Elasticity

Suppose that market demand in an economy is given by

$$D(p) = 100 - 10p.$$

It is clear that increasing the price of the commodity in question will reduce demand by 10 units; in this particular setting, this is true no matter what particular price we start at. However, depending on the current price the implications of reducing demand by 10 units will vary drastically.

Consider starting at $p = 1$; then $q = D(1) = 90$. Raising the price to $p' = 2$ leads to $q' = D(2) = 80$, a not insignificant but also not large change in demand. If instead we start at $p = 8$, then $q = D(1) = 20$. Raising the price to $p' = 9$ leads to $q' = D(9) = 10$. That is, increasing the price by the same amount $(p' - p = 1)$ has resulted in a *halving* of demand.

Elasticity captures this notion. We define the elasticity of a good x, with respect to its own price, by

$$\mathcal{E}_x(p_x) = \frac{\partial D(p_x)/\partial p_x}{D(p_x)/p_x} = \frac{\partial D(p_x)}{\partial p_x}\left(\frac{p_x}{D(p_x)}\right) = \frac{\partial q_x}{\partial p_x}\left(\frac{p_x}{q_x}\right).$$

That is, elasticity is the derivative of demand divided by the ratio of demand to price (alternatively: the derivative of demand times the ratio of price to demand). In a less easy-to-calculate notation, we may write this as

$$\mathcal{E}_x(p_x) = \frac{\partial q_x}{q_x}\left(\frac{\partial p_x}{p_x}\right)^{-1};$$

that is, elasticity is the change in quantity divided by quantity, all divided by the change in price divided by price. Change in a variable divided by its value simply represents percent change — e.g., if some variable moves from x to x', $(x' - x)/x$ is the percentage by which the quantitiy increased — so we generally say that elasticity is the percentage change in quantity divided by the percentage change in price. This will capture by how much demand will change in response to a particular change in price.

Since, in general, consumption of a good should *decrease* in response to an increase in price, the derivative of demand with respect to price should be negative. We often talk only about the absolute value of elasticity (if you downloaded the draft of these notes, you'll see this in the absolute value floating around everywhere), but in general it is a negative quantity.

- If $\mathcal{E}_x(p_x) > -1$, then demand for x is *inelastic* at p_x.
- If $\mathcal{E}_x(p_x) = -1$, then demand for x is *unit elastic* at p_x.
- If $\mathcal{E}_x(p_x) < -1$, then demand for x is *elastic* at p_x.

Problem: *suppose that demand for x is $D(p_x) = 100 - 10p_x$. Find the elasticity of x as a function of its price.*

Solution: we know $\mathcal{E}_x(p_x) = (\partial q_x/\partial p_x)(p_x/q_x)$. Our first step is then to compute the derivative of quantity with respect to price,

$$\frac{\partial q_x}{\partial p_x} = D'(p_x) = -10.$$

We also can see that

$$\frac{p_x}{q_x} = \frac{p_x}{100 - 10p_x}.$$

Hence elasticity is given by

$$\mathcal{E}_x(p_x) = -10\left(\frac{p_x}{100 - 10p_x}\right) = -\frac{p_x}{10 - p_x}.$$

□

Problem: *suppose that $\mathcal{E}_x(p_x) = -1$ when $p_x = 5$ and $q_x = 50$. Use elasticity to estimate demand when prices change to $p'_x = 5.5$.*

Solution: we know that elasticity captures percentage change in quantity in response to percentage change in price. When prices move from 5 to 5.5, they have increased by $(5.5-5)/5 = 10\%$. For simple estimation, an elasticity of 1 (demand is unit elastic) says that in response to a 10% change in prices demand should also change by 10%.

In particular, since prices have increased demand should decrease, so the new quantity demanded should be $50(90\%) = 45$.

We can view this somewhat more concretely using the loose definition of

$$\mathcal{E}_x(p_x) = \frac{\Delta q_x}{q_x}\left(\frac{p_x}{\Delta p_x}\right)^{-1} \quad \Longrightarrow \quad \frac{\Delta q_x}{q_x} = \mathcal{E}_x(p_x)\left(\frac{\Delta p_x}{p_x}\right).$$

In this case, elasticity is -1 and prices have increased by 10%. Hence

$$\frac{\Delta q_x}{q_x} = -1(10\%) \quad \Longrightarrow \quad \Delta q_x = -10\%(q_x) = -5.$$

Since $\Delta q_x = q'_x - q_x$ is the change in x, it follows that $q'_x = 45$. □

Problem: *suppose that demand is unit elastic when $p_x = 5$ and $q_x = 50$. You know that demand for x is linear, $D(p_x) = a - bp_x$. What is the demand function?*

Solution: in this case, we know that

$$\frac{\partial q_x}{p_x} = D'(p_x) = -b.$$

Since elasticity is defined by

$$\mathcal{E}_x(p_x) = \frac{\partial q_x}{\partial p_x}\left(\frac{p_x}{q_x}\right),$$

we know (since demand is unit elastic, $\mathcal{E}_x(p_x) = -1$)

$$-1 = -b\left(\frac{p_x}{q_x}\right) = -b\left(\frac{5}{50}\right) \quad \Longrightarrow \quad b = 10.$$

With $b = 10$, we know

$$D(p_x) = a - 10p_x.$$

Substituting in for given prices and quantities, we have

$$50 = a - 10(5) \quad \Longrightarrow \quad a = 100.$$

Hence demand is given by

$$D(p_x) = 100 - 10p_x.$$

□

Exchange economies

If economics were only the study of one person making one choice, it would have gone the way of phrenology a long time ago: if we would like to pretend to describe interactions in the real world, we had better start accounting for the fact that in the real world, there is more than one person.

