Mixed-strategy Nash equilibrium

Let's open with the simultaneous-move variant of the sequential game from last week, shown in Figure 1. To

	L_2	R_2
L_1	$4, 3$	$11, 0$
R_1	$8, 0$	$10, 2$

Figure 1: the sequential game from last week, recast as a simultaneous-move game

find Nash equilibrium, we consider what one agent does given what the other agent is doing. For example, if agent 1 is playing L_1, agent 2 prefers playing L_2 to playing R_2 as his payoff will be 3 rather than 0. So that we remember this, we underline the 3 in the box corresponding to (L_1, L_2) and obtain Figure 2. Now, if

	L_2	R_2
L_1	$4, \underline{3}$	$11, 0$
R_1	$8, 0$	$10, 2$

Figure 2: agent 2's optimal action — contingent on agent 1 playing L_1 — has been underlined

agent 1 is playing R_1, agent 2 preferes playing R_2 to playing L_2 as his payoff will be 2 rather than 0; again, we underline the chosen payoff and obtain Figure 3.

	L_2	R_2
L_1	$4, \underline{3}$	$11, 0$
R_1	$8, 0$	$10, \underline{2}$

Figure 3: agent 2's optimal action — contingent on agent 1 playing R_1 — has been underlined

We take the same approach for agent 1: if agent 2 is playing L_2, agent 1 prefers R_1 to L_1 as her payoff will be 8 rather than 0. Similarly, if agent 2 is playing R_2, agent 1 prefers L_1 to R_1 as her payoff will be 11 rather than 10. Hence all the underlines together give us Figure 4

We know that any box with two underlines represents a Nash equilibrium outcome: each agent is doing as well as they can given what the other agent is doing, so no one wants to change and the system is at rest. However, here there is no such outcome! Does this mean that game theory has no predictions about this game?

Indifference

Keep in mind that economics likes to think that it has answers/predictions about *everything*, so we're going to need some tools to figure out how to predict play in this case. Consider this: since there is no equilibrium in single actions (what we call a *pure-strategy Nash equilibrium*, although if *pure-strategy* is not specified we're still talking about this case), agent 2 might as well randomly select his action. If he randomizes just right, it is possible that agent 1 will be perfectly indifferent between playing L_1 and R_1; and if she is indifferent, she is *also* willing to randomize. If she randomizes just right, it is possible that agent 2 will be perfectly indifferent between playing L_2 and R_2, and hence agent 2 is willing to randomize, as we guessed.

How do we structure this? Suppose that agent 1 plays L_1 with probability p_1, and agent 2 plays L_2 with probability p_2; this implies that agent 1 is playing R_1 with probability $(1-p_1)$ and agent 2 is playing R_2 with probability $(1-p_2)$.[1] We know that agent 2 must randomize so that agent 1 is indifferent between playing

[1] If this does not bring back memories of Econ 41 or Stats 10, the underlying math is simple enough. Suppose that agent 1 is playing L_1 with probability p_1 and R_1 with probability q_1. It is a fact that adding up all the probabilities in an outcome space should result in 1; since here the only two possible outcomes are L_1 and R_1, it must be that $p_1 + q_1 = 1$, or $q_1 = 1 - p_1$.

	L_2	R_2
L_1	$4, \underline{3}$	$\underline{11}, 0$
R_1	$\underline{8}, 0$	$10, \underline{2}$

Figure 4: agent 1's optimal actions have been underlined

L_1 and R_1: agent 1 must receive identical expected utility from each outcome. Recalling our discussion of expected utility,

$$
\begin{aligned}
& & \mathbb{E}\left[u_1\left(\mathrm{L}_1\right)\right] &= \mathbb{E}\left[u_1\left(\mathrm{R}_1\right)\right] \\
&\iff & \Pr\left(\mathrm{L}_2\right)(4) + \Pr\left(\mathrm{R}_2\right)(11) &= \Pr\left(\mathrm{L}_2\right)(8) + \Pr\left(\mathrm{R}_2\right)(10) \\
&\iff & p_2(4) + (1-p_2)(11) &= p_2(8) + (1-p_2)(10) \\
&\iff & 1 &= 5p_2 \\
&\iff & p_2 &= \frac{1}{5}
\end{aligned}
$$

Therefore if agent 2 plays L_2 with probability $1/5$, agent 1 is indifferent between playing L_1 and R_1.

