This is a draft; email me with comments, typos, clarifications, etc.

The intuitive criterion

There has been some confusion recently regarding MWG's definition of the intuitive criterion as compared to Riley's. It is inarguable that Riley's definition is simpler (and, punnily, more intuitive); we can also show that, for the types of well-behaved problems we will be discussing in class, the two definitions are identical.

Definition

Riley's version: suppose we have a perfect Bayesian equilibrium with payoffs denoted by $u_i^N(\theta)$ for an agent of type θ. Let $\hat{a}_i$ be a strategy which is played with zero probability in this equilibrium, and let $u_i(\hat{a}_i, \theta_i)$ be player i's utility from playing $\hat{a}_i$ if it is believed (by all others) that she is type $\theta_i \in \Theta_i$. This equilibrium fails the *intuitive criterion* if for some player i of type $\hat{\theta}_i \in \Theta_i$,

$$u_i\left(\hat{a}_i, \hat{\theta}_i\right) > u_i^N\left(\hat{\theta}_i\right)$$

and for all other types $\theta \in \Theta_i \setminus \{\hat{\theta}_i\}$,

$$u_i\left(\hat{a}_i, \theta\right) < u_i^N\left(\theta\right)$$

Riley's variant says that an equilibrium fails the intuitive criterion if some player can posit a strategy which leaves her type better off but all others worse off. Note that, provided preferences are continuous (which we are given in Riley's questions, as utility functions are assumed continuous and differentiable) the second inequality in the above statement of the intuitive criterion may be replaced with $\leq$.

Definition

MWG's version: a perfect Bayesian equilibrium violates the *intuitive criterion* if there exists a type θ and an action a such that

$$\min_{s \in S^*(\Theta^{**}(a), a)} u_i(a, s, \theta) > u_i^*(\theta)$$

where $u_1^*(\theta)$ is the payoff to player i of type θ resulting in the perfect Bayesian equilibrium.

Reading MWG (pp469–471) is highly recommended, as we are papering over quite a few definitions here.

Citing MWG's definition leaves open the definition of $S^*(\Theta^{**}(a), a)$. We can give this analytically, but it will suit our present purposes to appeal to hand-waving. $\Theta^{**}(a)$ is the set of types which can do better by playing a for *some* beliefs supporting *some* best response s on the part of the other players. $S^*(\hat{\Theta}, a)$ for $\hat{\Theta} \subset \Theta$ is the set of equilibrium responses by players $-i$ which may follow i playing a if it is believed that $\theta \in \hat{\Theta}$.

So in the MWG definition, a perfect Bayesian equilibrium fails the intuitive criterion if the *worst* player i can do in equilibrium if other players believe her type is among those who may construct a deviation to a which is profitable for *some* best-response opponent strategy, is better than she is doing in the particular perfect Bayesian equilibrium (this sentence may need to be read several times). This is more concordant with Riley's definition than may appear: the only apparent difference is that MWG does not restrict the new strategy to be perfectly revealing of the player's type.

How can we show that Riley's definition is sufficient? We appeal to single-crossing and well-behaved utility functions in the case with finite types; this is more of a sketch of an argument than a formal proof, but hopefully the points hit home (and generalize well to the continuous-type case). Organize types into those who can possibly do better at a and those that cannot; WLOG, we can assume that there is some type which is perfectly indifferent between this new action and that in perfect Bayesian equilibrium (otherwise, by niceness of utility we can increase the desirability of the bundle in question to the point that some agent who was previously unhappy at the new allocation is now indifferent to it). By single-crossing, agents should be well-ordered: higher types prefer the new action and lower types prefer the existing equilibrium payoffs. Slide the new action along the indifferent type's indifference curve until the next-highest type is indifferent. Now slide the new action along the higher of the two indifferent type's indifference curve until the next-highest type is indifferent; at this point the original indifferent type will prefer the perfect Bayesian equilibrium outcome to the "slid" action (by single-crossing). We can iterate this process until only the highest type prefers the "slid" action; this implies the existence of some action profile which uniquely identifies a player of being of the highest type. Thus Riley's definition is satisfied.

Again, this argument deserves to be fleshed out (to ensure that it's valid and misses nothing) but hopefully it serves as a clarification of why the two perspectives are identical in the world with well-behaved utility functions and single crossing. It is also worth remembering that Riley's definition is far easier to recall and apply than the MWG definition.

Incentive compatibility

Incentive compatibility is a more formal structure around a concept we've been freely abusing throughout this class: it must be [weakly] optimal for agents to either truthfully report their type, or to play as to [weakly][1] reveal their type (these are, of course, two sides of the same coin once we apply the revelation principle).

Definition

A mechanism $\{(q_t, r_t)\}_t$ is *incentive compatible* if, for all types t and all types s,

$$(q_t, r_t) \succeq_t (q_s, r_s)$$

Hopefully it's clear how this definition might generalize outside of the Riley context (i.e., where allocations are not explicitly given by (q_t, r_t)). Now, because I have a habit of reversing the definitions of the local incentive constraints, here are two more useful definitions.

Definition

[1] We say "weakly" here to account for the possibility that pooling is optimal in equilibrium; in the event of pooling, types are of course generally not perfectly revealed, but on the other hand this implies no ill effects for agents in the system.

Type t's *local downward incentive constraint* is

$$(q_t, r_t) \succeq_t (q_{t-1}, r_{t-1})$$

Her *local upward incentive constraint* is

$$(q_t, r_t) \succeq_t (q_{t+1}, r_{t+1})$$

The best way to keep these two straight is to consider which direction an agent is looking to see whether or not she'd prefer to deviate; when she's looking up (from t to $t+1$) she's considering the local upward constraint, and when she's looking down (from t to $t-1$) she's considering the local downward constraint[2].

We have seen in the lecture notes that local incentive compatibility and single crossing imply global incentive compatibility; a slightly less general version of this also appeared in one of the questions in the week 2 handout.

