Competition

When we first talked about firms, we asked ourselves what a single firm might do in isolation. This left us with two possibilities: either the firm is so small relative to the market that it cannot affect prices (this is the case of perfect competition), or the firm is the only firm in the market and it uniquely determines prices (this is the case of monopoly). While either case can give us some interesting stories and useful intuition about what might happen in the economy, in reality there are often neither an extremely large number of firms in an industry, nor just a single firm in an industry.

As an example, consider the market for soda; you may think this is a stretch, but also assume that you can't tell the difference between different brands of soda. How many types of cola are out there? Coke and Pepsi are the big ones, and if you're feeling up to the challenge you might be able to remember that Vons/Pavilions/Safeway has its own in-house brand (Refreshe) and there's RC Cola out there, too. So if we can't taste the difference between these kinds of soda, the products are essentially identical and since there is more than one firm in the market this isn't a monopoly. But since we'd be hard-pressed to think of more than, say, half a dozen brands of soda, it's also not perfectly competitive; if Coke changes its production or pricing tomorrow, it's natural to assume that Pepsi and the others will follow suit.

The case where two firms compete for market share is referred to as *duopoly* (the same basic structure as *monopoly*, but using a different prefix to indicate that there are now two firms). If there are more than two, but still relatively few, firms in the market we refer to the situation as *oligopoly* (fun fact: *oligo-* is a prefix which means "few" or "scanty"). We'll see later that as the number of firms increases, analysing the market becomes increasingly difficult and tedious. For this reason, we'll restrict most of our analyses to the case of duopoly.

While it's pretty obvious that firms *do* compete in the real world, it's not necessarily obvious *how* they compete. For this reason, we concern ourselves separately with three methods of competition:

- **Cournot competition**: firms compete on quantity; that is, the firms in the market each select a quantity and price is determined (usually) by the total output of all firms in the economy. A particular firm's production choice will then have to take into account how its own action affects market price in tandem with what the other firms in the market might be doing.

- **Stackelberg competition**: firms still compete on quantity, but they enter the market sequentially. For ease of notation, firm 1 enters first and chooses a quantity; then firm 2 enters and chooses a quantity. This continues until all firms have entered (in the case of duopoly, there are only two firms entering, but we'll keep it general for now).

- **Bertrand competition**: firms compete on price; that is, the firms in the market each select a price at which to sell their goods. The whole of the market purchases from the firm issuing the lower price (up to funky tiebreaking rules we have to invent for the case in which they issue the same price); in general, we assume that firms cannot price-discriminate and must charge the same price to all customers.

How best to remember these methods of competition? At the risk of sounding ridiculous, here is how I keep them straight (no, really):

Name	Method	Beginning sound
Cournot	Quantity	*"K"*
Stackelberg	Sequential (quantity)	*"S"*
Bertrand	Price	*"P"*

It is a little bit of a stretch to say that "Bertrand" and "Price" both start with a "*P*" sound; if you'd like to waste a little time while making yourself feel better about that particular classification, Google "voiceless consonant" and see what comes up.

For now, we're going to ignore the case of Bertrand competition and focus on Cournot and Stackelberg.

Cournot competition

When we set up a market for a duopoly problem, we retain the same elements that we would have in a standard perfect-competition or monopoly problem. We need a price-demand function and a cost function for each firm. In general, firms may face different cost functions, but since this makes analysis a little more long-winded we'll assume the firms are effectively identical for now (changing the cost functions wouldn't change the way we try to solve the questions, it just makes it difficult to keep track of which firm is choosing what quantity).

Let's assume we have a price-demand function $P(Q) = 1 - Q$ and identical cost functions $C_1(q_1) = q_1^2, C_2(q_2) = q_2^2$. Notice that price is a function of the total quantity available in the market, $Q = q_1 + q_2$.