We need to repeat the same exercise to see how agent 2 can be indifferent; remember that if agent 2 is not indifferent, he is not willing to randomize, so we need to know how agent 1 randomizes over her strategies to support agent 2's randomization. Again,

$$
\begin{aligned}
& & \mathbb{E}\left[u_2\left(\mathrm{L}_2\right)\right] &= \mathbb{E}\left[u_2\left(\mathrm{R}_2\right)\right] \\
&\iff & \Pr\left(\mathrm{L}_1\right)(3) + \Pr\left(\mathrm{R}_1\right)(0) &= \Pr\left(\mathrm{L}_1\right)(0) + \Pr\left(\mathrm{R}_1\right)(2) \\
&\iff & p_1(3) + (1-p_1)(0) &= p_1(0) + (1-p_1)(2) \\
&\iff & 5p_1 &= 2 \\
&\iff & p_1 &= \frac{2}{5}
\end{aligned}
$$

Therefore if agent 1 plays L_1 with probability $2/5$, agent 2 is indifferent between playing L_2 and R_2.

Equilibrium

We can see then that if agent 1 plays $[L_1, R_1]$ with probability $[2/5, 3/5]$ and agent 2 plays $[L_2, R_2]$ with probability $[1/5, 4/5]$ *neither agent can be strictly better off by pursuing another strategy.* Somewhat curiously, neither agent has a strict incentive to play in exactly this fashion: since they are indifferent between actions, they could just as well choose one or another. However, if they did so the other agent would be able to improve their payoff by *not* randomizing, and we would be out of equilibrium.

This is a Nash equilibrium in mixed strategies, or a *mixed-strategy Nash equilibrium* (MSNE). This differs conceptually from "regular" Nash equilibrium only in that we allow agents to randomize — or, mix — over their available single actions — or, pure strategies. Together with pure-strategy Nash equilibria, it is good to keep in mind the following result:

Theorem

Let N be the number of (pure-strategy) Nash equilibria of a two-player game, and let M be the number of mixed-strategy Nash equilibria of the same game. Then if $N + M$ is finite, it is odd; that is, $N + M = 1$ is possible, as is $N + M = 3$, but we cannot have $N + M = 0$ or $N + M = 2$.

An example

Consider the simultaneous-move game of Figure 5. To find Nash equilibrium, we consider the following:

	B_2	S_2
B_1	$4, 1$	$0, 0$
S_1	$-1, 0,$	$2, 2$

Figure 5: a two-player simultaneous-move game

- If agent 1 plays B_1, agent 2 would rather play B_2 than S_2, since $1 > 0$.
- If agent 1 plays S_1, agent 2 would rather play S_2 than B_2, since $2 > 0$.
- If agent 2 plays B_2, agent 1 would rather play B_1 than S_1, since $4 > -1$.
- If agent 2 plays S_2, agent 2 would rather play S_1 than B_1, since $2 > 0$.

Putting these together, we underline the payoffs corresponding to the chosen actions, shown in Figure 6. As

	B_2	S_2
B_1	$\underline{4}, \underline{1}$	$0, 0$
S_1	$-1, 0,$	$\underline{2}, \underline{2}$

Figure 6: chosen payoffs have been underline

usual, any outcome with two underlines is a Nash equilibrium; we gently box these in Figure 7. In particular, the Nash equilibrium strategies are (B_1, B_2) and (S_1, S_2), and the Nash equilibrium outcomes are $(4, 1)$ and $(2, 2)$.

	B_2	S_2
B_1	$\boxed{\underline{4}, \underline{1}}$	$0, 0$
S_1	$-1, 0,$	$\boxed{\underline{2}, \underline{2}}$

Figure 7: Nash equilibria (in pure strategies) correspond to the cells with two underlines

Mixed-strategy Nash equilibria

Now, in light of the theorem given above, we should look for mixed-strategy Nash equilibrium. Why? We found two pure-strategy Nash equilibria; since the number of pure-strategy Nash equilibria plus the number of mixed-strategy Nash equilibria must be odd, there is a mixed-strategy Nash equilibrium hiding out there somewhere.

Let p_1 be the probability with which agent 1 plays B_1, and p_2 be the probability with which agent 2 plays

B_2.[2] Recall that agent 2's randomization must keep agent 1 indifferent between playing B_1 and S_1; that is,

$$\begin{aligned} & \mathbb{E}\left[u_1\left(B_1\right)\right] = \mathbb{E}\left[u_1\left(S_1\right)\right] \\ \iff & p_2(4) + (1-p_2)(0) = p_2(-1) + (1-p_2)(2) \\ \iff & 7p_2 = 2 \\ \iff & p_2 = \frac{2}{7} \end{aligned}$$

So if agent 2 plays B_2 with probability 2/7, agent 1 is indifferent between playing B_1 and S_1.