In working through the problems below, there seems to be a general principle regarding the relationship between the intuitive criterion and incentive compatibility. While it is certainly possible for an incentive compatible mechanism to fail the intuitive criterion (consider strange Bayesian Nash equilibria; by off-path beliefs truthful reporting is generally optimal, but a different set of equilibrium beliefs could make the agents better off), it seems to hold that *not* being incentive compatible implies that the intuitive criterion is *not* satisfied. This may seem to be fairly obvious: if truthful reporting is not optimal, we should be able to alter the mechanism so that truthful reporting is optimal, which must yield a higher payoff for the player who was previously lying.

Let's put some concreteness around the concept: in many Riley questions, we are asked to show that any mechanism which satisfies the intuitive criterion must satisfy certain incentive compatibility requirements, assuming single crossing. In particular, it is generally asked to ensure that one of the local constraints must be binding. The argument universally proceeds as follows: if a particular constraint is not binding, we may alter one dimension of the allocation by ε, up or down (context-dependent) while retaining all relevant incentive constraints[3]. It follows then that the agent can credibly report off the equilibrium path in such a way as to respect incentive constraints, certainly revealing his type; but when this is the case, the intuitive criterion is violated. Thus violation of the intuitive criterion is often sufficient to assert that one of the local incentive constraints binds, assuming single crossing. Of course this argument is not particularly rigorous, but most Riley arguments regarding binding incentive constraints will follow this path.

Continuous types

As we move analysis to continuous-type cases, the essential underlying notion of incentive compatibility will remain the same; however, we can apply a system of well-developed analytical shortcuts to make analysis more succinct. This will, in general, require well-behavedness of the underlying functions (in particular, continuity and continuous differentiability) but this is a small price to pay for a sizable amount of economic intuition.

Riley has claimed in class that $V'(\theta) = u_\theta(q(\theta), r(\theta); \theta)$ is a sufficient condition (along with monotonicity of $q(\cdot)$, single crossing, and well-behavedness) for incentive compatibility in continuous types. We'll reproduce

[2] Yes, this is ridiculous; but if this saves even one person from screwing it up somewhere down the line it's done its job. As I've told undergraduates in the past, it's not my intention to treat anyone like a kindergartener, odd mnemonics just really help me (and therefore, by symmetry, others).

[3] Generally speaking, it is necessary to make some mention of "daisy-chain" type arguments to ensure that incentives don't unravel one direction or another; for an example of this, see *Essential Microeconomics* question 11.2-3 below. Still, arguments regarding unravelling generally follow the same set of steps that arguments which don't consider unravelling do, so they're uninteresting to consider at this high level.

his proof here, expanding the later sections slightly, and introduce some graphical intuition. In contrast to Riley's notation, we'll use $U(\theta;\theta_i)$ as the utility of an agent of type θ_i who *reports* type θ; I apologize for the switch but it is more intuitive in my eyes to parameterize utility by an agent's own type, and this avoids confusion as to which θ is reported and which is the true type.

Suppose $\theta_2 > \theta_1$. By incentive compatibility (and here regardless of the order of the types), we require

$$U(\theta_2;\theta_2) \geq U(\theta_1;\theta_2), \qquad U(\theta_1;\theta_1) \geq U(\theta_2;\theta_1)$$

That is, truthful reporting is weakly optimal. We can subtract $U(\theta_1;\theta_1)$ from both sides of the first inequality and subtract $U(\theta_2;\theta_2)$ from both sides of the second inequality; this yields

$$U(\theta_2;\theta_2) - U(\theta_1;\theta_1) \geq U(\theta_1;\theta_2) - U(\theta_1;\theta_1)$$

$$U(\theta_1;\theta_1) - U(\theta_2;\theta_2) \geq U(\theta_2;\theta_1) - U(\theta_2;\theta_2) \qquad \iff \qquad U(\theta_2;\theta_2) - U(\theta_1;\theta_1) \leq U(\theta_2;\theta_2) - U(\theta_2;\theta_1)$$

Of course, these inequalities will still hold if we divide by $\theta_2 - \theta_1$ (since we assume $\theta_2 > \theta_1$), so we have

$$\frac{U(\theta_2;\theta_2) - U(\theta_1;\theta_1)}{\theta_2 - \theta_1} \geq \frac{U(\theta_1;\theta_2) - U(\theta_1;\theta_1)}{\theta_2 - \theta_1}$$
$$\frac{U(\theta_2;\theta_2) - U(\theta_1;\theta_1)}{\theta_2 - \theta_1} \leq \frac{U(\theta_2;\theta_2) - U(\theta_2;\theta_1)}{\theta_2 - \theta_1}$$

Let's restrict our attention to the first equation, since the second will follow symmetrically. Under the assumption that truthful reporting is optimal, we have

$$\frac{V(\theta_2) - V(\theta_1)}{\theta_2 - \theta_1} \geq \frac{U(\theta_1;\theta_2) - U(\theta_1;\theta_1)}{\theta_2 - \theta_1}$$

Now let $\theta_1 \to \theta_2$; that is, let the types become arbitrarily close as we may in a continuous type space. We have, in the limit

$$\lim_{\theta_1 \to \theta_2} \frac{V(\theta_2) - V(\theta_1)}{\theta_2 - \theta_1} = V'(\theta_2)$$
$$\lim_{\theta_1 \to \theta_2} \frac{U(\theta_1;\theta_2) - U(\theta_1;\theta_1)}{\theta_2 - \theta_1} = U_2(\theta_2;\theta_2)$$

The latter limit is the derivative of U with respect to the agent's *true type*; we know, spanning the notational divide

$$U_2(\theta_2;\theta_2) = u_\theta(q(\theta_2), r(\theta_2);\theta_2)$$

Properties of limits then give us

$$V'(\theta_2) \geq u_\theta(q(\theta_2), r(\theta_2);\theta_2)$$

It should be evident that we can follow the same process with the second inequality, obtaining an analogous inequality with $\leq$. Piecing these two together, we obtain that incentive compatibility implies some shape on the value function,

$$V'(\theta) = u_\theta(q(\theta), r(\theta);\theta)$$

Graphically, what's going on here? Suppose that (q_t, r_t) is the equilibrium allocation of type θ_t. We can plot the function $u(q_t, r_t;\theta)$, representing the utility that an agent of type θ receives from the allocation (q_t, r_t); that is, this is the utility that an agent of type θ receives from *playing as if* he were type θ_t. In this graph, we are considering utility as a function of θ and ignoring all dynamics due to changes in (q, r).