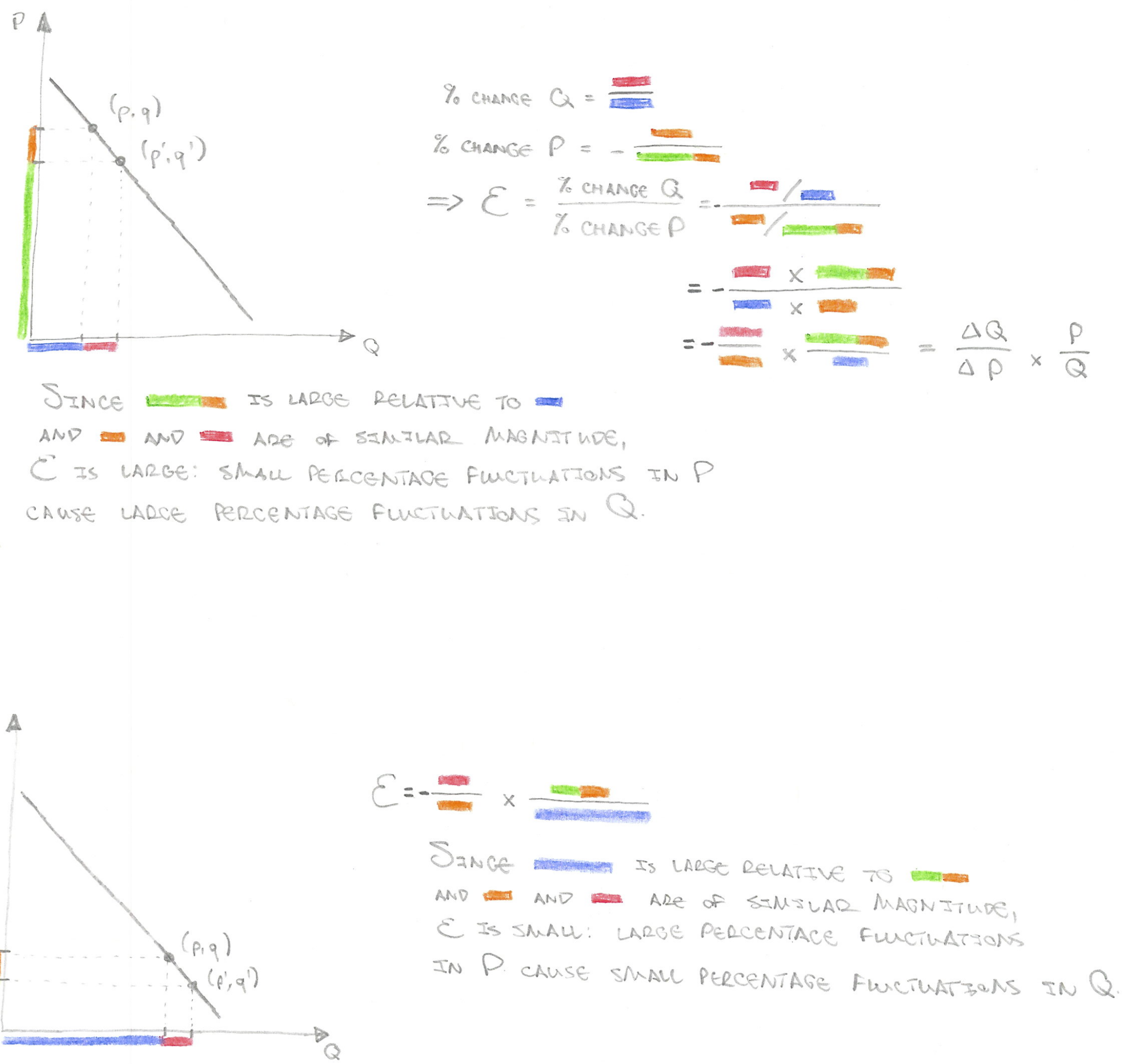

Figure 1: a graphical description of where elasticity comes from. Even with a constant, linear demand curve elasticity changes with prices and quantities.

Although we have the capacity now to think about both consumption and production, the simplest case to analyze is one without firms: we consider a world with only consumers. They may differ in their tastes and what they enter the world with — their *endowment* — but once together they come to some agreement as to how they should trade their goods to their mutual benefit.

As a concrete example, imagine a world where I have 20 Ferraris and no hamburgers, and you have no Ferraris and 10000 hamburgers. Evidently, you would probably like a means of transportation and I would prefer to have at least some food, so there are *gains from trade*. Studying exchange economies provides us with a baseline prediction of just how many Ferraris I give you and how many hamburgers you give me in return.

Wealth

While we are familiar with the concept of maximizing utility subject to a budget constraint, we need to step back and consider how a budget arises. In the context of an exchange economy, wealth comes from the ability of each agent to sell her endowment. This can be conceptualized by thinking about the agent selling everything they own for purchasing power, then spending all this purchasing power to buy their consumption; of course, they may be buying back some of their initial endowment, but in the abstract this is simply endowments leading to wealth leading to consumption.

In an economy with two goods, x and y, we therefore consider the underlying prices of these commodities, p_x and p_y. If the agent's endowment is $e = (e_x, e_y)$, her effective wealth is

$$w = p_x e_x + p_y e_y.$$

This is what will enter into the constraint as her budget when she is maximizing her utility.

