*This is a draft; email me with comments, typos, clarifications, etc.*

## Timing: ex ante, interim, ex post

In section, we unintentionally ran into the concepts of ex ante, interim, and ex post expectations. While these may not formally be of much value, the intuition they generate is quite useful in considering the nuances between various auction optimization strategies (those of the seller and of the buyers). To this end, we'll state some simple definitions below — mostly taken from MWG (p893) — and then use intuition to flesh out the meaning behind the math. These terms are intended with respect to the type space underlying the game.

*Definition*

Let $\theta_i$ represent the type of agent $i$ and $\theta_{-i}$ represent the types of all other agents. Agent $i$'s *ex post* expected utility is

$$u_i\left(\sigma_i(\theta_i), \sigma_{-i}(\theta_{-i})\right)$$

Her *interim* expected utility is

$$E_{\theta_i}\left[u_i\left(\sigma_i(\theta_i), \sigma_{-i}(\theta_{-i})\right)\right]$$

Her *ex ante*[1] expected utility is

$$E_{\theta}\left[u_i\left(\sigma_i(\theta_i), \sigma_{-i}(\theta_{-i})\right)\right]$$

Note that the term *ex post expected utility* is a little inappropriate: there is no expectation with respect to types! It is, of course, possible that based on the construction of the game there are still some elements of chance once types are revealed, but we'll table that for now.

The intuition underlying these concepts is not particularly difficult: ex post expected utility is taken once *all* types are known in the mechanism; interim expected utility is taken once *my* type is known to *me*; ex ante utility is taken before *I* even know *my* type. In this sense, it is natural to think about these expectations as issues of timing. We can support this with a throwaway story: an agent is driving around LA and sees a sign for a yard sale; she's bored so she goes to check it out. She gets out of her car and starts perusing the available goodies, and to her surprise discovers a mint-condition, original-release LP of *Purple Rain*. Excited, she approaches the homeowner and haggles for the item. Of course the homeowner did not intend to put such a valuable item up for sale and immediately takes it inside. Our agent does not receive the item.

Here is how these timing concepts play out in the above story:

- Before the agent has been to the yard sale, she is evaluating her ex ante expected utility (of the yard sale; the LP here is a red herring). When we discuss participation constraints, we are in general discussing ex ante expected utility incentives; here, the expected utility from checking out the yard sale was presumably greater than the expected utility from more-immediately getting where she was going, so she checked out the yard sale.

- Once the agent has valued the yard sale but before she knows how others value the yard sale — here, she's discovered something worth buying but doesn't know how much it might be sold for — she is evaluating her interim expected utility. That is, she knows that the yard sale holds something good, but she does not know yet whether or not she can acquire it as this is a function of the types of others (in particular, the type of the seller).

- After the agent has haggled with the seller over the item and lost, she is evaluating her ex post expected utility (here, we make the not-quite-correct presumption that haggling reveals types). Once it is known that she does not value the LP as much as the owner, her now-known value of the yard sale drops to 0.

We could of course have set this up as an auction proper but this would not have the staying power of *When Doves Cry*.

How does this tie to our optimal auction developments? We've been discussing the win probability function $w_i(\theta)$; this function is different from $\Pr(\theta_i \text{wins})$ in an important way: the win probability function $w_i(\cdot)$ is evaluated *once types are known*. That is, in a direct-revelation mechanism the seller should, at some point, know the types of the potential buyers. He is then free to allocate the item (or not) as he sees fit. So $w_i(\theta)$ is $i$'s probability of winning the item *once all types are known*; how this function is nondegenerate becomes clearer once we've solved through a few questions. Since it is evaluated with full information of types, it is an ex post concept.

A particular buyer does not know the other buyer's valuations, but knows her own. We express her value function as

$$V_i(\theta_i) = \int w_i(\theta) \left(\theta_i - r_i(\theta)\right) dF_{-i}(\theta_{-i}) = E_{\theta_{-i}} \left[w_i(\theta) \left(\theta_i - r_i(\theta)\right)\right]$$

That is, her value is evaluated as an expectation over the types of others; incentive compatibility — through first-order conditions and the envelope theorem — is constructed to support this. Therefore the buyer's problem is evaluated at the interim stage[2].

The seller must construct a mechanism without any knowledge of types. Thus when the seller attempts to maximize expected revenue over all possible realizations, $E_\theta[R(\theta)]$ he is optimizing the mechanism ex ante. It may seem odd that he is ex ante-optimizing over ex post actions (the win probability $w_i(\cdot)$); but once he knows types, he is free to allocate the good according to $w_i(\cdot)$ however he sees fit. This optimization over actions serves to cause agents to behave in a manner which is consistently optimal for the seller.

## Optimal auction subsidies

This discussion will [unintentionally] roughly parallel *Essential Microeconomics* exercise 12.3-3. Hopefully the fact that the two are slight variants of one another will make the underlying concepts abundantly clear.

Suppose we have an auction with two buyers, $i \in \{1, 2\}$, whose values are distributed according to $F_i(\cdot)$ on $[0, 1]$. Value functions, as usual, are given by

$$V_i(\theta_i) = \Pr(\theta_i \text{wins})\theta_i - r(\theta_i)$$

where $r(\theta_i)$ is the expected payment of type $\theta_i$.

Previously (i.e., earlier in the quarter), we had assumed $\Pr(\theta_i \text{wins}) = F_{-i}(\theta_i)$; that is, the probability that type $\theta_i$ wins is essentially the probability that she has a higher type than her opponent. In optimal auctions, we make this probability more generic; the probability that $i$ wins is now expressed as $w_i(\theta)$, a function of both agents' types. It is not immediate why this buys the seller power in optimization, but that should be made clear. Note that this transformation has the effect of making the $r(\theta_i)$ notation potentially insufficient; we then change expected payment to $r_i(\theta)$, so that payments may now depend on the type of the other player. The agent's value function then becomes

$$V_i(\theta_i) = E_{\theta_{-i}} \left[w_i(\theta)\theta_i - r_i(\theta)\right]$$

[2] It is also evaluated ex ante, when the buyer is deciding whether or not to participate; assuming participation constraints hold, the buyer's problem is fully in the interim stage.

