Production with multiple firms

Econ 11, Lecture 15

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Aggregate production

We know what the world looks like when there is a single firm operating in a vacuum.

- Costs are minimized subject to quantity constraints
- Overall quantity is selected on the basis of whether or not the firm is a price-taker

Of course, this isn't incredibly realistic: in the real world there is more than one firm. As we did with consumers, we generalize our viewpoint to consider a world with *two* firms. The tools we develop will apply with larger numbers, but we have to start somewhere.

What to look for

As a first pass, we'll consider what firms are doing together without considering what consumers are doing.

- An economy is constrained by the total amount of capital K and labor L available
- There are implicit tradeoffs: if firm A uses more capital, then firm B must use less
- Factor/input market clearing:

$$k_A + k_B = K, \qquad \ell_A + \ell_B = L$$

Later, this will be generalized to consider the amount of labor and capital *supplied* in an economy, not just the raw amount available. In the view of the world we have now, labor may be impressed into service against its will.

Efficiency of production: math

- Firm A uses inputs (k_A, ℓ_A) to produce output $f_A(k_A, \ell_A) = q_A$
- Firm B uses inputs (k_B, ℓ_B) to produce output $f_B(k_B, \ell_B) = q_B$

Definition

Production (q_A, q_B) is *efficient* if there is no reallocation of inputs (k'_A, ℓ'_A) and (k'_B, ℓ'_B) with

$$k'_A + k'_B \leq k_A + k_B \quad \text{and} \quad \ell'_A + \ell'_B \leq \ell_A + \ell_B$$

and

$$f_A(k'_A, \ell'_A) = q'_A \geq q_A \quad \text{and} \quad f_B(k'_B, \ell'_B) = q'_B \geq q_B,$$

with *strict* inequality for either firm A or firm B.

Efficiency of production: words

Definition

Production (q_A, q_B) is *efficient* if aggregate production cannot be increased without increasing the aggregate use of capital and labor in the economy. Alternatively, producton (q_A, q_B) is *efficient* if every reallocation of labor and capital — keeping aggregate supplies constant — results in lower production of one commodity.

This looks a lot like Pareto optimality: efficiency of production is to firms as Pareto optimality is to consumers.

Efficiency of production: meaning

Suppose that (q_A, q_B) is *inefficient*, is produced from (k_A, ℓ_A) and (k_B, ℓ_B), and that firms are price-takers.

- There is (q'_A, q'_B) with $q'_A > q_A$ and $q'_B \geq q_B$ (or vice-versa) and

$$k'_A + k'_B \leq k_A + k_B \quad \text{and} \quad \ell'_A + \ell'_B \leq \ell_A + \ell_B$$

- By definition, aggregate revenues are higher:

$$p_A q'_A + p_B q'_B > p_A q_A + p_B q_B$$

- (continued)

Efficiency of production: meaning

- Aggregate cost does not change:

$$\begin{aligned} C(q'_A) + C(q'_B) &= (p_k k'_A + p_\ell \ell'_A) + (p_k k'_B + p_\ell \ell'_B) \\ &= p_k(k'_A + k'_B) + p_\ell(\ell'_A + \ell'_B) \\ &\geq p_k(k_A + k_B) + p_\ell(\ell_A + \ell_B) \\ &= C(q_A) + C(q_B) \end{aligned}$$

Then aggregate costs have (weakly) fallen and aggregate revenues have strictly increased: aggregate profits are higher! In some sense, when production is inefficient then *society* may be made better off via an alternate production plan.

Example

- *Firm A has production function* $f_A(k_A, \ell_A) = \min\{k_A, \ell_A\}$
- *Firm B has production function* $f_B(k_B, \ell_B) = \min\{k_B, \ell_B\}$
- *Inputs are* $(k_A, \ell_A) = (1, 0)$ *and* $(k_B, \ell_B) = (0, 1)$

Is production efficient? If not, what is one efficient production plan in this economy?

What can be produced?

If production of (q_A, q_B) is efficient, then this pair of outputs lies on the *production possibility frontier.*

Definition

The *production possibility frontier* is the set of all outputs which are efficient in production, given aggregate capital and labor.

This may look like a contract curve for firms, but it is not! The frontier is not the set of *inputs* which lead to efficient production, but rather the set of efficient productions themselves.