Market clearing

A particular concern in an economy with agents bringing their own commodities is that consumption is fundamentally bounded: since all of the, e.g., x in the system is brought in initially by the agents themselves, they logically cannot consume more than the amount they (together) started with. If we denote agent 1's consumption of x by x_1 and agent 2's consumption of x by x_2, this may be stated as

$$x_1 + x_2 \leq e_{x_1} + e_{x_2}.$$

We generally also assume that they cannot throw away any of a commodity, so we use the stronger condition

$$x_1 + x_2 = e_{x_1} + e_{x_2}.$$

This is called the *market clearing constraint* for good x. The language suggests a world in which all of the x available is put into some central marketplace; once all of the x has been purchased, the market is empty, or clear.

Of course, we need to consider this constraint for each commodity in the system. In a world with only two goods x and y, the market clearing constraints together are

$$x_1 + x_2 = e_{x_1} + e_{x_2}, \qquad y_1 + y_2 = e_{y_1} + e_{y_2}.$$

Equilibrium

What behavior/outcomes do we expect to see in an exchange economy? There is a natural concept akin to stability: we should observe outcomes which the agents have no direct means of improving. If all agents are doing as well as they can in a system, there is no reason that anything should change; we refer to this notion generically as *equilibrium.*

In the particular case of an exchange economy with two goods, we have a very concise definition.

Definition

Given an exchange economy, an *exchange equilibrium* is prices (p_x, p_y) and a consumption bundle for each agent $c_i = (x_i, y_i)$ such that

- agents utility-maximize, given prices
- markets clear.

It should be clear how this generalizes to a case with more than two commodities. If a markets clear, we will alternativeely refer to consumption as *feasible.*

Solution method

The definition of equilibrium in an exchange economy gives some clear hints as to how to determine equilibrium outcomes; in particular, we will need to maximize utility and find prices such that markets clear. This can be achieved in the following steps.

① **Determine the budget constraints.** In order to make any claims about agents maximizing utility subject to their budget constraints, we need to know what their budget constraints are!

② **Solve for agent demand as a function of prices and budgets.** This of course involves its own set of steps; keep in mind that since we want to *find* prices, we cannot simply substitute in for them. The results here will be demand for goods as a function of p_x and p_y (since an agent's budget is a function of p_x and p_y, these will be the only variables in the system).

③ **Find prices which clear markets.** We have equations for demand for each agent above, and we know that together consumptions cannot exceed endowments. By substituting demand into the market clearing constraints, we can find *relative* prices. *In general, we cannot find explicit prices, so it is safe to normalize $p_x = 1$ and then determine a numeric value for p_y. It is also acceptable to leave p_x and p_y as variables and catalog their relationship in your answer.*

④ **Use prices to determine consumption.** Substitute the prices determined in equilibrium into each agent's demand function to find consumption levels.

Examples

Problem: *suppose there are two agents, with $u_1(x_1, y_1) = x_1^{1/3} y_1^{2/3}$ and $u_2(x_2, y_2) = x_2^{2/3} y_2^{1/3}$; endowments are given by $e_1 = (2, 1)$ and $e_2 = (1, 2)$. Find equilibrium in this exchange economy.*

Solution: we will follow the steps above precisely.

① With prices p_x and p_y, the budgets of the two agents are given by

$$w_1 = 2p_x + p_y, \quad w_2 = p_x + 2p_y.$$

② In a more general setting, we would solve for optimal demand by equating marginal utilities per unit cost across all goods. However, since these demand functions are Cobb-Douglas, we are well-set to use our familiar shortcut:

$$x_1 = \frac{1}{3}\left(\frac{w_1}{p_x}\right), \qquad y_1 = \frac{2}{3}\left(\frac{w_1}{p_y}\right),$$
$$x_2 = \frac{2}{3}\left(\frac{w_2}{p_x}\right), \qquad y_2 = \frac{1}{3}\left(\frac{w_2}{p_y}\right).$$

③ The market clearing constraints in this system are

$$x_1 + x_2 = e_{x_1} + e_{x_2} = 2 + 1 = 3, \qquad y_1 + y_2 = e_{y_1} + e_{y_2} = 1 + 2 = 3.$$

We substitute our x-demand equations from step ② into the market clearing equation for good x,

$$\begin{aligned} & x_1 + x_2 = 3 \\ \iff \quad & \frac{1}{3}\left(\frac{w_1}{p_x}\right) + \frac{2}{3}\left(\frac{w_2}{p_x}\right) = 3 \\ \iff \quad & w_1 + 2w_2 = 9p_x \\ \iff \quad & (2p_x + p_y) + 2(p_x + 2p_y) = 9p_x \\ \iff \quad & 5p_y = 5p_x \\ \iff \quad & p_y = p_x. \end{aligned}$$

At this point, we *could* substitute into the market clearing constraint for good y. However, since we can only ever determine relative prices we are actually done; as an exercise, try working through the above logic but with tyhe market clearing constraint for good y: you will obtain the same result.

It is fair to say simply "$p_x = p_y$," or as economists prefer, "$p_x/p_y = 1$." That being said, there is nothing wrong with normalizing $p_x = 1$ (unless, of course, you are explicitly told not to, or are told to normalize p_x to some other value, or are told to normalize p_y instead); when we make this assumption, we have $p_x = p_y = 1$.