To construct an optimal auction, we are concerned with maximizing the seller's revenue[3]. The seller receives $r_i(\theta)$ from each of $i \in \{1,2\}$; however, the seller is unaware of agents' types — that is why an auction mechanism is being run in the first place! — so the evaluation of revenue must take place as an ex ante expectation. The seller's expected revenue is then

$$E_\theta[R(\theta)] = E_\theta[r_1(\theta) + r_2(\theta)] = E_\theta\left[w_1(\theta)\theta_1 - V_1(\theta_1) + w_2(\theta)\theta_2 - V_2(\theta_2)\right]$$

At this point, we need to obtain an expression for $E_{\theta_i}[V_i(\theta_i)]$. From the integral form, we have

$$\begin{aligned} E_{\theta_i}\left[V_i(\theta_i)\right] &= \int_0^1 V_i(\theta_i)dF(\theta_i) \\ &= -\left. V_i(\theta_i)(1-F(\theta_i))\right|_{\theta_i=0}^1 + \int_0^1 V_i'(\theta_i)(1-F(\theta_i))d\theta_i \\ &= \int_0^1 V_i'(\theta_i)\left(\frac{1-F(\theta_i)}{f(\theta_i)}\right)f(\theta_i)d\theta_i \\ &= E_{\theta_i}\left[V_i'(\theta_i)\left(\frac{1-F(\theta_i)}{f(\theta_i)}\right)\right] \end{aligned}$$

Notice that we have applied here the fact that $V_i(0) = 0$ without much fanfare; that the low type receives 0 payoff is not the point of this exercise, and follows from standard arguments.

By standard envelope theorem arguments, we have

$$V_i'(\theta_i) = E_{\theta_{-i}}\left[w_i(\theta)\right]$$

Then the expectation derived above becomes

$$\begin{aligned} E_{\theta_i}\left[V_i(\theta_i)\right] &= E_{\theta_i}\left[E_{\theta_{-i}}\left[w_i(\theta)\right]\left(\frac{1-F(\theta_i)}{f(\theta_i)}\right)\right] \\ &= E_{\theta_i}\left[E_{\theta}-i\left[w_i(\theta)\left(\frac{1-F(\theta_i)}{f(\theta_i)}\right)\right]\right] \\ &= E_\theta\left[w_i(\theta)\left(\frac{1-F(\theta_i)}{f(\theta_i)}\right)\right] \end{aligned}$$

From this, we obtain an expression for the expected revenue of the seller,

$$E_\theta[R(\theta)] = E_\theta\left[w_1(\theta)\left(\theta_1 - \left(\frac{1-F_1(\theta_1)}{f_1(\theta_1)}\right)\right) + w_2(\theta)\left(\theta_2 - \left(\frac{1-F_2(\theta_2)}{f_2(\theta_2)}\right)\right)\right]$$

Transforming this into Riley's $J_i(\cdot)$ function, this may be expressed succinctly as

$$E_\theta[R(\theta)] = E_\theta\left[\sum_{i=1}^2 w_i(\theta)J_i(\theta_i)\right]$$

The summation notation here is of course a little overkill, but helps to demonstrate just how cleanly this should generalize to the case of many agents.

How does this tie to maximization of seller revenue? The above becomes, in integral form,

$$E_\theta[R(\theta)] = \int_0^1\int_0^1\sum_{i=1}^2 w_i(\theta)J_i(\theta_i)dF_2(\theta_2)dF_1(\theta_1)$$

[3]Presumably, the denotation "optimal" here refers to the fact that, in the real world, the seller is generally the mechanism designer (or, in the case of things like eBay, at least the mechanism selector). In this sense, if we are to act like reasonable economists it behooves us to think like the designer should in practice; our concern for optimization is then the seller's revenue.

The Mirrlees trick tells us that maximization on the left-hand side may be attained by pointwise-maximization of the integrand on the right-hand side. So the firm is actually solving (for each $\theta = (\theta_1, \theta_2)'$)

$$\max_{w_i(\theta)} \sum_{i=1}^{2} w_i(\theta) J_i(\theta_i)$$

But this maximization is linear! Since $w_i(\cdot)$ is constrained to be a valid probability distribution (possibly including the opportunity for no allocation), it is immediate that

$$w_i(\theta) = \chi\left(\forall \theta_{-i},\ J_i(\theta_i) > J_{-i}(\theta_{-i}) \wedge J_i(\theta_i) > 0\right)$$

That is, the object is allocated to the bidder with the highest $J_i(\theta_i)$, provided this value is positive. If no buyers have positive $J_i(\theta_i)$, the object remains with the seller.

At this point, it is reasonable to ask why we are even bothering to construct optimal auctions, given the revenue equivalence theorem. But we need to be careful when discussing this result: revenue equivalent only holds for ex post efficient outcomes, where the highest-typed bidder wins the item for sure. Here, we are causing a higher-typed buyer to lose some fraction of the time in order to extract more expected revenue from him. Thus the antecedents of revenue equivalence do not hold, and optimization is not hamstrung by this mathematical necessity.

It is this inefficiency of outcomes that leads to the notion of subsidy. Of course, we are not in this case actually providing a monetary subsidy to the less-powerful bidder, but in tilting the odds in their favor we are offering an implicit subsidy to their actions. While it is unclear that this causes the weaker bidder to bid more strongly, intuitively it makes sense that it should cause the stronger bidder to bid even higher than before. This can be seen by solving through the particular bid functions (using explicit distributions, if necessary), but the algebra is nasty and not particularly informative.

# Ironing

In problems of consumer demand from a monopolist, we have seen over the past few weeks that monotonicity of the quantity function $q(\cdot)$ is necessary to ensure incentive compatibility. In Riley's lecture notes, we saw an example of when, following the standard method of solving for $q(\cdot)$, the function is nonmonotonic. We'll solve through that example here and discuss the technique for fixing the nonmonotonicity problem.

Suppose we are given a standard monopolist setup: there is a firm which produces quantity $q$ at cost $c(q)$, and a buyer with value $\theta$ distributed with CDF $F(\cdot)$ on $[\alpha, \beta]$. The agent's value is as usual,

$$V(\theta) = B(q(\theta); \theta) - r(\theta)$$

For incentive compatibility, we have

$$V'(\theta) = B_\theta\left(q(\theta); \theta\right)$$

Ex ante utility is

$$E\left[V(\theta)\right] = \int_\alpha^\beta V(\theta) F'(\theta) d\theta$$

Integrating by parts, we find

$$\begin{aligned}
E_i\left[V(\theta)\right] &= V(\alpha) + \int_\alpha^\beta V'(\theta)(1 - F_i(\theta))d\theta \\
&= V(\alpha) + \int_\alpha^\beta B_\theta\left(q(\theta);\theta\right)\left(1 - F_i(\theta)\right)d\theta \\
&= V(\alpha) + \int_\alpha^\beta B_\theta\left(q(\theta);\theta\right)\left(\frac{1 - F_i(\theta)}{F_i'(\theta)}\right)dF_i(\theta) \\
&= V(\alpha) + E\left[B_\theta\left(q(\theta);\theta\right)\left(\frac{1 - F_i(\theta)}{F_i'(\theta)}\right)\right]
\end{aligned}$$