To establish that the value function $V(\cdot)$ must be tangent to $u(q_t, r_t;\cdot)$ at θ_t, consider what happens if this tangency condition is not fulfillied; in particular, suppose that $V'(\theta_t) < u_\theta(q_t, r_t;\theta_t)$ (the opposing inequality will follow identically). Then by increasing θ_t infinitesimally to $\theta' = \theta_t + \varepsilon$, we know from local linearity

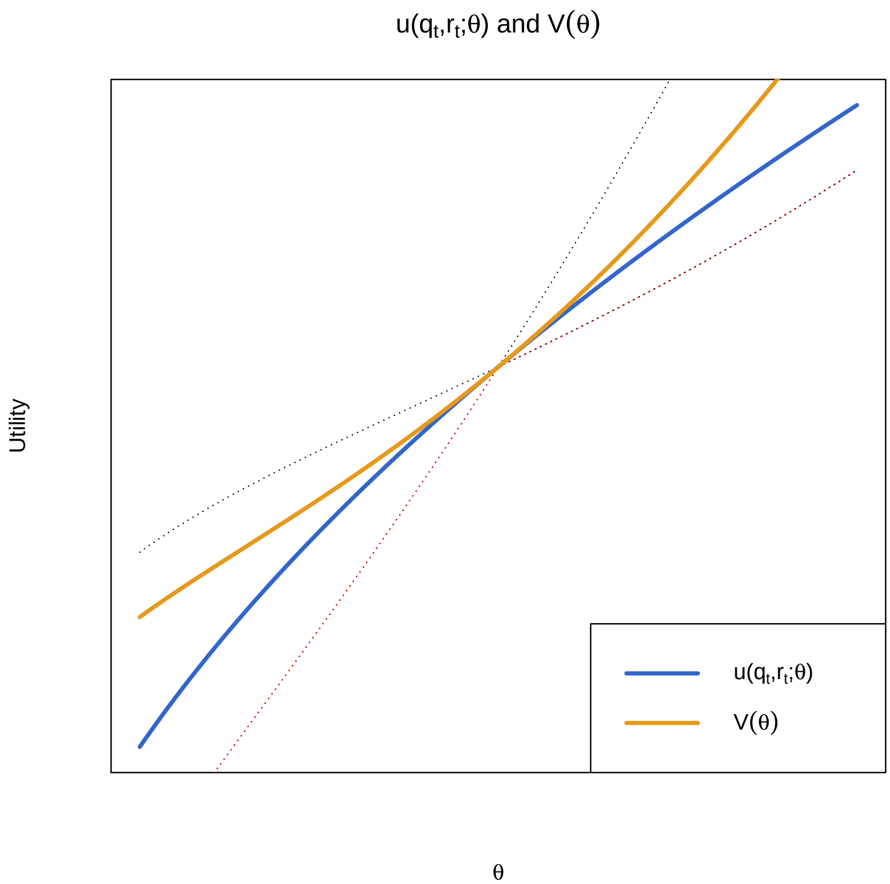

Figure 1: the value function $V(\cdot)$ must be tangent to $u(q(\theta), r(\theta); \cdot)$ for all types θ; otherwise, the utility available to neighboring types from deviating is greater than their optimal utility in the mechanism (the dashed red lines), a violation of incentive compatibility.

that $V(\theta') < u(q_t, r_t; \theta')$. That is to say, type θ''s *optimal* utility in the constrained problem is *less* than the utility she can get by playing as if she were type θ_t. So she has a positive incentive to misreport, a violation of incentive compatbility.

Since this logic will show us as well that $V'(\theta_t)$ cannot be greater than $u_\theta(q_t, r_t; \theta_t)$ we then have that incentive compatibility gives us $V'(\theta) = u_\theta(q(\theta), r(\theta); \theta)$ for all θ[4].

Essential Microeconomics, exercise 11.1-3

A type-θ consultant has a marginal product of $m(\theta) = k\theta$ where $\Theta = [0, \beta]$. The cost of accumulating educational credential q is $C(\theta, q) = \frac{q}{\theta^2}$ so that the payoff to a type-θ consultant if she receives a payment r for her services is

$$u(\theta, q, r) = r - \frac{q}{\theta^2}$$

[4]This last step — that establishing this for θ_t establishes it for all θ — may make implicit use of the envelope theorem; regardless, it's kosher.

(a) Show that a necessary condition for incentive compatibility in a separating equilibrium is $q'(\theta) = k\theta^2$.

Solution: to retain incentive compatibility, agents must utility-maximize. Then from the first-order conditions on the utility function,

$$\frac{\partial}{\partial \theta_i} u(\theta_i; \theta) = r'(\theta_i) - \frac{q'(\theta_i)}{\theta^2}$$

We know this quantity should equal 0. Further, from the firm's side we have $r(\theta_i) = m(\theta_i)$, so $r'(\theta_i) = k$. This gives us

$$k - \frac{q'(\theta_i)}{\theta^2} = 0$$

Under incentive compatibility, agents truthfully report; we obtain

$$k\theta^2 = q'(\theta)$$

(b) Let $V(\theta) = k\theta - \frac{q(\theta)}{\theta^2}$ be the equilibrium payoff. Show that $V'(\theta) = \frac{2q(\theta)}{\theta^3}$.

Solution: using the envelope theorem, we find the first derivative of the value function to be

$$\begin{aligned} V'(\theta) &= r'(\theta) - \frac{q'(\theta)}{\theta^2} + \frac{2q(\theta)}{\theta^3} \\ &= \frac{2q(\theta)}{\theta^3} \end{aligned}$$

(c) Hence show that the equilibrium payoff must satisfy the following differential equation:

$$2\theta V(\theta) + \theta^2 V'(\theta) = 2k\theta^2$$

Solution: substituting in for the first derivative of the payoff function,

$$V(\theta) = k\theta - \frac{\theta}{2} V'(\theta)$$

It follows that

$$2\theta V(\theta) = 2k\theta^2 - \theta^2 V'(\theta)$$

Rearranging,

$$2\theta V(\theta) + \theta^2 V'(\theta) = 2k\theta^2$$

(d) Solve this differential equation and hence show that $V(\theta) = \frac{2}{3}k\theta$.