④ Having found $p_x = p_y = 1$, we know

$$w_1 = 2p_x + p_y = 3, \qquad w_2 = p_x + 2p_y = 3.$$

Hence demand is given by

$$x_1 = \frac{1}{3}\left(\frac{3}{1}\right) = 1, \qquad y_1 = \frac{2}{3}\left(\frac{3}{1}\right) = 2,$$
$$x_2 = \frac{2}{3}\left(\frac{3}{1}\right) = 2, \qquad y_2 = \frac{1}{3}\left(\frac{3}{1}\right) = 1.$$

Having followed these steps, we say that equilibrium is prices $(p_x, p_y) = (1, 1)$ and consumption $(x_1, y_1) = (1, 2)$, $(x_2, y_2) = (2, 1)$. □

Problem: *suppose that there are two agents, with* $u_1(x_1, y_1) = x_1 + \ln y_1$ *and* $u_2(x_2, y_2) = x_2 + 2\ln y_2$*; endowments are given by* $e_1 = (1, 3)$*,* $e_2 = (1, 3)$*. Find equilibrium in this exchange economy.*

Solution: we will follow the steps above precisely.

① With prices p_x and p_y, the budgets of the two agents are given by

$$w_1 = p_x + 3p_y, \qquad w_2 = p_x + 3p_y.$$

② To find demand, we need to find marginal utilities.

$$\text{MU}_{x_1} = 1, \qquad \text{MU}_{y_1} = \frac{1}{y},$$
$$\text{MU}_{x_2} = 1, \qquad \text{MU}_{y_2} = \frac{2}{y}.$$

Equating marginal utility per unit cost for each agent, we have

$$\frac{\text{MU}_{x_1}}{p_x} = \frac{\text{MU}_{y_1}}{p_y} \rightsquigarrow \frac{1}{p_x} = \frac{1}{y_1 p_y} \implies y_1 = \frac{p_x}{p_y},$$
$$\frac{\text{MU}_{x_2}}{p_x} = \frac{\text{MU}_{y_2}}{p_y} \rightsquigarrow \frac{1}{p_x} = \frac{2}{y_2 p_y} \implies y_2 = \frac{2p_x}{p_y}.$$

Demand for the good x will come from the budget constraint, as usual:

$$p_x x_1 + p_y y_1 = w_1 \rightsquigarrow p_x x_1 + p_x = w_1 \implies x_1 = \frac{w_1 - p_x}{p_x},$$
$$p_x x_2 + p_y y_2 = w_2 \rightsquigarrow p_x x_2 + 2p_x = w_2 \implies x_2 = \frac{w_2 - 2p_x}{p_x}.$$

③ The market clearing constraints in this system are

$$x_1 + x_2 = e_{x_1} + e_{x_2} = 1 + 1 = 2, \qquad y_1 + y_2 = e_{y_1} + e_{y_2} = 3 + 3 = 6.$$

We substitute our y-demand equations from step ② into the market clearing constraint for y,

$$\begin{aligned} & y_1 + y_2 = 6 \\ \iff & \frac{p_x}{p_y} + \frac{2p_x}{p_y} = 6 \\ \iff & 3p_x = 6p_y \\ \iff & p_x = 2p_y. \end{aligned}$$

Again, we can say either "$p_x = 2p_y$," or "$p_x/p_y = 2$." However, it will be algebraically simplest to normalize $p_x = 1$, which gives us $p_x = 1$, $p_y = 1/2$.

④ Having found $p_x = 1$, $p_y = 1/2$, we know

$$w_1 = p_x + 3p_y = \frac{5}{2}, \qquad w_2 = p_x + 3p_y = \frac{5}{2}.$$

Hence demand is given by

$$x_1 = \frac{w_1 - p_x}{p_x} = \frac{3}{2}, \qquad y_1 = \frac{p_x}{p_y} = 2,$$
$$x_2 = \frac{w_2 - 2p_x}{p_x} = \frac{1}{2}, \qquad y_2 = \frac{2p_x}{p_y} = 4.$$

Having followed these steps, we say that equilibrium is prices $(p_x, p_y) = (1, 1/2)$ and consumption $(x_1, y_1) = (3/2, 2)$, $(x_2, y_2) = (1/2, 4)$.

Follow-up question: what happens if each agent's initial endowment of x is significantly smaller, say $e_1 = e_2 = (1/4, 3)$? Though the change is easy to state, the question is significantly more complicated! Why is this? □

Pareto optimality

Equilibrium describes a particular outcome of an economy. However, economists are often concerned with what *might* happen, regardless of the initial state of the system. There has been a robust historical debate on the matter,[1] but we have roughly settled on the following criterion: given an outcome, if there is no other available outcome which leaves *every* agent in the system weakly better off and at least one agent strictly better off, this outcome is not unreasonable.

This of course has some drawbacks. Nearly the entire world would be better off if George Lucas quit remastering the *Star Wars* movies, everyone except for George Lucas himself. Since ceasing his endless re-edits would reduce George Lucas' utility, economists cannot unequivocally say that putting the kibosh on is a superior outcome. This example, albeit ridiculous, indicates that this criterion for predicting outcomes is extremely weak.

More robustly, a *social allocation* is a vector c^* consisting of a consumption bundle c_i for each agent i in the economy; that is, if there are N people in an economy, a social allocation is $c^* = (c_1, c_2, \ldots, c_N)$. A social allocation is *feasible* if markets clear for each good, in the sense defined above.

Definition

A feasible social allocation c^* is *Pareto optimal* (or *Pareto efficient*) if there is no feasible social allocation c' such that $c'_i \succeq_i c^*_i$ for all agents i, and $c'_i \succ_i c^*_i$ for some i. That is, c^* if Pareto optimal if there is no feasible social allocation which makes every agent weakly better off while making some agent strictly better off.