We can obtain an expression for the expected revenue of the seller,

$$\begin{aligned}
E\left[r(\theta)\right] &= \int_\alpha^\beta r(\theta)dF(\theta) \\
&= \int_\alpha^\beta \left(B\left(q(\theta);\theta\right) - V\left(\theta\right)\right)dF(\theta) \\
&= \int_\alpha^\beta B\left(q(\theta);\theta\right)dF(\theta) - E\left[V(\theta)\right] \\
&= \int_\alpha^\beta \left(B\left(q(\theta);\theta\right) - B_\theta\left(q(\theta);\theta\right)\left(\frac{1 - F(\theta)}{F'(\theta)}\right)\right)dF(\theta)
\end{aligned}$$

Now, the firm would like to maximize expected revenue. Assuming the functions involved are well-behaved, we can pursue a pointwise maximization strategy; that is, for each $\theta$ we maximize

$$R(q;\theta) \equiv B\left(q;\theta\right) - B_\theta\left(q;\theta\right)\left(\frac{1 - F(\theta)}{F'(\theta)}\right)$$

according to the supplied quantity $q$. Recall that

$$B\left(q;\theta\right) = \int_0^q p(t;\theta)dt$$

If we are free to switch the order of the derivatives, we can obtain from first-order conditions

$$\frac{\partial}{\partial q}R(q;\theta) = p(q;\theta) - p_\theta(q;\theta)\left(\frac{1 - F(\theta)}{F'(\theta)}\right)$$

To proceed any further, it will serve us well to specify the problem more completely. Suppose types are distributed in $\Theta = [0, 2]$,

$$F(\theta) = \begin{cases} \frac{3}{4}\theta & \text{if } \theta < 1 \\ \frac{1}{4}\theta + \frac{1}{2} & \text{otherwise} \end{cases}$$

We assume a linear cost function, $c(q) = kq$, and linear price demand, $p(q;\theta) = \theta - q$ (that is, $B(q;\theta) = \theta q - \frac{q^2}{2}$).

From standard analysis, we have that at the optimum marginal revenue should equal marginal cost. Then $q$ is defined by

$$q = \theta - k - \left(\frac{1 - F(\theta)}{F'(\theta)}\right)$$

In a more-closed form, this gives

$$q = \begin{cases} 2\theta - k - \frac{4}{3} & \text{if } \theta \in [0, 1) \\ 2\theta - k - 2 & \text{if } \theta \in [1, 2] \end{cases}$$

Of course, we still need to do some edge correcting to make sure the quantity supplied is never negative, but this is a nonissue[4].

Notice that this $q$ is nonmonotonic (check the jump at 1). Intuitively, this violates incentive compatibility: to maintain continuity of the value function, the firm will need to offer two prices for the same quantity. (we have seen this before: that $q$ must be monotone)

How then can we transform $q$ to respect incentive compatibility? Using Riley logic, we were able to see that it only makes sense to introduce a flat region into the $q$ function; that is, deviation to a nonconstant quantity will cause the firm to sacrifice more revenue relative to the optimum than it must otherwise. The question then becomes, where should we locate this flat region?

One key principle is that this region must be anchored at both ends to the existing $q$ function. That is, let $[\underline{\theta}, \overline{\theta}]$ represent the region which has been flattened; to properly specify that a region is flat, we need $q(\underline{\theta}) = q(\overline{\theta})$. Let's assume that this relationship can be modeled as $\overline{\theta} = m(\underline{\theta})$ for some $m(\cdot)$.

We claim that the best the seller can do is minimize lost profit (relative to the opimum obtained from $q$), and this is fairly intuitive: if we need a new device to be incentive compatible but we are solving in the context of the seller's optimum, we should minimize deviations from the unconstrained optimization problem. Since $q$ will remain unchanged outside of the range $[\underline{\theta}, \overline{\theta}]$, we can constrain all of our reasoning to this particular range.

At the optimum, the seller's profit from buyers in this range is

$$\int_{\underline{\theta}}^{m(\underline{\theta})} \left(R\left(q(\theta);\theta\right) - c\left(q(\theta)\right)\right) dF(\theta)$$

Since the quantity supplied in the flattened, incentive compatible mechanism is anchored to $q(\underline{\theta})$, we can represent the seller's profit from flattening as

$$\int_{\underline{\theta}}^{m(\underline{\theta})} \left(R\left(q\left(\underline{\theta}\right);\theta\right) - c\left(q\left(\underline{\theta}\right)\right)\right) dF(\theta)$$

The optimum level of profit must always lie [weakly] above the flattened level of profit, so minimization as stated is equivalent to[5]

$$\min_{\underline{\theta}} \int_{\underline{\theta}}^{m(\underline{\theta})} \left(R\left(q(\theta);\theta\right) - c\left(q(\theta)\right)\right) dF(\theta) - \int_{\underline{\theta}}^{m(\underline{\theta})} \left(R\left(q\left(\underline{\theta}\right);\theta\right) - c\left(q\left(\underline{\theta}\right)\right)\right) dF(\theta)$$

To locate the minimum, we take first-order conditions with respect to $\underline{\theta}$ and obtain

$$\begin{aligned} 0 = &\left(R\left(q\left(m\left(\underline{\theta}\right)\right);m\left(\underline{\theta}\right)\right) - c\left(q\left(m\left(\underline{\theta}\right)\right)\right)\right) F'\left(m\left(\underline{\theta}\right)\right) m'\left(\underline{\theta}\right) - \left(R\left(q\left(\underline{\theta}\right);\underline{\theta}\right) - c\left(q\left(\underline{\theta}\right)\right)\right) F'\left(\underline{\theta}\right) \ldots \\ &- \left(\left(R\left(q\left(\underline{\theta}\right);m\left(\underline{\theta}\right)\right) - c\left(q\left(\underline{\theta}\right)\right)\right) F'\left(m\left(\underline{\theta}\right)\right) m'\left(\underline{\theta}\right) - \left(R\left(q\left(\underline{\theta}\right);\underline{\theta}\right) - c\left(q\left(\underline{\theta}\right)\right)\right) F'\left(\underline{\theta}\right)\right) \ldots \\ &- \int_{\underline{\theta}}^{m(\underline{\theta})} \left(R_q\left(q\left(\underline{\theta}\right);\theta\right) - c'\left(q\left(\underline{\theta}\right)\right)\right) q'\left(\underline{\theta}\right) dF(\theta) \end{aligned}$$

From Riley logic, we were able to see that $q(\underline{\theta}) = q(\overline{\theta}) = q(m(\underline{\theta}))$. It follows that the leading four terms in the above expression cancel, and we are left with

$$\int_{\underline{\theta}}^{m(\underline{\theta})} \left(R_q\left(q\left(\underline{\theta}\right);\theta\right) - c'\left(q\left(\underline{\theta}\right)\right)\right) q'\left(\underline{\theta}\right) dF(\theta) = 0$$

[4] It was mentioned in section that the easiest way to get around a large part of this is simply to assume $k = 0$, all production is free of charge. Our analysis will hold up with or without this assumption, but if you're graphing along at home $k = 0$ will make things reasonably nice.