Solution: we see that the above is equivalent to

$$\frac{\partial}{\partial \theta}\left[\theta^2 V(\theta)\right] = \frac{\partial}{\partial \theta}\left[\frac{2k}{3}\theta^3\right]$$

Integrating up from 0, we find

$$\theta^2 V(\theta) = \frac{2k}{3}\theta^3$$

Dividing out,

$$V(\theta) = \frac{2k\theta}{3}$$

(e) Let $r = R(q)$ be the equilibrium wage offered to a worker who chooses signal level q. Show that

$$R'(q) = \frac{1}{\theta^2}$$

Solution: this is equivalent to part (a) above; however, instead of differentiating with respect to type θ_i, we differentiate with respect to q:

$$\begin{aligned}\frac{\partial}{\partial q} u\left(q, R(q); \theta\right) &= \frac{\partial}{\partial q}\left[R(q) - \frac{q}{\theta^2}\right] \\ &= R'(q) - \frac{1}{\theta^2}\end{aligned}$$

It follows from maximization that

$$R'(q) = \frac{1}{\theta^2}$$

(f) Hence show that $R(q)^2 R'(q) = k^2$.

Solution: suppose that a play of q indicates type θ; then the equilibrium payment upon play of q is $R(q) = k\theta$. It follows that

$$R(q)^2 R'(q) = (k\theta)^2 \left(\frac{1}{\theta^2}\right) = k^2$$

(g) Solve for the equilibrium payment function.

Solution: the above is an apparent differential equation. Then integrating up from 0 (hand-waving over $R(0) = 0$), we see

$$\frac{1}{3} R(q)^3 = k^2 q$$

It follows that

$$R(q) = \sqrt[3]{3k^2 q}$$

Note that this may also be obtained from the value function. With $V(\theta) = \frac{2k\theta}{3}$ and $r(\theta) = k\theta$, we know that $\frac{q(\theta)}{\theta^2} = \frac{k\theta}{3}$. This gives us $q(\theta) = \frac{k\theta^3}{3}$; then substituting into the equation $r(\theta) = k\theta$ we can obtain

$$R(q) = \sqrt[3]{3k^2 q}$$

This method seems easier and more straightforward to me.

Essential Microeconomics, exercise 11.2-3

A type-θ worker's marginal product is $m(\theta)$ where $\theta \in \Theta = \{\theta_1, \ldots, \theta_T\}$. Her outside opportunity wage is r_0. The signalling cost function is $C(\theta, q)$.

Further, suppose $C(\theta, 0) = 0$, cost is strictly increasing in q, and that single crossing is satisfied.

(a) Show that to satisfy the intuitive criterion, the lowest type that signals, θ_t, must satisfy the following conditions: $m(\theta_t) - C(\theta_{t-1}, q_t) = r_0$, and $m(\theta_{t-1}) \leq r_0$.

Solution: suppose $m(\theta_t) - C(\theta_{t-1}, q_t) > r_0$. Then type θ_{t-1} is better off by misreporting type θ_t than by staying out of the mechanism (our presumption) so this cannot be a perfect Bayesian equilibrium. Now suppose $m(\theta_t) - C(\theta_{t-1}, q_t) < r_0$; type θ_{t-1} is now *strictly* opposed to entry. Under the intuitive criterion, type θ_t may be able to abuse this to capture more surplus.

Let $\hat{q}$ be such that $m(\theta_t) - C(\theta_{t-1}, \hat{q}) = 0$; clearly, $\hat{q} < q_t$. Let $q^* \in (\hat{q}, q_t)$, so

$$m(\theta_t) - C(\theta_t, q_t) < m(\theta_t) - C(\theta_t, q^*)$$

Further, since C is strictly increasing it follows that for all $t' < t$,

$$m(\theta_t) - C(\theta_{t'}, q^*) < r_0$$

If $(q_{t+1}, r_{t+1}) \succ_{t+1} (q_t, r_t)$, it is obvious from single crossing that we can pick some q^* in the appropriate range so that incentive compatibility still holds in all directions and type t is better off; thus the intuitive criterion is not validated. So suppose instead that $(q_{t+1}, r_{t+1}) \sim_{t+1} (q_t, r_t)$. Let $\bar{t}$ denote the lowest type so that $(q_{\bar{t}}, r_{\bar{t}}) \succ_{\bar{t}} (q_{\bar{t}-1}, r_{\bar{t}-1})$ (or T if there is none such). If $\bar{t} = T$, we know by single crossing that $(q_T, r_T) \prec_{T-1} (q_{T-1}, r_{T-1})$; then we can reduce q_T slightly to increase type θ_T's payoff while retaining incentive compatibility. Thus the intuitive criterion is still not satisfied.

Lastly, suppose $\bar{t} < T$. By single crossing, we know $(q_{\bar{t}-1}, r_{\bar{t}-1}) \prec_{\bar{t}-2} (q_{\bar{t}-2}, r_{\bar{t}-2})$. Following from this and our definition of $\bar{t}$, we can reduce $q_{\bar{t}-1}$ slightly to increase type $\theta_{\bar{t}-1}$'s payoff while retaining incentive compatibility. Thus the intuitive criterion is violated.