We will interchange *Pareto optimal* and *Pareto efficient* with simply *optimal* and *efficient*, as suits our needs and the constraints of time. There is a related concept of *weak* Pareto efficiency, which holds when it is not possible for *every* agent to be made better off. In the context of Econ 11, these concepts are identical; proving this is beyond the scope of class.

Mathematical characterization

It is convenient to think about Pareto optima in terms of marginal rates of substitution. Suppose that $\text{MRS}^1_{x,y} > \text{MRS}^2_{x,y}$; that is, agent 1's marginal rate of substitution between x and y is higher than that of agent 2. To make this concrete, let $\text{MRS}^1_{x,y} = 2$ and $\text{MRS}^2_{x,y} = 1/2$, so that agent 1 is willing to give up two units of y for one unit of x, and agent 2 is willing to give up two units of x for one unit of y.

Consider the following trade: agent 1 gives agent 2 one infinitesimal unit of y for one infinitesimal unit of x. Since agent 1 effectively likes x twice as much as y, she is better off from this trade; similarly, since agent 2 likes y twice as much as x, he is also better off. The fact that this trade exists implies that where the agents started cannot be a Pareto optimum: there was a feasible allocation (resulting from trade) that left them both better off!

Since the only thing we used in this example was that agents have different marginal rates of substitution[2] it follows that at a Pareto optimum, marginal rates of substitution must be equal across agents. In general, we will write this in somewhat longer form as equal marginal utility ratios; since the marginal rate of substituion *is* the ratio of marginal utilities, there is no issue here.

[1] For a not inconsiderable period of time, the utilitarians held the predominant position: if punching your TA makes you happier than getting punched makes him sad, then you should punch your TA in equilibrium. With the advent of modern undergraduate education, it was apparent that this criterion would cause significant grief and the predominant view shifted.

[2] Technically, we also assumed that the trade described is feasible; that is, that is it possible for agent 1 to give agent 2 an infinitesimal unit of y and for agent 2 to give agent 1 an infinitesimal unit of x. At a corner solution, this may not hold.

Solution method

① **Determine the market clearing constraints.** At a Pareto optimum, we know that we cannot improve both agents' utilities at any feasible social allocation. To make this claim, we need to know feasibility, which is defined by the market clearing constraints.[3]

② **Find quantities which solve $\mathbf{MRS}^1_{x,y} = \mathbf{MRS}^2_{x,y} \rightsquigarrow \mathbf{MU}_{x_1}/\mathbf{MU}_{y_1} = \mathbf{MU}_{x_2}/\mathbf{MU}_{y_2}$.** Since indifference curves should be tangent to one another at an efficient allocation, we can find Pareto optima by tracing out the set of tangencies of indifference curves of the two agents.

③ **Use the above equations to find an expression for Pareto optima.** From the above steps, you should have three equations in four unknowns: x_1, y_1, x_2, and y_2. You will be able to vary one variable — say, x_1 — and see what it implies about the remaining variables. Since we are often concerned with the *set* of Pareto optima — not a particular optimum — it is important to express how consumptions interact with one another.

Example

Problem: *suppose there are two agents, with $u_1(x_1, y_1) = x_1^{1/3} y_1^{2/3}$ and $u_2(x_2, y_2) = x_2^{2/3} y_2^{1/3}$; endowments are given by $e_1 = (2, 1)$ and $e_2 = (1, 2)$. Find the set of Pareto optimal allocations in this economy.*

Solution: we will follow the steps above precisely.

① Market clearing is specified by

$$x_1 + x_2 = e_{x_1} + e_{x_2} = 2 + 1 = 3, \qquad y_1 + y_2 = e_{y_1} + e_{y_2} = 1 + 2 = 3.$$

② To equate marginal utility ratios across agents, we need to find marginal utilites. Notice that while we can use the Cobb-Douglas shortcut with regard to finding optimal consumption under a budget constraint (as in the example above), we don't [yet] have a shortcut for equating marginal utility ratios for two agents with Cobb-Douglas utility.

$$\mathrm{MU}_{x_1} = \frac{1}{3} x_1^{-2/3} y_1^{2/3}, \qquad \mathrm{MU}_{y_1} = \frac{2}{3} x_1^{1/3} y_1^{-1/3},$$
$$\mathrm{MU}_{x_2} = \frac{2}{3} x_2^{-1/3} y_2^{1/3}, \qquad \mathrm{MU}_{y_2} = \frac{1}{3} x_2^{2/3} y_2^{-2/3}.$$

Hence marginal utility ratios are

$$\frac{\mathrm{MU}_{x_1}}{\mathrm{MU}_{y_1}} = \frac{\frac{1}{3} x_1^{-2/3} y_1^{2/3}}{\frac{2}{3} x_1^{1/3} y_1^{-1/3}} = \frac{y_1}{2x_1}, \qquad \frac{\mathrm{MU}_{x_2}}{\mathrm{MU}_{y_2}} = \frac{\frac{2}{3} x_2^{-1/3} y_2^{1/3}}{\frac{1}{3} x_2^{2/3} y_2^{-2/3}} = \frac{2y_2}{x_2}.$$

Equating marginal utility ratios, we have

$$\frac{y_1}{2x_1} = \frac{2y_2}{x_2} \qquad \Longrightarrow \qquad x_2 y_1 = 4 x_1 y_2.$$

③ Using the market clearing constraints, we can see

$$x_2 = 3 - x_1, \qquad y_2 = 3 - y_1.$$

[3] As you saw on the midterm (see the special section notes on corner solutions), sometimes feasibility is a little broader than "nonnegative consumption." For the purposes of economies with more than one agent, I doubt we'll introduce these complexities as they make the math considerably ugly.