Aside: any *inefficient* level of production lies strictly within the production possibility frontier, and may be obtained by misallocating input factors between the two firms.

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Marginal rate of technical substitution

Definition

The *marginal rate of technical substitution* (alternately, *rate of technical substitution*) for a firm is

$$\text{MRTS}_{KL} = \frac{\text{MPK}}{\text{MPL}}.$$

Remember that for consumers the marginal rate of substitution is $\text{MRS}_{xy} = \text{MU}_x/\text{MU}_y$; the marginal rate of technical substitution is an analogue for firms.

MRS tells how much y is needed to make up from losses of x in order to hold utility constant; MRTS tells how much labor is needed to make up for losses in capital to hold output constant: it is the *rate* at which the firm can *substitute* between inputs without reducing output.

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Finding the PPF: story

Suppose

- Firm A has $\text{MRTS}^A_{KL} = 1$
- Firm B has $\text{MRTS}^B_{KL} = 2$

Consider the following reallocation:

- Firm A gives firm B one unit of capital
- Firm B gives firm A one unit of labor

Since $\text{MRTS}^A_{KL} = 1$, firm A's output does not change; but since $\text{MRTS}^B_{KL} = 2$ and firm B has given up only one unit of labor in exchange for one unit of capital, its output has increased.

Finding the PPF: short story

If firm B's output can increase while holding firm A's output constant, without introducing more capital or labor into the system, then original production is inefficient. To have efficient production, then, we need[1]

$$\text{MRTS}^A_{KL} = \text{MRTS}^B_{KL}.$$

This is almost exactly how we found Pareto optima.

[1] Ignoring ugly corner solutions.

Example

- *Firm A has production function* $f_A(k_A, \ell_A) = k_A^{1/3} \ell_A^{2/3}$
- *Firm B has production function* $f_B(k_B, \ell_B) = k_B^{1/4} \ell_B^{1/2}$
- *There is* $K = 1$ *unit of capital in the economy*
- *There are* $L = 2$ *units of labor in the economy*

Find the production possibility frontier.

What about prices?

This is not the full story: in an economy, firms form production plans based on market prices for capital and labor.

- Remember that to minimize cost (and depending on your TA),

$$\frac{\text{MPK}}{p_k} = \frac{\text{MPL}}{p_\ell} \quad \text{or} \quad \frac{\text{MPK}}{\text{MPL}} = \frac{p_k}{p_\ell}$$

- If there are two firms, A and B, both minimizing costs,

$$\text{MRTS}^A_{KL} = \frac{\text{MPK}^A}{\text{MPL}^A} = \frac{p_k}{p_\ell} = \frac{\text{MPK}^B}{\text{MPL}^B} = \text{MRTS}^B_{KL}$$

Then when both firms minimize costs subject to market prices, the resulting outcomes satisfy the conditions for efficiency of production.

The magic of prices

This tells us something about equilibrium:

- Production will be efficient
- Ergo, prices in equilibrium must be such that production is efficient and input factor markets clear

Countries as firms

There is no reason[2] that firms should have unique, differentiable outputs.

- Coca-Cola produces several types of sodas
- Cadillac produces more than one type of car

We can conceptualize countries as firms with multiple outputs.

- A country allocates resources to the production of different goods (sometimes implicitly)
- The nation as a whole faces some cost function regarding aggregate production

[2] Other than keeping our math simple.

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An example: setup

There are two countries, the US and France, and two commodities, hamburgers (H) and fries (R). There is one input to production, labor. Production functions are given by

$$f_H^{US}(\ell_H) = 4\ell_H, \qquad f_R^{US}(\ell_R) = 8\ell_R,$$
$$f_H^{F}(\ell_H) = \ell_H, \qquad f_R^{F}(\ell_R) = 6\ell_R.$$

In a world without trade, the US must produce all the hamburgers and fries its population would like to consume, and France must produce all the hamburgers and fries its population would like to consume. Let's assume that nobody in either country is happy consuming 0 hamburgers or 0 fries, so both the US and France produce both hamburgers and french fries.

An example: trade

What happens when we allow for trade of commodities?

- Labor is stuck in the country in which it starts: there is no immigration
- Notice that the US is better at producing *both* hamburgers and french fries, in terms of labor productivity

If the US is better at everything, will France produce anything?