From these cases, we find that we must have $m(\theta_t) - C(\theta_{t-1}, q_t) = r_0$. This directly implies that $m(\theta_t) \geq r_0$. We now demonstrate that $m(\theta_{t-1}) \leq r_0$. Suppose to the contrary that $m(\theta_{t-1}) > r_0$. Then there is some q_{t-1} such that $C(\theta_{t-1}, q_{t-1}) = m(\theta_{t-1}) - r_0$; it is apparent that at such $(q_{t-1}, m(\theta_{t-1}))$, type θ_{t-1} is indifferent between her allocation and type θ_t's. By the single-crossing property, type θ_t must then strictly prefer his allocation to this new allocation for θ_{t-1}. From this strictness, we can reduce q_{t-1} slightly so that type θ_{t-1} strictly prefers $\theta_{t-1} - \varepsilon$ to the outside option r_0 while type θ_t still prefers his original bundle. Further, by C being increasing in the θ we know that lower types will also not see fit to deviate (for ε sufficiently small). Then θ_{t-1} has an available deviation which reveals her type, a violation of the intuitive criterion. We must then have $m(\theta_{t-1}) \leq r_0$.

(b) Taking the limit as the difference between types approaches zero, show that $q_t \to 0$ and $m(\theta_t) \to r_0$.

Solution: we begin by defining what it means (or rather, might mean) to have the difference between types go to 0. While the results seem to hold for a broad class of limiting tendencies, let $\Theta_1 = \Theta$ as above; recursively define

$$\Theta_{i+1} = \Theta_i \cup \left\{ \frac{\theta_j + \theta_{j+1}}{2} : 1 \leq j < |\Theta_i| \right\}$$

That is, Θ_{i+1} is Θ_i along with all consecutive means. Roughly speaking, as $i \to i+1$, $\Theta_{i \to i+1}$ becomes twice as dense.

Suppose $\theta_{t-1} \to \theta^*$; since $\theta_t - \theta_{t-1} \to 0$, it follows that $\theta_t \to \theta^*$, as well. Now having seen $m(\theta_t) \geq r_0$ and $m(\theta_{t-1}) \leq r_0$, so long as $m(\cdot)$ is continuous we should have

$$m(\theta_{t-1}) \to m(\theta^*), \qquad m(\theta_t) \to m(\theta^*)$$

By properties of limits, we know $m(\theta^*) \leq r_0$ and $m(\theta^*) \geq r_0$; hence $m(\theta^*) = r_0$, and $m(\theta_t) \to r_0$.

Again appealing to continuity, we know

$$m(\theta_t) - C(\theta_{t-1}, q_t) = r_0$$

It follows that, in the limit,

$$m(\theta^*) - C\left(\theta^*, \lim_{i \to \infty} q_t\right) = r_0$$

Subtracting known values,

$$C\left(\theta^*, \lim_{i \to \infty} q_t\right) = 0$$

Assumptions on the shape of the cost function then give us

$$\lim_{i\to\infty} q_t = 0$$

More succinctly, $q_t \to 0$.

Notice that these results are fairly independent of the precise manner in which types become dense; it is necessary above to state a particular mechanism only to establish some baseline for intuition.

Essential Microeconomics, exercise 11.2-6

A worker of type θ has a marginal product of $m(\theta, q) = 2\theta q^{\frac{1}{2}}$ if he achieves education level q. His cost of education is $C(\theta, q) = \frac{q}{\theta}$. Types are continuously distributed on the interval $[0, 4]$. There is no outside opportunity.

(a) With full information, show that type t will choose $q^*(\theta) = \theta^4$ and that his wage will be $m(\theta, q^*(\theta)) = 2\theta^3$.

Solution: under full information, the firm is aware of the agent's type as well as the signalled level of education. Then the equilibrium payment level is $r = 2\theta q^{\frac{1}{2}}$. The agent's optimization is

$$\max_q 2\theta q^{\frac{1}{2}} - \frac{q}{\theta}$$

First-order conditions give us

$$\theta q^{-\frac{1}{2}} = \frac{1}{\theta}$$

Rearranging, it is immediate that

$$q^*(\theta) = \theta^4$$

Substituting in for the agent's value to the firm, we find

$$\begin{aligned} m(\theta, q^*) &= 2\theta\sqrt{\theta^4} \\ &= 2\theta^3 \end{aligned}$$

(b) With asymmetric information, extend the argument above to show that the equilibrium wage function $r(q)$ must satisfy the following ordinary differential equation:

$$2r(q)\frac{\partial r}{\partial q} = 4q^{\frac{1}{2}}$$

Solution: under asymmetric information, we can consider the agent as optimally revealing a type; however we can remove the direct revelation aspect from the question and consider simply signalling a particular level of education. Following our usual approach, we see

$$\frac{\partial}{\partial q}u(q;\theta) = r'(q) - \frac{1}{\theta}$$

At the optimum, then,

$$r'(q) = \frac{1}{\theta}$$

According to the firm's optimal payout, $r(q) = 2\theta q^{\frac{1}{2}}$. Then multiplying both sides of the above by this quantity, we find

$$r(q)r'(q) = 2q^{\frac{1}{2}}$$

Multiplying by 2 we can obtain the desired result.

Note that the difference here between the present result and that in part (a) is completely due to whether or not the firm's response is endogenized by the agent (i.e., whether we are solving the planner's problem or the second-best allocation).

(c) Solve for the equilibrium level of education $q(\theta)$ and the wage function $r(q)$.

Solution: we may transform the above to obtain

$$\frac{\partial}{\partial q}\left[r(q)^2\right] = 4q^{\frac{1}{2}}$$

Integrating up, we find

$$r(q)^2 = \left(\frac{8}{3}\right)q^{\frac{3}{2}} \qquad \Longrightarrow \qquad r(q) = \sqrt{\frac{8}{3}}q^{\frac{3}{4}}$$

Recall from part (b) that, at the optimum $r'(q) = \frac{1}{\theta}$. It follows that

$$\left(\frac{3}{4}\right)\sqrt{\frac{8}{3}}q^{-\frac{1}{4}} = \sqrt{\frac{3}{2}}q^{-\frac{1}{4}} = \frac{1}{\theta}$$

Rearranging, this gives us

$$q^*(\theta) = \left(\frac{9}{4}\right)\theta^4$$

As is often the case in problems of asymmetric information, the agent is incurring an extra cost (in fact, 125% more cost) to signal her type to the firm; of course, this effect here is somewhat confounded as the agent is becoming simultaneously more useful to the firm, but the intuition provided should prove fairly general (in all circumstances where the agent finds it worth her while to report her type; asymmetric information could prove powerful enough — a la Akerloff's lemons model — to destroy any incentive whatsoever to signal).