Substituting into the result from step ② above, we can compute

$$\begin{aligned}
& & x_2 y_1 &= 4x_1 y_2 \\
&\rightsquigarrow & (3 - x_1)y_1 &= 4x_1(3 - y_1) \\
&\implies & 3y_1 - x_1 y_1 &= 12x_1 - 4x_1 y_1 \\
&\implies & 3y_1 &= 12x_1 - 3x_1 y_1 \\
&\implies & y_1 &= 4x_1 - x_1 y_1 \\
&\implies & y_1 &= \frac{4x_1}{x_1 + 1}, \\
&\rightsquigarrow & y_2 &= \frac{3 - x_1}{x_1 + 1}.
\end{aligned}$$

Considering that $x_1 \in [0, 3]$, the rigorous expression of Pareto optima is then

$$\left\{ \left(x_1, \frac{4x_1}{x_1 + 1} \right), \left(3 - x_1, \frac{3 - x_1}{x_1 + 1} \right) : x_1 \in [0, 3] \right\}.$$

□

The Edgeworth box

Throughout, we will appeal to the Cobb-Douglas example we used above in finding equilibrium and the set of Pareto optima.

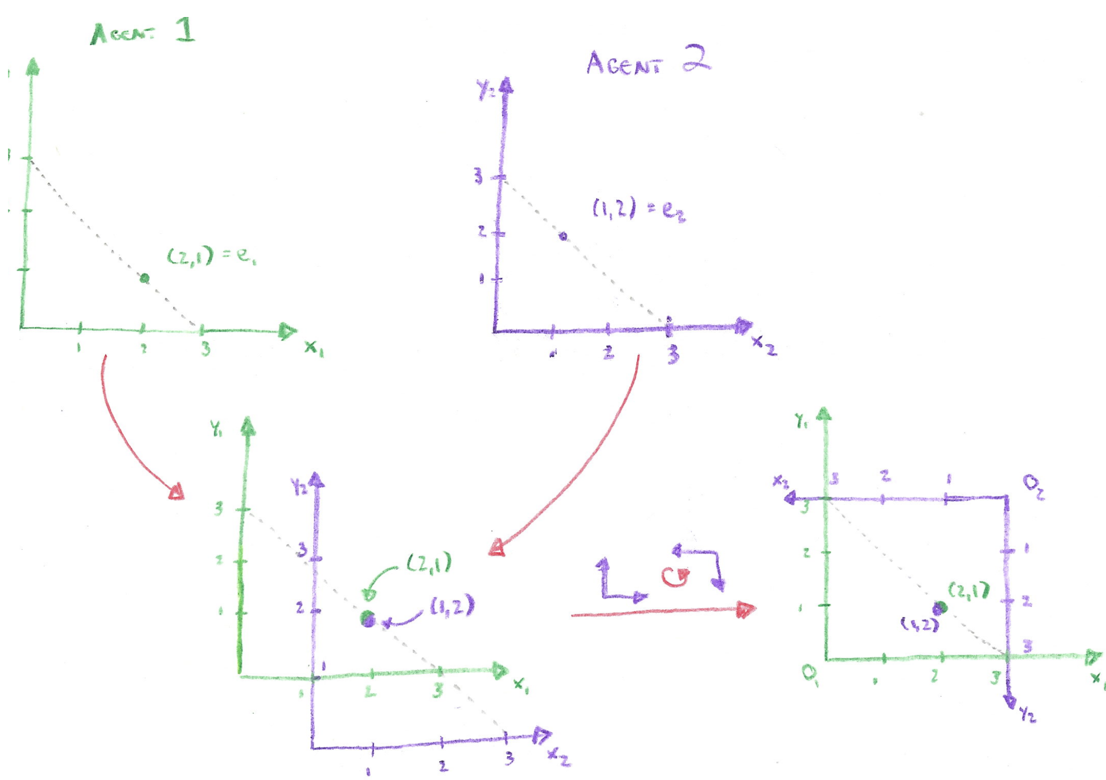

Figure 2: one way of getting from a standard pair of axes for each agent to an Edgeworth box.

Illustration of equilibrium in an exchange economy is somewhat more involved than illustrating the solution to a single agent's maximization problem. In particular, there are tradeoffs in equilibrium — allocation of more of a good to one agent implies allocation of less of the same good to another agent — that we would like to capture in a diagram. This is the role of the Edgeworth box.

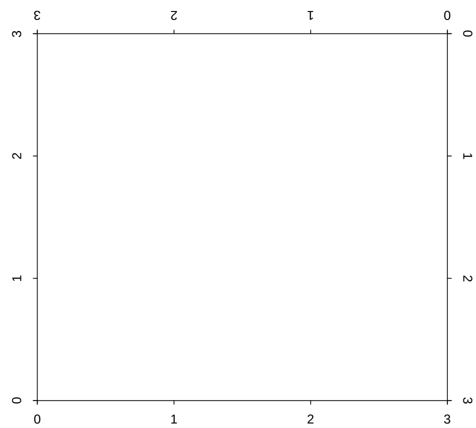

Figure 3: as described in the text: the axes for agent 1 are standard, and the axes for agent 2 are inverted and placed to intersect with agent 1's axes at the points at which agent 1 receives all of the available units of a particular good. Since there are 3 units of x in the economy, one intersection should be $(3, 0)$; since there are 3 units of y in the economy, the other intersection should be $(0, 3)$.