[5] In section, we either reversed the order of subtraction, or inappropriately used min when we should have used max (depending on how you want to look at it). Since we didn't bother checking second-order conditions none of our subsequent math was affected by this mistake.

Marginal cost pricing tells us $c'(q(\underline{\theta})) = R_q(q(\underline{\theta});\underline{\theta})$, so the above becomes

$$\int_{\underline{\theta}}^{m(\underline{\theta})} \left(R_q\left(q\left(\underline{\theta}\right);\theta\right) - R_q\left(q\left(\underline{\theta}\right);\underline{\theta}\right)\right) q'\left(\underline{\theta}\right) dF(\theta) = 0$$

Here, it becomes painful to proceed without substituting in for the existing problem specification[6]. Note first that we can explicitly calculate $m(\underline{\theta})$,

$$\begin{aligned} & 2\underline{\theta} - k - \frac{4}{3} = 2\overline{\theta} - k - 2 \\ \iff & \overline{\theta} = \underline{\theta} + \frac{1}{3} \end{aligned}$$

Further, we have a clear boundary point delimiting where the integrand changes signs; in particular, when $q(\underline{\theta}) < q(\theta)$ we should have $R_q(q(\underline{\theta});\theta) > c(q(\underline{\theta}))$ ($q(\theta)$ should increase to the maximum) and when $q(\underline{\theta}) > q(\theta)$ we should have $R_q(q(\underline{\theta});\theta) < c(q(\underline{\theta}))$. Since we have a single point of nonmonotonicity (in some sense), we can restate the problem as

$$\int_{\underline{\theta}}^{1} \left(R_q\left(q\left(\underline{\theta}\right);\theta\right) - R_q\left(q\left(\underline{\theta}\right);\underline{\theta}\right)\right) q'\left(\underline{\theta}\right) dF(\theta) = \int_{1}^{\underline{\theta}+\frac{1}{3}} \left(R_q\left(q\left(\underline{\theta}\right);\theta\right) - R_q\left(q\left(\underline{\theta}\right);\underline{\theta}\right)\right) q'\left(\underline{\theta}\right) dF(\theta)$$

On the interior of both of these ranges, $q'(\cdot) = 2$ is well-defined. In the left-hand range, $dF(\theta) = \frac{3}{4}$ and in the right-hand range, $dF(\theta) = \frac{1}{4}$. So we may simplify further

$$3\int_{\underline{\theta}}^{1} \left(R_q\left(q\left(\underline{\theta}\right);\theta\right) - R_q\left(q\left(\underline{\theta}\right);\underline{\theta}\right)\right) d\theta = \int_{1}^{\underline{\theta}+\frac{1}{3}} \left(R_q\left(q\left(\underline{\theta}\right);\underline{\theta}\right) - R_q\left(q\left(\underline{\theta}\right);\theta\right)\right) d\theta$$

Substituting in for known equations above, we have (for a particular $\gamma$ depending on the side of the equation in discussion)

$$\begin{aligned} R_q\left(q\left(\underline{\theta}\right);\theta\right) - R_q\left(q\left(\underline{\theta}\right);\underline{\theta}\right) &= \theta - q\left(\underline{\theta}\right) - (\gamma - \theta) - \left(\underline{\theta} - q\left(\underline{\theta}\right) - \left(\frac{4}{3} - \underline{\theta}\right)\right) \\ &= 2\theta - 2\underline{\theta} + \left(\frac{4}{3} - \gamma\right) \end{aligned}$$

Then our overall optimum simplifies quite nicely to

$$3\int_{\underline{\theta}}^{1} \theta - \underline{\theta} d\theta = \int_{1}^{\underline{\theta}+\frac{1}{3}} \underline{\theta} + \frac{1}{3} - \theta d\theta$$

Solving through,

$$\begin{aligned} & 3\left(\frac{\theta^2}{2} - \theta\underline{\theta}\right)\Bigg|_{\theta=\underline{\theta}}^{1} = \left(\theta\left(\underline{\theta} + \frac{1}{3}\right) - \frac{\theta^2}{2}\right)\Bigg|_{\theta=1}^{\underline{\theta}+\frac{1}{3}} \\ \iff & \frac{3}{2}\underline{\theta}^2 - 3\underline{\theta} + \frac{3}{2} = \left(\left(\underline{\theta} + \frac{1}{3}\right)\left(\underline{\theta} + \frac{1}{3}\right) - \frac{1}{2}\left(\underline{\theta} + \frac{1}{3}\right)^2\right) - \left(\underline{\theta} + \frac{1}{3}\right) + \frac{1}{2} \\ \iff & \frac{3}{2}\underline{\theta}^2 - 2\underline{\theta} + \frac{8}{6} = \frac{1}{2}\left(\underline{\theta}^2 + \frac{2}{3}\underline{\theta} + \frac{1}{9}\right) \\ \iff & \underline{\theta}^2 - \frac{7}{3}\underline{\theta} + \frac{23}{18} = 0 \\ \implies & \underline{\theta} = \frac{7 - \sqrt{3}}{6} \end{aligned}$$

[6] Indeed, it seems Myerson's specification leaves roughly this form as the final solution.

Then we have a full characterization of quantities in the optimal incentive compatible mechanism,

$$q(\theta) = \begin{cases} 2\theta - k - \frac{4}{3} & \text{if } \theta \in \left[\frac{3k+4}{6}, \frac{7-\sqrt{3}}{6}\right] \\ \max\left\{\frac{3-\sqrt{3}}{3} - k, 0\right\} & \text{if } \theta \in \left[\frac{7-\sqrt{3}}{6}, \frac{9-\sqrt{3}}{6}\right] \\ \max\{2\theta - k - 2, 0\} & \text{if } \theta \in \left[\frac{9-\sqrt{3}}{6}, 2\right] \\ 0 & \text{otherwise} \end{cases}$$

## Ostroy topics

At this point, we have not yet covered much "economics" proper in Ostroy's class; this will change in the coming weeks. Still, we should address some of the topics that he will be covering and engage in our usual comp review. It is necessary to formally introduce two new concepts to cover the selected question this week, but do not use this as the go-to reference: Ostroy will cover this far more thoroughly than we can hope to here, and always take his word over the notes'.