If this method of solution (from parts (b) and (c)) seems unfamiliar, this question can also be solved (or verified, if need be) by the following process:

- Take first-order conditions of $u(\theta_i; \theta)$ with respect to θ_i.
- Use the above along with first-order conditions for optimality and the revelation principle to obtain an equation in θ alone (this includes the functions $r(\cdot)$ and $q(\cdot)$).
- Substitute in for the firm's response $r(\theta, q)$.
- Express the above as a differential equation in θ; this takes some serious rearrangement.
- Integrate up and obtain the desired result.

This series of steps is more in line with what we've seen this year. Still, it is useful to be versed in a number of suitable approaches to any particular question.

Essential Microeconomics, exercise 11.3-3

Each consumer purchases either one unit from a firm's product line or nothing at all. Different consumers place different values on product quality. A type θ_t consumer's value of a unit of quality q is

$$B(\theta_t, q) = \theta_t\left(10q - \frac{1}{2}q^2\right)$$

This is private information. The cost of producing a unit of quality q is $6q$. There are three types, $\theta_t \in \{1, 2, 3\}$ and there are equal numbers of each type.

(a) Solve for the optimal quality levels.

Solution: we interpret this question to mean that we should find the *first-best* (i.e., full information) quality levels. Assuming that the firm charges its cost to the agents, the maximization is

$$q_t = \operatorname*{argmax}_q \theta_t \left(10q - \frac{1}{2}q^2\right) - 6q$$

First-order conditions give us

$$q_t = 10\theta_t - 6$$

(b) Solve for the profit-maximizing quality levels.

Solution: we assume that we are no longer in the first-best solution, and that types are private knowledge. Further, it seems reasonable given the demands and type distributions that it will be optimal to provide three distinct packages; we'll see later that this causes trouble.

From the single-crossing property, we know that local downward constraints will bind (for a more thorough take on this, check out part (d) of the 2010 Spring comp question below). This gives us the following equalities:

$$\begin{aligned} r_1 &= B(\theta_1, q_1) \\ r_2 - r_1 &= B(\theta_2, q_2) - B(\theta_2, q_1) \\ r_3 - r_2 &= B(\theta_3, q_3) - B(\theta_3, q_2) \end{aligned}$$

These may be restated to give us maximizing price levels of

$$\begin{aligned} r_1 &= B(\theta_1, q_1) \\ r_2 &= B(\theta_2, q_2) - B(\theta_2, q_1) + B(\theta_1, q_1) \\ r_3 &= B(\theta_3, q_3) - B(\theta_3, q_2) + B(\theta_2, q_2) - B(\theta_2, q_1) + B(\theta_1, q_1) \end{aligned}$$

The firm's problem is then stated as (with equal numbers of each type, we can ignore distributional terms)

$$\begin{aligned} &\max_{r_i, q_i} r_1 + r_2 + r_3 - 6q_1 - 6q_2 - 6q_3, \qquad \text{s.t. IC} \\ =&\max_{q_i} 3B(\theta_1, q_1) + 2B(\theta_2, q_2) - 2B(\theta_2, q_1) + B(\theta_3, q_3) - B(\theta_3, q_2) - 6(q_1 + q_2 + q_3) \end{aligned}$$

We know

$$B_q(\theta, q) = \theta(10 - q)$$

Then first-order conditions from the firm's problem give us

$$\begin{aligned} \frac{\partial}{\partial q_1} &: \qquad 0 = 3\theta_1(10 - q_1) - 2\theta_2(10 - q_1) - 6 \\ \frac{\partial}{\partial q_2} &: \qquad 0 = 2\theta_2(10 - q_2) - \theta_3(10 - q_2) - 6 \\ \frac{\partial}{\partial q_3} &: \qquad 0 = \theta_3(10 - q_3) - 6 \end{aligned}$$

Solving these equations we obtain

$$q_1 = 10 - \frac{6}{3\theta_1 - 2\theta_2}$$
$$q_2 = 10 - \frac{6}{2\theta_2 - \theta_3}$$
$$q_3 = 10 - \frac{6}{\theta_3}$$

Plugging in for known values of θ_t,

$$q_1 = 16$$
$$q_2 = 4$$
$$q_3 = 8$$

However, it is evident that q_1 is not amenable to incentive compatibility! The firm now has two options: it can either pool some types together, or it can cut type θ_1 out of the market. In the latter case, the firm will extract full surplus from type θ_2, leading to an optimization of the form

$$\max_{q_i} 2B(\theta_2, q_2) + B(\theta_3, q_3) - B(\theta_3, q_2) - 6(q_2 + q_3)$$

First-order conditions will give us

$$q_1 = 0$$
$$q_2 = 4$$
$$q_3 = 8$$

We can see that it will not be optimal to have types θ_2 and θ_3 pool (θ_3's incentive constraint will cause headaches in this case); what if types θ_1 and θ_2 pool? Intuitively, this will result in lower surplus for the firm per-agent, but since this allows more agents to participate the firm may be better off. Type θ_1's surplus will be fully-extracted, and type θ_3's downward constraint will bind; the firm's optimization is then

$$\max_{q_i} 3B(\theta_1, q_1) + B(\theta_3, q_3) - B(\theta_3, q_1) - 6(2q_1 + q_3)$$

First-order conditions here give us $q_1 = \pm\infty$, a clear problem.

It follows that optimal quantities are $(q_1, q_2, q_3) = (0, 4, 8)$.

(c) Solve for the price charged for each product in the monopolist's product line.