That the Edgeworth box is a box rather than a pair of axes represents a point of fundamental importance in an exchange economy: resources are limited. In particular, if agent 1 receives more good x then agent 2 must receive less. By placing points in a box rather than on axes, we can show that moving to the right from agent 1's perspective — increasing her allocation of good x — is identical to moving to the left from agent 2's perspective — decreasing his allocation of good x.

Understanding that this is what we would like to represent, constructing the bounds of the Edgeworth box is pretty simple: draw a pair of axes for agent 1; then draw an *inverted* pair of axes for agent 2 which intersect with the axes for agent 1 at the points at which agent 1 receives all of the available units of a particular good. This is illustrated in Figures 2 and 3.

The box itself is not inherently descriptive; what matters, of course, is what we draw within its confines. Conveniently, since we already know how to plot most things we are interested in on a standard pair of axes, plotting them in an Edgeworth box is only a matter of remembering to rotate if we are drawing for agent 2. To begin, let's add the endowment point to the Edgeworth box. Note that by plotting agent 1's endowment on her axes, we have already plotted agent 2's endowment on his axes! Since any point in the box represents allocations (x_1, y_1) and (x_2, y_2) such that

$$x_1 + x_2 = e_{x_1} + e_{x_2}, \quad y_1 + y_2 = e_{y_1} + e_{y_2},$$

if we plot the point $(x_1, y_1) = (e_{x_1}, e_{y_1}) = e_1$ it equally represents the point $(x_2, y_2) = (e_{x_2}, e_{y_2}) = e_2$. This is shown in Figure 4.

Adding indifference curves through the endowment point is barely any more complicated: drawing agent 1's indifference curve is straightforward; for agent 2, drawing the indifference curve amounts to rotating the piece of paper you are working on by 180 degrees and then drawing his indifference curve in the standard way. In this context, notice that the area that is above both agent 1's indifference curve (from her perspective)

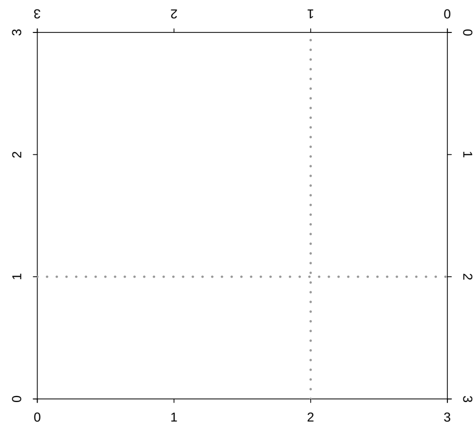

Figure 4: notice that the point $(x_1, y_1) = (2, 1)$ is identical to the point $(x_2, y_2) = (1, 2)$, since we are perfectly capturing allocational tradeoffs within the Edgeworth box.

and above agent 2's indifference curve (from his perspective) represents feasible allocations which leave both agents better off. This is demonstrated in Figure 5.

We may also be asked to show equilibrium an prices; again, we can proceed by plotting for agent 1 and letting agent 2 follow. In particular, for the same reasons that we need only plot a single point to show the initial endowments, we will only need to plot a single point to show the equilibrium allocation. With regards to the budget frontier, consider a bundle (x_1, y_1) which is affordable for agent 1. Then we know

$$\begin{aligned} & & p_x x_1 + p_y y_1 &= p_x e_{x_1} + p_y e_{y_1} \\ \Longleftrightarrow & & p_x x_1 + p_y y_1 &= p_x(e_{x_1} + e_{x_2} - e_{x_2}) + p_y(e_{y_1} + e_{y_2} - e_{y_2}) \\ \Longleftrightarrow & & p_x e_{x_2} + p_y e_{y_2} &= p_x(e_{x_1} + e_{x_2} - x_1) + p_y(e_{y_1} + e_{y_2} - y_1) \\ \Longleftrightarrow & & p_x e_{x_2} + p_y e_{y_2} &= p_x x_2 + p_y y_2. \end{aligned}$$

It follows that if (x_1, y_1) is perfectly affordable for agent 1, then the implicit (x_2, y_2) is perfectly affordable for agent 2! Thus we need only concern ourselves with plotting a single budget frontier, as anything strictly affordable for agent 1 lies outside agent 2's budget set, and vice-versa. This is demonstrated in Figure 6.

The last thing ever necessary to plot in an Edgeworth box is the set of Pareto optimal allocations. While the mathematical description of the set of Pareto optima is a little daunting, notice that the bundle for agent 1 is given by

$$\left(x_1, \frac{4x_1}{x_1 + 1} \right).$$

Again, since any bundle for agent 1 tells us exactly what agent 2's bundle is, we need only concern ourselves with this set. And in particular, this phrasing has a nice feature: y_1 is a simple function of x_1,

$$y_1 = \frac{4x_1}{x_1 + 1}.$$

Plotting this by hand is of course nonobvious but can be done by selecting a few points. This function — and hence the set of Pareto optima — is graphed in Figure 7.

[5] Although perhaps seeing that equilibrium prices go through the corners of the figure indicates a part of why this question has "nice" answers.

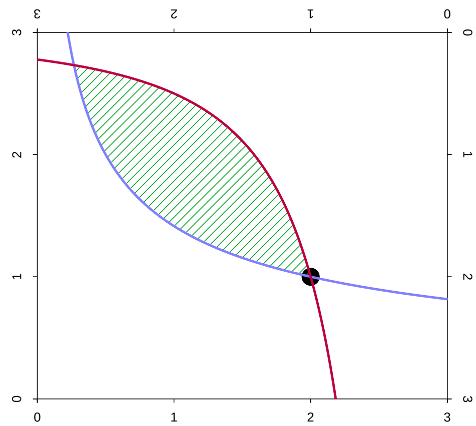

Figure 5: indifference curves for agents 1 (blue) and 2 (red); notice that although agent 2's indifference curve looks like it pokes outward, it actually slopes inward like a standard indifference curve when viewed from agent 2's perspective. The shaded green area represents social allocations in which both agents are better off than they are at the endowment point.