*Definition*

A *price-taking equilibrium* for a given vector of value functions $v$ consists of a normalized price vector $(p, 1)$ and an allocation $(z_i, m_i)$ for each $i \in I$ such that $(z_i, m_i) \in d(v_i, p)$ and markets clear,

$$\sum_i z_i = 0, \qquad \sum_i m_i = 0$$

That is, a price-taking equilibrium in the quasilinear model is precisely the idea of equilibrium that we are accustomed to: agents utility-maximize given prices, and markets clear. This definition points to one of the key features of the quasilinear model: all endowments may be normalized to 0 and encoded in the value function (that is, rather than having a utility function and an endowment floating around for each agent, we alter the reference point for utility and place all information into the value function) and budgets disappear. Market clearing is then fully described by all interesting quantities summing to 0, or that one agent's increased consumption relative to his endowment must be balanced by another agent's lowered consumption relative to hers.

*Definition*

Given an economy $\mathcal{E} = \langle I, \{X_i\}, \{\succeq_i\}, \{e_i\}\rangle$ (where this formulation accommodates the quasilinear case), the *k-replicate* $\mathcal{E}^k$ is defined by

$$\mathcal{E}^k = \bigcup_{h=1}^{k} \langle I_h, \{X_{ih}\}, \{\succeq_{ih}\}, \{e_{ih}\}\rangle$$

where $I_h = I$, $X_{ih} = X_i$, $\succeq_{ih} = \succeq_i$, and $e_{ih} = e_i$. That is, $\mathcal{E}^k$ is the original economy $\mathcal{E}$ replicated $k$ times, with $k$ copies of each agent $i$ and her attached preferences, endowments, and consumption opportunities.

In Ostroy's class, replicates will eventually be used to make the case for competition; as more agents are introduced to the system, sets which were once nonconvex become approximately convex on an appropriate scale. This leads to nice properties for optimization and equilibrium-attainment. The rationale behind replicating an existing economy rather than introducing new agents and creating a new one is that in replication, we have a set of properties which transfer from the original economy to its replicates (that is, we know something about the underlying agents in a proof-friendly way).

## *Essential Microeconomics*, exercise 12.3-3

Two bidders have values that are continuously distributed on $[0, 1]$. The CDF for bidder $i$ is $F_i(\theta)$. The hazard rate is higher for the second bidder; that is,

$$\frac{f_1(\theta)}{1 - F_1(\theta)} < \frac{f_2(\theta)}{1 - F_2(\theta)}$$

(a) Show that $F_1(\theta) < F_2(\theta)$, $\theta \in (0, 1)$, so that buyer 1 is the "strong" bidder.

**Solution:** note that at $\theta = 0$, the hazard rate inequality gives us $f_1(0) < f_2(0)$. Thus within a neighborhood of 0, for $\varepsilon > 0$ sufficiently small we should have $F_1(\varepsilon) < F_2(\varepsilon)$.

Suppose there exists some $\theta$ such that $F_1(\theta) \geq F_2(\theta)$. By continuity of the underlying distributions, there must exist some $\theta^*$ such that $F_1(\theta^*) = F_2(\theta^*)$; let $\theta^*$ be such that

$$\theta^* = \min \{\theta : F_1(\theta) = F_2(\theta)\}$$

By continuity, we know that this minimum is well defined. Knowing that $F_1(\varepsilon) < F_2(\varepsilon)$, we may assume that $\theta^* < \theta$ (that is, we are analyzing the "first" crossing of $F_2$ by $F_1$).

Again following from the hazard rate inequality, we see that $f_1(\theta^*) < f_2(\theta^*)$; then for $\delta > 0$ sufficiently small, a Taylor expansion around $\theta^*$ will give us $F_1(\theta^* - \delta) > F_2(\theta^* - \delta)$. But if this is the case, continuity tells us that there must exist some $\theta^{**} < \theta^* - \delta$ such that $F_1(\theta^{**}) = F_2(\theta^{**})$, a contradiction of our definition of $\theta^*$.

Hence there cannot be $\theta$ such that $F_1(\theta) \geq F_2(\theta)$, and so $F_1(\theta) < F_2(\theta)$ for all $\theta \in (0, 1)$.

(b) Show that the buyer's payoff from any selling scheme can be expressed simply as a function of the allocation rule $w_i(\theta_1, \theta_2)$, $i \in \{1, 2\}$, where these are the probabilities that the item is assigned to bidder $i$.

**Solution:** we assume that the question intends, "any *incentive compatible* selling scheme."[7] We have, as usual

$$V_i(\theta_i) = E_{\theta_{-i}} [w_i(\theta_i, \theta_{-i}) (\theta_i - b(\theta_i))]$$

Following incentive compatibility, we take first-order conditions to obtain

$$V_i'(\theta_i) = E_{\theta_{-i}} [w_i(\theta_i, \theta_{-i})]$$

Knowing that $V_i(0) = 0$, we can then see

$$V_i(\theta_i) = \int_0^{\theta_i} E_{\theta_{-i}} [w_i(\theta, \theta_{-i})] \, d\theta$$

(c) Show that the expected seller revenue is

$$\int_0^1 \int_0^1 [w_1(\theta) J_1(\theta_1) + w_2(\theta) J_2(\theta_2)] \, dF_1(\theta_1) dF_2(\theta_2), \qquad J_i(v) \equiv v - \frac{1 - F_i(v)}{f_i(v)}$$

[7] Although, in a way, an incentive *in*compatible selling scheme does not make much sense.