Solution: using the above incentive constraints, we have

$$\begin{aligned} r_1 &= 0 \\ r_2 &= 2\left(10(4) - \frac{1}{2}4^2\right) \\ &= 64 \\ r_3 &= 3\left(10(8) - \frac{1}{2}8^2\right) - 3\left(10(4) - \frac{1}{4}4^2\right) + r_2 \\ &= 3(48) - 3(32) + 64 \\ &= 112 \end{aligned}$$

The firm's profits are then

$$\pi = 64 + 112 - 6(4 + 8) = 104$$

2010 practice midterm, question 4

Consider an educational signalling model where the marginal product of a type-θ worker is θ and her cost of signalling is

$$\text{(i)} \quad C_1(\theta, x) = \frac{x}{1+\theta}$$
$$\text{(ii)} \quad C_2(\theta, y) = \frac{y^2}{1+\theta}$$
$$\text{(iii)} \quad C_3(\theta, z) = \frac{z^3}{2+\theta}$$

It is unclear whether these technologies are meant as different subproblems or as technologies to the same problem; we take the latter approach although, in hindsight, the former would prove significantly easier.

(a) If there are two types, $\theta \in \{\theta_1, \theta_2\}$, where $0 < \theta_1 < \theta_2$ characterize the equilibrium payoff in each case *if the equilibrium satisfies the intuitive criterion.*

Solution: already, we know that at least one type should signal 0 education. Since $\theta_1 < \theta_2$, the cost to type θ_1 for obtaining a particular level of education is always greater than the cost to type θ_2; it follows that type θ_1 should obtain $q = 0$ and receive $r = \theta_1$.

What will θ_2 do in an equilibrium satisfying the intuitive criterion? We need to solve for type θ_1's indifference along all three mechanisms to see which performs best for type θ_2[5]. Solving generically, we have

$$\theta_1 = \theta_2 - \frac{q^k}{b_k + \theta_1}$$

It is apparent that for indifference, we need

$$q^k = (\theta_2 - \theta_1)(b_k + \theta_1)$$

To satisfy the intutive criterion, type θ_2 must select the mechanism which minimizes his cost. We then check

$$\min_k \frac{q^k}{b_k + \theta_2} = \min_k \frac{(\theta_2 - \theta_1)(b_k + \theta_1)}{b_k + \theta_2}$$

It is evident that this is minimized when we minimize

$$\min_k \frac{b_k + \theta_1}{b_k + \theta_2}$$

With $\theta_1 < \theta_2$, the mechanism selected will have $b_k = 1$ and so education will be obtained from either mechanism 1 or mechanism 2. In either event, the payoffs to type θ_2 are identical. Fully-described equilibrium payoffs are then

$$u(\theta_1) = \theta_1, \qquad u(\theta_2) = \theta_2 - \frac{(\theta_2 - \theta_1)(1 + \theta_1)}{1 + \theta_2}$$

Notice that when θ_1 is close to θ_2, type θ_2 does not need to obtain much education to keep type θ_1 from copying; this is because the rewards from deviation are greatly reduced.

[5] There is an embedded assumption here that is not generally made clear: not only is the level of education observable, but so is the mechanism by which this education is obtained; that is, the signal is now not only quantity of education but also the signalling mechanism. Although we can abstract away from this if necessary (consider, for example, a single technology which allows agents to choose the minimum from a menu of costs; we could also apply incomplete information to the situation) it is useful in this context to consider education as, say, getting a PhD or a Bachelor's from Harvard or DeVry.

(b) Repeat the analysis if types are continuously-distributed on $[0, 1]$.

Solution: we know

$$V(\theta) = r(\theta) - \frac{q(\theta)^k}{b_k + \theta}$$

It follows from the envelope theorem that

$$V'(\theta) = \frac{q(\theta)^k}{(b_k + \theta)^2}$$

We then see

$$V(\theta) = r(\theta) - V'(\theta)(b_k + \theta)$$

Knowing $r(\theta) = \theta$, we may integrate up and find

$$V(\theta) = \frac{\theta^2}{2\,(b_k + \theta)}$$

Notice that this implies that there is no difference between signalling technologies 1 and 2! Further, signalling technology 3 is dominated for all $\theta > 0$. We perform one further check to see if there are any possible tangencies between the signalling technologies (focusing only on technologies 1 and 3),

$$V_1'(\theta) = \frac{\theta^2 + 2\theta}{2(1 + \theta)^2}$$
$$V_3'(\theta) = \frac{\theta^2 + 4\theta}{2(2 + \theta)^2}$$

Rearranging, we have

$$V_1'(\theta) = \frac{\theta}{2(1 + \theta)} + \frac{\theta}{2(1 + \theta)^2}$$
$$V_3'(\theta) = \frac{\theta}{2(2 + \theta)} + \frac{2\theta}{2(2 + \theta)^2}$$

It is easy to see algebraically that $V_1'(\theta) > V_3'(\theta)$ for all $\theta > 0$, so there are no possible points of identical slope along the two value functions. It follows that there is now possible method to substitute to an alternate technology in a profitable way. Then to satisfy the intuitive criterion, all agents use signalling technology 1, or all agents use signalling technology 2 and obtain values of

$$V(\theta) = \frac{\theta^2}{2 + 2\theta}$$

2010 Spring comp, question 5

(a) A monopoly offers different "plans," where a plan is a payment r for q units. That is, a q-pack costs r. What is the single-crossing property? Confirm that it holds if a type-t agent's demand price function $p_t(q)$ is greater for higher types.

Solution: the single-crossing property claims that higher types have a stronger preference for the commodity good. In particular, for all $s < t$ and $q' < q$ we have

$$(q, r) \succeq_s (q', r') \qquad \Longrightarrow \qquad (q, r) \succ_t (q', r')$$

Buyer surplus is defined as

$$B_t(q, r) = \int_0^q p_t(x)dx - r$$

Now suppose we have $(q,r),(q',r')$ with $q'<q$ so that

$$B_s(q,r) \geq B_s(q',r')$$

It follows that

$$\int_{q'}^{q} p_s(x)dx \geq r' - r$$

Assuming $s<t$, we have $p_s(q) < p_t(q)$. Then we must have

$$\int_{q'}^{q} p_t(x)dx > r' - r$$

and it is immediate that

$$B_t(q,r) > B_t(q',r')$$

Then when $(q,r) \succeq_s (q',r')$ and $p_s(q) < p_t(q)$, $(q,r) \succ_t (q','r')$, and the single-crossing property holds.

(b) Let (q_t, r_t) be the plan selected by type-t buyers. That is, a type-t buyer pays r_t for q_t units. Show that it is necessarily the case that if $s<t$, then $q_s < q_t$.

