (2001), LiCalzi and Pavan (2005), McAdams (2007)) have shown that elastic or adjustable supply is sufficient to eliminate low revenue equilibria, reserve prices may be substantially simpler to implement, and have the same power to eliminate low revenue equilibria. In multi-unit uniform price auctions, reserve prices are a simple yet potentially powerful tool for obtaining auctioneer-preferred outcomes.

References

Back, Kerry, Zender, Jaime F., 1993. Auctions of divisible goods: on the rationale for the treasury experiment. Rev. Financ. Stud. 6 (4), 733–764.
Back, Kerry, Zender, Jaime F., 2001. Auctions of divisible goods with endogenous supply. Econ. Lett. 73 (1), 29–34.
Blume, Andreas, Heidhues, Paul, 2004. All equilibria of the Vickrey auction. J. Econ. Theory 114 (1), 170–177.
Blume, Andreas, Heidhues, Paul, Lafky, Jonathan, Münster, Johannes, Zhang, Meixia, 2009. All equilibria of the multi-unit Vickrey auction. Games Econ. Behav. 66 (2), 729–741.
Burkett, Justin, Woodward, Kyle, 2020. Uniform price auctions with a last accepted bid pricing rule. J. Econ. Theory.
Chassang, Sylvain, Ortner, Juan, 2015. Collusion in Auctions with Constrained Bids: Theory and Evidence from Public Procurement.
Graham, Daniel A., Marshall, Robert C., 1987. Collusive bidder behavior at single-object second-price and English auctions. J. Polit. Econ. 95 (6), 1217–1239.
Kremer, Ilan, Nyborg, Kjell G., 2004a. Divisible-good auctions: the role of allocation rules. Rand J. Econ., 147–159.
Kremer, Ilan, Nyborg, Kjell G., 2004b. Underpricing and market power in uniform price auctions. Rev. Financ. Stud. 17 (3), 849–877.
LiCalzi, Marco, Pavan, Alessandro, 2005. Tilting the supply schedule to enhance competition in uniform-price auctions. Eur. Econ. Rev. 49 (1), 227–250.
McAdams, David, 2003. Isotone equilibrium in games of incomplete information. Econometrica 71 (4), 1191–1214.
McAdams, David, 2007. Adjustable supply in uniform price auctions: non-commitment as a strategic tool. Econ. Lett. 95 (1), 48–53.
McAfee, R. Preston, McMillan, John, 1992. Bidding rings. Am. Econ. Rev., 579–599.
Reny, Philip J., 2011. On the existence of monotone pure-strategy equilibria in Bayesian games. Econometrica 79 (2), 499–553.
Sade, Orly, Schnitzlein, Charles, Zender, Jaime F., 2005. Competition and cooperation in divisible good auctions: an experimental examination. Rev. Financ. Stud. 19 (1), 195–235.
Tsuchihashi, Toshihiro, 2016. A revenue-enhancing effect of a buyout price in multi-unit uniform-price auctions.