**Solution:** from the seller's perspective, we have

$$\begin{aligned} E_{\theta_i}\left[V_i\left(\theta_i\right)\right] &= E_{\theta_i}\left[E_{\theta_{-i}}\left[w_i(\theta_i,\theta_{-i})\left(\theta_i - b(\theta_i)\right)\right]\right] \\ &= E_\theta\left[w_i(\theta_i,\theta_{-i})\theta_i\right] - E_\theta\left[w_i(\theta_i,\theta_{-i})b(\theta_i)\right] \\ &= E_\theta\left[w_i(\theta_i,\theta_{-i})\theta_i\right] - E_\theta\left[r_i(\theta)\right] \end{aligned}$$

Thus we have

$$E_\theta\left[r_i(\theta)\right] = E_\theta\left[w_i(\theta_i,\theta_{-i})\theta_i\right] - E_{\theta_i}\left[V_i\left(\theta_i\right)\right]$$

Following our standard tricks, we see

$$\begin{aligned} E_{\theta_i}\left[V_i\left(\theta_i\right)\right] &= \int_0^1 V_i\left(\theta_i\right)dF_i(\theta_i) \\ &= -\left.V_i\left(\theta_i\right)\left(1-F_i(\theta_i)\right)\right|_{\theta_i=0}^1 + \int_0^1 V_i'\left(\theta_i\right)\left(1-F_i(\theta_i)\right)d\theta_i \\ &= \int_0^1 E_{\theta_{-i}}\left[w_i(\theta_i,\theta_{-i})\right](1-F_i(\theta_i))d\theta_i \\ &= \int_0^1 E_{\theta_{-i}}\left[w_i(\theta_i,\theta_{-i})\right]\left(\frac{1-F_i(\theta_i)}{F_i'(\theta_i)}\right)dF(\theta_i) \\ &= E_{\theta_i}\left[E_{\theta_{-i}}\left[w_i(\theta_i,\theta_{-i})\left(\frac{1-F_i(\theta_i)}{F'(\theta_i)}\right)\right]\right] \end{aligned}$$

We then see

$$\begin{aligned} E_\theta\left[r_i(\theta)\right] &= E_\theta\left[w_i(\theta_i,\theta_{-i})\theta_i\right] - E_{\theta_i}\left[E_{\theta_{-i}}\left[w_i(\theta_i,\theta_{-i})\left(\frac{1-F_i(\theta_i)}{F'(\theta_i)}\right)\right]\right] \\ &= E_\theta\left[w_i(\theta_i,\theta_{-i})\left(\theta_i - \left(\frac{1-F_i(\theta_i)}{F'(\theta_i)}\right)\right)\right] \\ &= E_\theta\left[w_i(\theta_i,\theta_{-i})J_i(\theta_i)\right] \end{aligned}$$

The seller's expected revenue is the sum of the expected revenue from each of the two bidders,

$$E_\theta[R(\theta)] = E_\theta\left[w_1(\theta_1,\theta_2)J_1(\theta_1)\right] + E_\theta\left[w_2(\theta_1,\theta_2)J_2(\theta_2)\right] = E_\theta\left[w_1(\theta)J_1(\theta_1) + w_2(\theta)J_2(\theta_2)\right]$$

Expressing this expectation in integral form gives the desired result,

$$E_\theta[R(\theta)] = \int_0^1\int_0^1 \left[w_1(\theta)J_1(\theta_1) + w_2(\theta)J_2(\theta_2)\right]dF_2(\theta_2)dF_1(\theta_1)$$

(d) Hence comment on whether or not the optimal selling scheme is symmetric, and whether the playing field should be tilted in favor of the weak player.

**Solution:** using the Mirrlees trick, maximization of expected revenue is identical to pointwise maximization of the integrand. That is, for each $\theta = (\theta_1, \theta_2)$ the seller should maximize the quantity

$$w_1(\theta)J_1(\theta_1) + w_2(\theta)J_2(\theta_2)$$

The quantity which the seller is optimizing is the vector of win probabilities, $w(\theta)$. We know that the win probabilities are constrained so that $0 \leq w_1(\theta) + w_2(\theta) \leq 1$ (that is, they represent valid probabilities and the good does not necessarily need to be allocated). Under this constraint, the firm's optimum decision is intuitive: if both $J_1(\theta_1)$ and $J_2(\theta_2)$ are less than 0, let $w(\theta) = 0$; otherwise, set $w_i(\theta) = 1$ if $J_i(\theta_i) > J_{-i}(\theta_{-i})$. We ignore the possibility of ties, as they are zero-probability events and will not enter into agent decisions.

What is key here is the second element of the optimum decision. Suppose that the two players draw the same type, $\theta_1 = \theta_2$; for sake of argument, assume both $\theta_i$ are so that $J_i(\theta_i) > 0$. From the hazard rate inequality, we see

$$\begin{aligned} & \theta_1 = \theta_2 \\ \implies \quad & \theta_1 - \left(\frac{1 - F_1(\theta_1)}{f_1(\theta_1)}\right) < \theta_2 - \left(\frac{1 - F_2(\theta_2)}{f_2(\theta_2)}\right) \\ \iff \quad & J_1(\theta_1) < J_2(\theta_2) \end{aligned}$$

Then even though both bidders have the same valuation, bidder 2 should be allocated the object! As long as the hazard rate is smooth, there will be a neighborhood where $\theta_1 > \theta_2$ and yet bidder 2 is still allocated the object. That is, at the optimum the odds are tilted in bidder 2's favor. This is done to encourage bidder 1 to bid higher than he would otherwise, in an incentive-compatible way; it so happens that the [opportunity] losses from allocating the good inefficiently to bidder 2 are more than offset by the gains from bidder 1 increasing his bid strategy.

## *Essential Microeconomics*, exercise 12.3-4

There are two buyers, and values are independent. Each buyer has a value of 1 with probability $1 - p$ and a value of 2 with probability $p$.

(a) Show that the equilibrium strategy in the sealed, high-bid auction is for a low-value buyer to bid his value and for a high-value buyer to bid a mixed strategy $b$ with support $[1, 1 + p]$.

**Solution:** the rationale behind the low-value buyer bidding his value is gone over thoroughly in the week 1 section notes. Intuitively, the low-value buyer will never bid beyond his valuation due to negative expected utility; if we assume monotonicity of the bid functions, it follows that he is unwilling to bid below his valuation *in equilibrium*, due to the positive probability associated with tiebreaking.

The rationale behind assuming a nice, well-behaved bidding CDF for the high-value buyer is also fully covered in the week 1 section notes, so here we will merely solve the indifference conditions to demonstrate the desired claim. By limiting indifference, we know that the high-value buyer's expected utility (in the limit; recall the "bid $1 + \varepsilon$" argument) from bidding 1 is

$$(1 - p)(2 - 1) = 1 - p$$

At the upper bound $\bar{b}$ of her mixture, she wins with probability 1; therefore by indifference we have

$$(1 - p) = (2 - \bar{b}) \qquad \implies \qquad \bar{b} = 1 + p$$

It follows that the high-value bidder is randomizing over support $[1, 1 + p]$.

(b) Confirm that the payoff to each buyer type is the same in the sealed, high-bid and open, ascending-bid auction.

**Solution:** from above, we know that in the first-price auction (sealed, high-bid) expected payoffs are

$$V_1^{\text{fp}} = 0, \quad V_2^{\text{fp}} = 1 - p$$

In the second-price auction (open, ascending-bid) each buyer bids his type. We know

$$V_1^{\text{sp}} = 0$$

We may compute the expected payoff of the high-value buyer as

$$\begin{aligned} V_2^{\text{sp}} &= (2-1)(1-p) + \frac{1}{2}(2-2)p \\ &= 1-p \end{aligned}$$

Thus we have revenue equivalence between the two auction mechanisms.