Solution: for incentive compatibility, we need $(q_s,r_s) \succeq_s (q_t,r_t)$. As argued above, if $q_t < q_s$ we must have $(q_s,r_s) \succ_t (q_t,r_t)$, a violation of incentive compatibility. Now suppose $q_t = q_s$; for $(q_s,r_s) \succeq_s (q_t,r_t)$ we must have $r_s \leq r_t$ (provided utility is decreasing in r). Assuming that we are representing a separating equilibrium, we cannot have $r_s = r_t$, so $r_s < r_t$. But if this is the case, we have $(q_s,r_s) \succ_t (q_t,r_t)$, a violation of single crossing. Therefore $s<t$ implies $q_s<q_t$.

(c) Prove that for any $\{(q_t,r_t)\}$ satisfying the above monotonicity condition, revenue is maximized if and only if the local downward constraints are binding.

Solution: suppose that the local downward constraints do not bind, $(q_t,r_t) \succ_t (q_s,r_s)$ for $s<t$. Then since $(q_s,r_s) \succeq_s (q_t,r_t)$, we may increase r_t slightly while maintaining the local upward constraint for s and the local downward constraint for t. But this has the effect of increasing the firm's revenue while maintaining all constraints.

The only *real* concern here is that by increasing r_t we will violate type-t's upward constraint. But we can increase $r_{t'}$ along with r_t (for any $t'>t$) to maintain the local upward constraints for all higher types; this further increases firm revenue. Moreover, we are assured that participation constraints will not be an issue since the lower types weakly prefer entry and we have single crossing.

Thus if the local downward constraint is not binding, we have the ability to raise the price r_t of a q_t pack without affecting an agent's willingness to purchase. This implies that we may freely increase revenue at least slightly, and thus the firm is not revenue-maximizing.

Showing the other direction is problematic: it is not hard to construct a monotonic separating equilibrium in which the local downward constraints are binding but revenue is not maximized (for example, take the quantities in any such equilibrium and let $q_t' = \frac{1}{2}q_t$; set r_t' so that the local downward constraints bind; revenue here *should be* either lower or higher than in the previous case, indicating that revenue in one case or the other is not maximized). We take this, then, to intend that *fixing* $\{q_t\}_{t=1}^T$, binding constraints imply revenue maximization; however this is apparent from the methods above. The only way to obtain more revenue, keeping q_t fixed, is to raise r_t. Since local downward constraints are binding, raising r_t will violate incentive compatibility, establishing the impossibility of the firm doing any better.

It follows that revenue is maximized if and only if the local downward constraints bind, fixing $\{q_t\}_{t=1}^T$.

(d) In the two-type case, suppose that the number of each type is the same ($N_1 = N_2 = N$). Total cost is a strictly convex function $C(Nq_1 + Nq_2)$. The demand price functions are $p_q = 100 - \frac{1}{2}q_1$ and

$p_2 = 120 - \frac{1}{2}q_2$. If it is profit-maximizing to offer two plans, what can you say about the difference in the number of units in each plan?

Solution: to begin, notice that one agent or the other must receive 0 net utility; otherwise, by our standard arguments, the firm is not profit-maximizing. Suppose that type θ_2 receives 0 utility; then we have

$$r_2 = \int_0^{q_2} 120 - \frac{1}{2}q dq = 120q_2 - \frac{1}{4}q_2^2$$

By binding local downward constraints,

$$\begin{aligned} r_1 &= \int_0^{q_1} 120 - \frac{1}{2}q dq - \int_0^{q_2} 120 - \frac{1}{2}q dq + r_2 \\ &= -\int_{q_1}^{q_2} 120 - \frac{1}{2}q dq + 120q_2 - \frac{1}{4}q_2^2 \\ &= 120q_1 - \frac{1}{4}q_1^2 \end{aligned}$$

Type θ_1's utility is then

$$\int_0^{q_1} 100 - \frac{1}{2}q dq - r_1 = -20q_1$$

So if $q_1 > 0$, type θ_1's participation constraint is violated!

It follows then that type θ_1's participation constraint must bind. Then we have

$$r_1 = \int_0^{q_1} 100 - \frac{1}{2}q dq$$

From binding local downward constraints, we know

$$\int_0^{q_2} 120 - \frac{1}{2}q dq - r_2 = \int_0^{q_1} 120 - \frac{1}{2}q dq - r_1$$

This gives us

$$r_2 - r_1 = \int_{q_1}^{q_2} 120 - \frac{1}{2}q dq$$

We can substitute these constrained-optimal prices into the firm's maximization problem (recalling that there are N agents of each type),

$$\max_{q_1, q_2} N \int_0^{q_1} 100 - \frac{1}{2}q dq + N \left(\int_{q_1}^{q_2} 120 - \frac{1}{2}q dq + \int_0^{q_1} 100 - \frac{1}{2}q dq \right) - C(Nq_1 + Nq_2)$$

First-order conditions yield

$$\begin{aligned} \frac{\partial}{\partial q_1}: &\quad 0 = 2N\left(100 - \frac{1}{2}q_1\right) - N\left(120 - \frac{1}{2}q_1\right) - NC'(Nq_1 + Nq_2) \\ \frac{\partial}{\partial q_2}: &\quad 0 = N\left(120 - \frac{1}{2}q_2\right) - NC'(Nq_1 + Nq_2) \end{aligned}$$

Equating both sides, the $C'(\cdot)$ terms will cancel; this gives us

$$\begin{aligned} & 2N\left(100 - \frac{1}{2}q_1\right) - N\left(120 - \frac{1}{2}q_1\right) = N\left(120 - \frac{1}{2}q_2\right) \\ \iff & 200 - q_1 - 120 + \frac{1}{2}q_1 = 120 - \frac{1}{2}q_2 \\ \iff & -40 - \frac{1}{2}q_1 = -\frac{1}{2}q_2 \\ \iff & q_2 = q_1 + 80 \end{aligned}$$

Then at the firm's optimum, the difference between the supplied quantities is $q_2 - q_1 = 80$.