As a last aside, notice that we have been iteratively adding and removing things from our Edgeworth box: as a graphical device with significant demonstrative capabilities, the number of things we can show within an Edgeworth box greatly exceed the number of things which can be placed into it in a legible way. Be careful when constructing your boxes! Choose details which are relevant to the question you are trying to answer and ignore those which are irrelevant; the above are not a perfect reference, but they are a start. For posterity — and because computers make it easy — an Edgeworth box "with everything" is given in Figure 8.

[7] There are of course technical caveats here, but they are few.

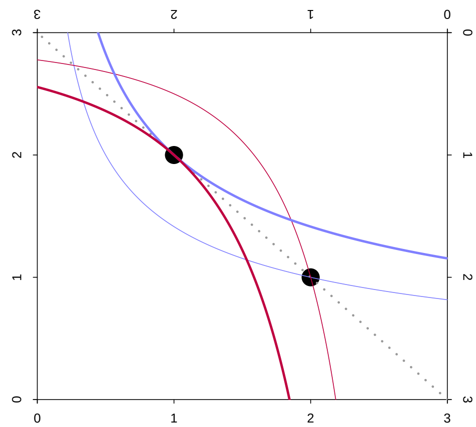

Figure 6: exchange equilibrium shown in an Edgeworth box; the original indifference curves are drawn lightly. Notice that the equilibrium allocation lies well within the set of allocations which are strictly preferred to the endowment by each agent. That the budget frontier goes through the corners of the figure is *not* a general feature and is particular to the endowments chosen in this question.[5]

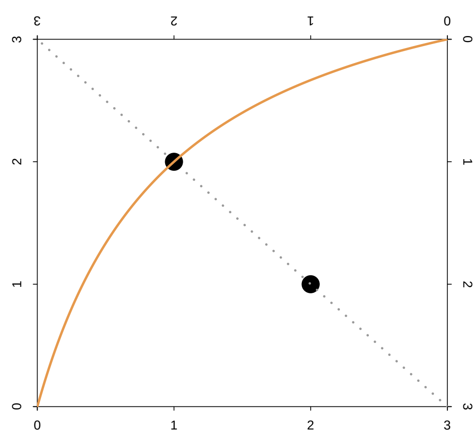

Figure 7: the set of Pareto optima in this exchange economy. Notice that the equilibrium allocation is precisely where the set of Pareto optima intersects the equilibrium budget frontier: this is due to a much more general feature, that exchange equilibria are Pareto efficient.[7]

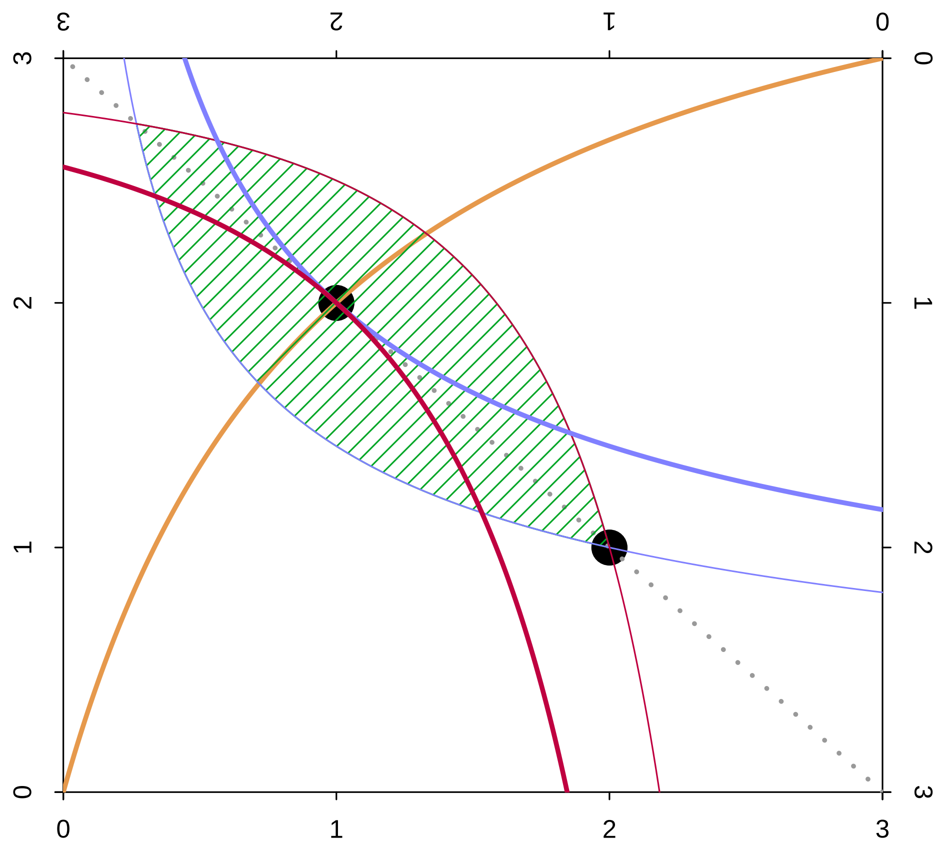

Figure 8: a very poor way to visualize the economy. Carefully choose what you place on a graph, rather than just dumping everything because you can.