Now, what probability must agent 1 assign to playing B_1 in order for agent 2 to be indifferent between B_2 and S_2? Again,

$$\begin{aligned} & \mathbb{E}\left[u_2\left(B_2\right)\right] = \mathbb{E}\left[u_2\left(S_2\right)\right] \\ \iff & p_1(1) + (1-p_1)(0) = p_1(0) + (1-p_1)(2) \\ \iff & 3p_1 = 2 \\ \iff & p_1 = \frac{2}{3} \end{aligned}$$

So if agent 1 plays B_1 wth probability 2/3, agent 2 is indifferent between playing B_2 and S_2, and hence is willing to randomize between the two.

Together, we see that agent 1 playing $[B_1, S_1]$ with probability $[2/3, 1/3]$ and agent 2 playing $[B_2, S_2]$ with probability [2/75/7] constitutes a mixed-strategy Nash equilibrium. Moreover, this is the *only* mixed-strategy Nash equilibrium, since all of our algebra implied that there was only one probability per agent to keep the opposite agent indifferent between actions. That is, if $p_2 \neq 2/7$, agent 1 is surely not indifferent between B_1 and S_1!

Pareto efficiency

We can see that both pure-strategy Nash equilibrium outcomes are Pareto efficient: we cannot make one agent better off without harming another. Is this also true of the mixed-strategy Nash equilibrium we have found?

Let's consider two methods of computing the agents' utilities in this mixed-strategy Nash equilibrium.

(a) *Direct method.* Agents are randomly selecting actions, and they are doing so independently. Hence if agent 1 plays B_1 with probability 2/3 and agent 2 plays B_2 with probability 2/7, the probability of outcome (B_1, B_2) is $(2/3)(2/7) = 4/21$. We can compute this for each possible outcome,

$$\begin{aligned} \Pr\left(B_1, B_2\right) &= \left(\frac{2}{3}\right)\left(\frac{2}{7}\right) & \Pr\left(B_1, S_2\right) &= \left(\frac{2}{3}\right)\left(1-\frac{2}{7}\right) \\ &= \frac{4}{21} & &= \frac{10}{21} \\ \Pr\left(S_1, B_2\right) &= \left(1-\frac{2}{3}\right)\left(\frac{2}{7}\right) & \Pr\left(S_1, S_2\right) &= \left(1-\frac{2}{3}\right)\left(1-\frac{2}{7}\right) \\ &= \frac{2}{21} & &= \frac{5}{21} \end{aligned}$$

A useful algebra check of these probabilities is $(4/21 + 10/21 + 2/21 + 5/21) = 21/21 = 1$.

Helpfully, we can write these results in something that looks like a payoff matrix, shown in Figure 8.

[2] We might more generally say that p_i is the probability with which agent i plays B_i.

	B_2	S_2
B_1	4/21	10/21
S_1	2/21	5/21

Figure 8: the probability of each outcome in MSNE

Computing each agent's utility is a matter of computing the expected utility over possible outcomes. In particular, for agent 1 we have

$$\begin{aligned}\mathbb{E}[u_1] &= \Pr(B_1, B_2)\, u_1(B_1, B_2) + \Pr(B_1, S_2)\, u_1(B_1, S_2) \\ &\quad + \Pr(S_1, B_2)\, u_1(S_1, B_2) + \Pr(S_1, S_2)\, u_1(S_1, S_2) \\ &= \frac{4}{21}(4) + \frac{10}{21}(0) + \frac{2}{21}(-1) + \frac{5}{21}(2) \\ &= \frac{24}{21} = \frac{8}{7}\end{aligned}$$

For agent 2, we find

$$\begin{aligned}\mathbb{E}[u_2] &= \frac{4}{21}(1) + \frac{10}{21}(0) + \frac{2}{21}(0) + \frac{5}{21}(2) \\ &= \frac{14}{21} = \frac{2}{3}\end{aligned}$$

Thus the expected utilities in this mixed-strategy Nash equilibrium are $(8/7, 2/3)$.

(b) *Shortcut.* Remember now that each agent is indifferent between their possible actions; therefore they receive the same payoff no matter which action they take. Since the agents' randomizations are independent, we know[3]

$$\mathbb{E}[u_1] = \Pr(B_1)\, \mathbb{E}[u_1(B_1)] + \Pr(B_2)\, \mathbb{E}[u_1(S_1)].$$

But since $\mathbb{E}[u_1(B_1)] = \mathbb{E}[u_1(S_1)]$, this implies

$$\mathbb{E}[u_1] = (\Pr(B_1) + \Pr(B_2))\, \mathbb{E}[u_1(B_1)].$$

Now, since these probabilities must add to 1 we have

$$\mathbb{E}[u_1] = \mathbb{E}[u_1(B_1)] = \mathbb{E}[u_1(S_1)]$$

That is, to know agent 1's expected payoff we need only compute the expected payoff from choosing either pure action; the same logic will obviously hold for agent 2.