- French employees are stuck in France: can't go to the US and be more productive
- Therefore French labor has to be allocated to French production somehow; doesn't make sense to throw it away
- Relatedly, the US is better at producing fries than hamburgers; will it produce any hamburgers?

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An example: outputs

The question we should ask is, *does it make sense for both countries to produce both outputs?*

- Suppose the US produces q_H^{US} hamburgers and q_R^{US} fries; France produces q_H^{F} hamburgers and q_R^{F} fries
- Global production is $q_H^{\text{US}} + q_H^{\text{F}}$ hamburgers and $q_R^{\text{US}} + q_R^{\text{F}}$ fries

1. Suppose that France substitutes 2 units of hamburger labor into fry production; French production of hamburgers falls by 2, but production of fries increases by 12
2. The US substitutes 1 unit of fry labor into hamburger production; US production of fries falls by 8, but production of hamburgers increases by 4
3. Global production of hamburgers has increased by $4 - 2 = 2$ and global production of french fries has increased by $12 - 8 = 4$; there is more to go around!

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Comparative advantage

France has a *comparative advantage* in production of french fries: even though it is less effective overall at producing each good, it is relatively more effective at producing fries relative to hamburgers than the US is.

1. France should produce some fries
2. Similarly, the US should produce some hamburgers
3. *At least one of* the US and France should produce *none* of the good that it is relatively less effective at producing

The actual outputs will depend on market size and consumer demand. Comparative advantage gives us a tool to asses how firms (countries) specialize toward production of one good or another.

Common technology

There is a difference between a Schwinn and a toothpick.

- One is a bicycle and one is a toothpick
- No other bicycles are Schwinns, but a toothpick is pretty much a toothpick no matter who makes it

We have focused so far on *product differentiation*: Trek and Cannondale also make bicycles, but you can tell which is which (if only from the logo). There is no such ability with toothpicks. However, there is another important feature of toothpicks:

- All toothpicks are made from logs
- There is only one way to make a toothpick from a log[3]

That is, all firms producing toothpicks are using the *same* technology; they have identical production functions.

[3] It involves a lot of sawdust.

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Maximum production

What happens when there are many firms producing a homogeneous (undifferentiable) good from identical technology? Suppose that an industry has two firms with identical technology $f(k,\ell) = k^r \ell^t$ and that there is $K = 1$ unit of capital and $L = 1$ unit of labor in the economy. The question of efficient production changes to, *what is the greatest amount of output that can be had from this economy?*

- We want to maximize $q_A + q_B$
- Could maximize q_A and q_B separately, as when deriving the production possibility frontier
- Could it be better to assign all production to one firm?

Solving

- Compute $\text{MRTS}_{KL} = (\ell/k)(r/t)$
- If equal across firms,

$$\rightsquigarrow \frac{\ell_A}{k_A} = \frac{\ell_B}{k_B} = \frac{1-\ell_A}{1-k_A} \quad \Longrightarrow \quad \ell_A = k_A$$

- Thus $q_A = k_A^{r+t}$ and $q_B = (1-k_A)^{r+t}$
- Total production: $k_A^{r+t} + (1-k_A)^{r+t}$

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Solving (cont.)

- To maximize total output,

$$\frac{\partial}{\partial k_A} : (r+t)\left(k_A^{r+t-1} - (1-k_A)^{r+t-1}\right) = 0$$

- ALGEBRA: $k_A = 1/2$
- Maximum or minimum?

$$\frac{\partial^2}{\partial k_A^2} :(r+t)(r+t-1)\left(k_A^{r+t-2} + (1-k_A)^{r+t-2}\right)$$
$$= \left(\frac{1}{2}\right)^{r+t-1}(r+t)(r+t-1)$$

- Second derivative is negative when $r+t<1$ and positive when $r+t>1$

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- Second derivative is negative when $r+t<1$ and positive when $r+t>1$

Solution

- When $r + t < 1$, want equal inputs to each firm (interior solution)
- When $r + t > 1$, want to assign all input to one firm (boundary solution)

Notice that when $r + t < 1$, the firms have decreasing returns to scale, and when $r + t > 1$ the firms have increasing returns to scale. Here, decreasing returns to scale means that efficient output comes from a competitive allocation between firms; increasing returns to scale means that efficient output comes from allocating all resources to a single firm — this looks troublingly like monopoly being good for society.

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