(c) For an optimal auction, argue that the local downward constraint must be binding. Characterize an optimal direct revelation scheme.

**Solution:** suppose that the local downward constraint is not binding. Then the high-value bidder strictly prefers bidding truthfully to representing as the low-value type. But, as usual, this means that we may increasing the high-value's expected payment slightly without affecting incentive compatibility. Since this action leaves the seller better off, a mechanism in which the local downward constraint does not bind cannot be optimal.

From this construction, we can see that we need the high-value buyer to be strictly indifferent between reporting a high type and reporting a low type. Necessarily — as in part (a) — the low-value buyer will pay 1 in an optimal, incentive compatible mechanism. To construct this as an equilibrium, we assume that a high-value buyer will truthfully report; then another high-value buyer's expected payoff from reporting as the low type is

$$V_{21} = \frac{1}{2}(1-p)(2-1)$$

The other high-value buyer's expected payoff from reporting as the high type, letting $r_2$ be his expected payment provisional on winning, is

$$V_{22} = \left((1-p) + \frac{1}{2}p\right)(2-r_2)$$

Equating the two valuations as we must to support binding local downward constraints, we have

$$\begin{aligned} & \frac{1}{2}(1-p) = \left(1 - \frac{1}{2}p\right)(2-r_2) \\ \iff \quad & \frac{2-2p}{4-2p} = 2 - r_2 \\ \iff \quad & r_2 = \frac{6-2p}{4-2p} \\ & \phantom{r_2} = \frac{3-p}{2-p} \end{aligned}$$

Then in an optimal mechanism, the low-value buyer pays 1 if she receives the good while the high-value buyer expects to pay $\frac{6-2p}{4-2p}$ if he receives the good. Outcomes follow efficiency criteria, with one of the high-value buyers receiving the item with probability 1 if one exists in the mechanism.

(d) Show that it is optimal to sell using a sealed second-price auction where buyers are only able to make one of two possible bids.

**Solution:** from the above argument, we may construct a sealed second-price auction in two possible bids. It is evident that one bid should be 1. The other, $b_2$, we solve using the previous expected payment equation. Using Bayes' rule, we see that the high-value buyer expects to pay the low-value buyer's bid with probability $\frac{2-2p}{2-p}$, conditional on winning; he expects to pay the high-value buyer's bid with probability $\frac{p}{2-p}$. Applying this to the equation above, we have

$$\begin{aligned} \left(\frac{2-2p}{2-p}\right)+\left(\frac{p}{2-p}\right)b_2 &= \frac{3-p}{2-p} \\ \iff \quad (2-2p)+pb_2 &= 3-p \\ \iff \quad pb_2 &= 1+p \\ \iff \quad b_2 &= \frac{1+p}{p} \end{aligned}$$

So we may construct a second-price auction in two possible bids, $b_1 = 1$ and $b_2 = \frac{1+p}{p}$, which is optimal and incentive compatible. Notice that, perhaps perversely, if the probability of meeting another high bidder is sufficiently low, $b_2$ is quite large (and well above 2, the bidder's valuation); the trick here is that we are supporting indifference between bids, not guaranteeing that the buyer is necessarily *happy* with the outcome ex post.

For pedagogy's sake, we'll ensure that indifference is achieved. The expected utility to the high-value buyer from bidding $b_1$ is

$$V_{21} = \frac{1}{2}(1-p)(2-1) = \frac{1-p}{2}$$

His expected utility from bidding $b_2$ is

$$\begin{aligned} V_{22} &= (1-p)(2-1)+\frac{1}{2}p\left(2-\frac{1+p}{p}\right) \\ &= (1-p)+\frac{1}{2}(p-1) \\ &= \frac{1-p}{2} \\ V_{22} &= V_{21} \end{aligned}$$

Thus the binding downward constraint is verified.

## 2010 Spring comp, question 6

Suppose $(\overline{z}_i)$ is a feasible allocation, $\sum_i \overline{z}_i = 0$, for the quasilinear model $v = (v_1, \ldots, v_N)$, where each $v_i$ is merely continuous.

In the following questions, determine if the statement is true or false. If true, demonstrate; if false, provide a counterexample.

(a) If there exists a positive integer $k$ and a feasible allocation $(z_{ih}^k)$ for the $k$-replica of $v$ such that

$$\sum_i \sum_h v_i(z_{ih}) > k \sum_i v_i(\overline{z}_i)$$

then $(\overline{z}_i)$ cannot be a price-taking equilibrium for $v$.

**Solution:** true[8].

Suppose that $(\overline{z}_i)$ is a price-taking equilibrium supported by prices $p$; then each consumer $i$ is solving $\overline{z}_i \in v_i^*(p)$. It follows that if we extend this allocation to the $k$-replicate, each agent is still solving

[8] This is the opposite of what I claimed in section.

$\overline{z}_{ih} \in v_i^*(p)$. Hence $(\overline{z}_{ih})$ is a price-taking equilibrium in the $k$-replicate, supported by prices $p$. Social welfare at this allocation is

$$\sum_h \sum_i v_i\left(\overline{z}_{ih}\right) = k \sum_i v_i\left(\overline{z}_i\right)$$

A price-taking equilibrium must be efficient[9]. If there exists some feasible allocation $(z_{ih}^k)$ such that $\sum_h \sum_i v_i(z_{ih}^k) > k \sum_i v_i(\overline{z}_i)$, the allocation $(\overline{z}_{ih})$ is not solving the maximization problem necessary for efficiency. Hence $(\overline{z}_{ih})$ cannot be a price-taking equilibrium, and so $(\overline{z}_i)$ is not a price-taking equilibrium; the claim in question is true.

(b) If for all positive integers $k$ and all feasible allocations $(z_{ih}^k)$, $h \in \{1, \ldots, k\}$ for the $k$-replica of $v$,

$$\sum_i \sum_h v_i(z_{ih}) \leq k \sum_i v_i\left(\overline{z}_i\right)$$

then $(\overline{z}_i)$ is a price-taking equilibrium for $v$.

**Solution:** true.

This is a version of the Second Welfare Theorem. That is, we can see that, since $z_{ih} = \overline{z}_i$ is a feasible allocation in every $k$ replica it must be that $(\overline{z}_i)$ maximizes social welfare in every $k$-replica; hence $(\overline{z}_i)$ is efficient, since efficiency is equivalent to maximization of social utility in the quasilinear model. From the Second Welfare Theorem and the principle of no wealth effects, it follows that $(\overline{z}_i)$ is a price-taking equilibrium.