We can then compute — much more simply than the above —

$$\begin{aligned}\mathbb{E}[u_1] &= \mathbb{E}[u_1(B_1)] \\ &= \Pr(B_2)\, u_1(B_1, B_2) + \Pr(S_2)\, u_1(B_1, S_2) \\ &= \frac{2}{7}(4) + \frac{5}{7}(0) \\ &= \frac{8}{7}\end{aligned}$$

For agent 2,

$$\begin{aligned}\mathbb{E}[u_2] &= \mathbb{E}[u_2(B_2)] \\ &= \frac{2}{3}(1) + \frac{1}{3}(0) \\ &= \frac{2}{3}\end{aligned}$$

[3] This is a little deeper in Econ 41 (and possibly Stats 10); if it rings a bell, it is the law of iterated expectations.

These answers correspond exactly to the answers above!

Note that we could just as well have used, for example, $\mathbb{E}[u_1] = \mathbb{E}[u_1(\mathrm{S}_1)]$ here and computed the same thing; the answer would have been no different. We used B_1 only because we needed to choose one or the other.

We know then that the expected utilities in this equilibrium are $(8/7, 2/3)$. Importantly, both pure-strategy Nash equilibria — $(4,1)$ and $(2,2)$ in outcome space — Pareto dominate this outcome! Why is this? In this equilibrium, there are sizable probabilities placed on undesirable outcomes such as $(0,0)$ and $(-1,0)$, since agents are randomizing independently. If they have the power to randomize in a different way (more on this later in the quarter) we might expect different outcomes.

In case you are wondering, yes, this does seem like an absurd prediction in this context. Not only do agents need to follow this needlessly-complex randomization strategy (needless since there are pure-strategy equilibria), but they are worse off than in the pure-strategy equilibria for doing so! All the same, this *is* an equilibrium of the game. We are only looking to quantify which strategies imply that no agent is better off doing something else, and this mixing approach satisfies the criterion.

Questions

Question 1: consider the version of the prisoner's dilemma in Figure 9. Suppose that agent 1 plays C_1 with

	C_2	D_2
C_1	$-5, -4$	$0, -10$
D_1	$-11, 0$	$-2, -1$

Figure 9: prisoner's dilemma for question 1

probability p_1 and agent 2 plays C_2 with probability p_2.

(a) What must p_1 be for agent 2 to be indifferent between C_2 and D_2?

(b) What must p_2 be for agent 1 to be indifferent between C_1 and D_1?

(c) Recalling the laws of probability, what does this say about the existence of mixed-strategy Nash equilibrium in this game?

(d) How many pure-strategy Nash equilibria are there in this game? Does this confirm or refute your answer to (c)? (is it possible in this game that there is more than one p_1 which leaves agent 2 indifferent between C_2 and D_2?)

(e) Is there a good intuitive reason for your answers to (c) and (d), in terms of the payoffs of the game?

Question 2: consider the sequential game in Figure 10.

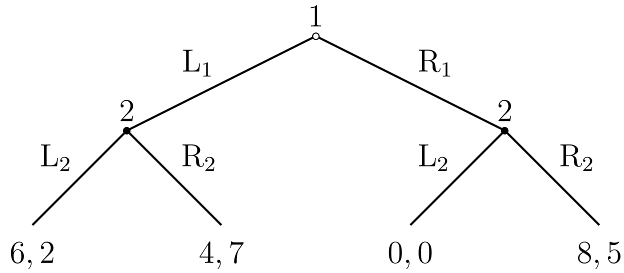

Figure 10: the sequential game for question 2

(a) What is the subgame-perfect Nash equilibrium of this game?

(b) Render this game as a payoff matrix. (remember that agent 2 has 4 possible strategies)

(c) What are the Nash equilibria of this game?

(d) Are there any mixed-strategy Nash equilibria of this game?

Question 3 (hard): consider any two-player sequential game with two actions for each player and distinct payoffs for each player across outcomes. How many subgame-perfect Nash equilibria are there? How many Nash equilibria can there be? Can there exist mixed-strategy Nash equilibria? If so, are they "interesting"?