Suppose that $(\overline{x}_i)$ is a feasible allocation for the ordinal preferences model of an exchange economy $\mathcal{E} = \{(X_i), (\succeq_i), (\omega_i)\}$, where $X_i = \mathbb{R}_+^\ell$ and each $\succeq_i$ is merely continuous.

(c) If there exists a positive integer $k$ and a feasible allocation $(x_{ih}^k)$ for the $k$-replica of $\mathcal{E}$ such that

$$\overline{x}_{ih}^k \succ_i \overline{x}_i, \qquad \forall i, h \in \{1, \ldots, k\}$$

then $(\overline{x}_i)$ cannot be a price-taking equilibrim for $\mathcal{E}$.

**Solution:** this depends on the definition of the demand function $d(\succeq_i, p)$ in the ordinal preferences model of an exchange economy without quasilinear utility[10]. In the quasilinear model, there is no free disposal (we must lie on the frontier of our budget set); we will presume the same thing here. In this case, the claim is demonstrably false, but in the case with free disposal, the claim is true.

Suppose there are 2 commodities, $\ell = 2$, and that there is a single agent $i$ with preferences $\succeq_i$ representable by

$$u_i(z; \kappa) = \max\left\{\min_{y \in \mathbb{N}^2}\left\{1 - \left|\left|\frac{z-y}{\kappa}\right|\right|\right\}, 0\right\}$$

Intuitively, for $\kappa$ sufficiently small this preference relation has the agent indifferent between most possible allocations, with steep "utility cones" around natural coordinates. In a sense, the agent wants to consume only whole amounts of goods and *hates* fractions; these preferences are continuous since all upper and lower contour sets are closed for any $\kappa > 0$. Suppose the agent's endowment is $\omega_i = (\frac{1}{2}, \frac{1}{2})$.

[9] This appears in the class notes, but the intuition isn't so bad in the differentiable case with nonsatiation: if we are not maximizing, we can make a differential change in allocations to improve social welfare. But this implies that agents' margins cannot equal the same price level, hence they are not utility-maximizing. Then the allocation is not an equilibrium. In particular, in the quasilinear framework preferences are necessarily locally nonsatiated since more of the money commodity is always a good thing; this is sufficient for the First Welfare Theorem.

[10] For which I cannot find a good definition in Ostroy's notes.

In the 1-replicate, we can support $\overline{x}_i = \omega_i$ as a price-taking equilibrium [roughly] with prices with a non-natural ratio, $\kappa$ sufficiently small. So let $p = (2, 3)$; then the agent's budget frontier is

$$z_2 = \frac{5 - 4z_1}{6}$$

Importantly, the intercepts are not natural numbers, so there must exist some $\kappa > 0$ which yields only 0 utility along the budget frontier; so given the price vector $p = (2, 3)$, the agent is maximizing utility by consuming his endowment.

Now consider the 2-replicate. By trading with himself (between replicas), the allocation

$$x_{i1} = (1, 0), x_{i2} = (0, 1)$$

may be attained (that is, this allocation is feasible). By construction, $x_{ih} \succeq_i \overline{x}_i$ for all $h$. Thus the premise of the question holds, but $\overline{x}_i$ may be supported as a price-taking equilibrium. Hence the statement is false.

*How does the statement change with free disposal?* Suppose there is some allocation $x^k_{ih}$ such that $x^k_{ih} \succ_i \overline{x}_i$ for all $i$, $h$. Assume that $\overline{x}_i$ is a price-taking equilibrium supported by prices $p$; we claim that at least one agent $ih$ is consuming somewhere which was initially feasible according to a budget constraint with free disposal. Suppose otherwise; then $p \cdot x^k_{ih} > p \cdot x_i$. It then follows that

$$\sum_h \sum_i p \cdot x^k_{ih} > k \sum_i p \cdot \overline{x}_i$$

$$\iff \qquad p \cdot \left( \sum_h \sum_i x^k_{ih} \right) > kp \cdot \sum_i \overline{x}_i$$

$$\iff \qquad p \cdot \left( k \sum_i e_i \right) > kp \cdot \sum_i e_i$$

But this is a contradiction! So we must have at least one agent with $p \cdot x^k_{ih} < p \cdot \overline{x}_i$. But since we have assumed $x^k_{ih} \succ_i \overline{x}_i$, this cannot hold with the agent utility-maximizing in a world with free disposal. It follows that if $\overline{x}_i$ is a price-taking equilibrium, there exists no $k$-replicate with feasible $x^k_{ih}$ such that $x^k_{ih} \succ_i \overline{x}_i$ for all $i$ and all $h$. Then the suggested claim is true.

(d) Suppose that for every $k$ and every feasible allocation $(x^k_{ih})$ for the $k$-replica of $\mathcal{E}$,

$$x^k_{ih} \succeq_i \overline{x}_i, \quad \forall i, h \in \{1, \ldots, k\} \qquad \Longrightarrow \qquad x^k_{ih} \sim_i \overline{x}_i$$

Can you conclude that $(\overline{x}_i)$ is a price-taking equilibrium for $\mathcal{E}$?

**Solution:** no (or, "false").

Suppose $\ell = 2$, and that preferences are representable by the utility function

$$u_i(x) = \sqrt{x_1} + \sqrt{x_2}$$

There is one agent, and his endowment is $(e_1, e_2) = (0, 1)$. In the unique feasible allocation in the 1-replica, $\overline{x}_i = (0, 1)$.

In all $k$-replicas of this economy, $[x^k_{ih}]_1 = 0$ by market clearing. Then if $x^k_{ih} \succeq_i \overline{x}_i$ for all $h$, we must have $x^k_{ih} = \overline{x}_i$; hence $x^k_{ih} \sim_i \overline{x}_i$ for all $h$. Thus the assumptions stated in the claim above hold.

However, $\overline{x}_i = (0, 1)$ is not a price-taking equilibrium. For $(0, 1)$ to possibly be supported as optimal consumption subject to prices, we must have $p_2 > 0$ — otherwise the demand for good 2 is infinite. Then since the agent is provided with a positive budget and the Inada conditions hold at 0 (the endowment of good 1), the agent will have strictly positive demand for good 1 *for any finite price level* $p_1$. Finite prices are an implicit requirement for equilibrium (the budget constraint is ill-defined if the agent has 0 units of good 1 but its price is infinite), so this cannot support a price-taking equilibrium.

Since the market clearing constraints tell us that $[\overline{x}_i]_1 = 0$, it follows that $\overline{x}_i$ is not a price-taking equilibrium.