Solving a Cournot problem is reasonably straightforward: firms choose quantities to maximize profits. We define the profit function of firm 1 as

$$\pi_1(q_1, q_2) = P(Q)q_1 - C_1(q_1) = (1 - (q_1 + q_2))\, q_1 - q_1^2$$

Notice that firm 1's profits depend on firm 2's choice of quantity! Still, if we want to maximize firm 1's profits we can follow the usual path and take first-order conditions.

$$\begin{aligned}
& \max_{q_1} \pi_1(q_1, q_2) = \max_{q_1} (1 - (q_1 + q_2))\, q_1 - q_1^2 \\
& \qquad\qquad\qquad\;\; = \max_{q_1} q_1 - q_1 q_2 - 2q_1^2 \\
\frac{\partial}{\partial q_1} : \quad & 0 = 1 - q_2 - 4q_1 \\
\implies \quad & q_1 = \frac{1 - q_2}{4}
\end{aligned}$$

There are two things to notice here:

- Where before (in the case of perfect competition) the market price may have entered into the firm's production decision, here the firm takes its effect on market price into account when maximizing its profits. Since this maximization depends on q_2, the quantity firm 1 chooses depends crucially on the quantity firm 2 chooses.

- q_1 depends negatively on q_2; that is, the higher q_2 is, the lower firm 1 will set q_1. The intuition here is that if firm 2 decides to give up some market share and lower q_2, firm 1 will want to scoop up at least some of it and raise q_1. If firm 2 makes an aggressive play and raises q_2, the market price is lowered and firm 1 needs to lower q_1 to shift downward on its marginal cost curve (of course, this particular feature is due to the particular cost functions we set up).

We've obtained a best-response function for firm 1, denoting how it sets q_1 according to how firm 2 sets q_2. The next step in the question is to work the other way around and determine how firm 2 sets q_2 according to how firm 1 has chosen q_1. Again, we maximize firm 2's profits,

$$\begin{aligned}
& \max_{q_2} \pi_2(q_2) = \max_{q_2} (1 - (q_1 + q_2))\, q_2 - q_2^2 \\
& \qquad\qquad\quad\; = \max_{q_2} q_2 - q_1 q_2 - 2q_2^2 \\
\frac{\partial}{\partial q_2} : \quad & 0 = 1 - q_1 - 4q_2 \\
\implies \quad & q_2 = \frac{1 - q_1}{4}
\end{aligned}$$

And this is firm 2's best-response function in terms of firm 1's choice of q_1. Notice that it has essentially the same features that firm 1's best-response function does; this is due to the fundametally symmetric nature of the problem: since both firms face the same cost curve and affect market price identically, their best-response functions should look reasonably similar.

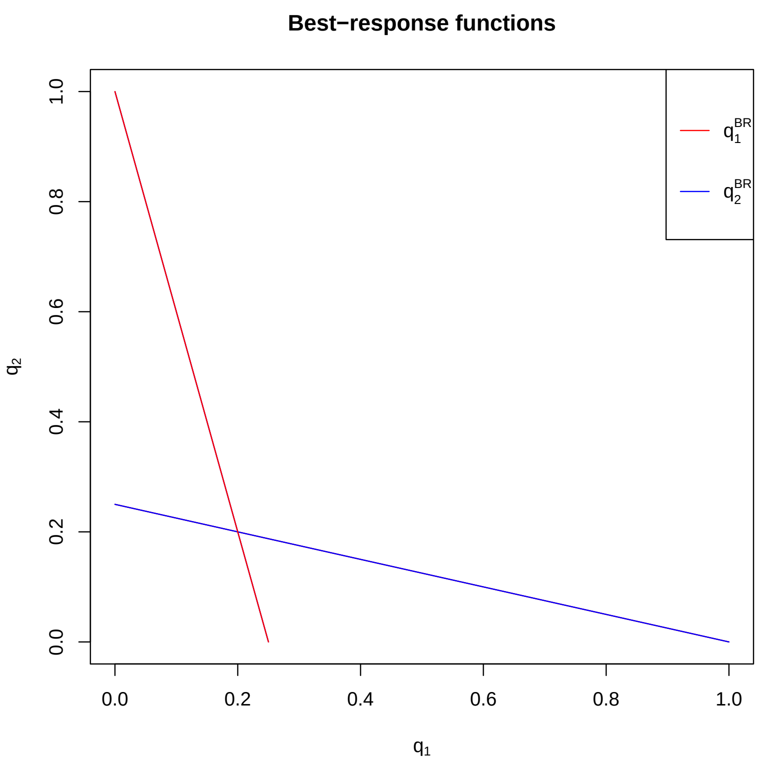

Figure 1: best response functions in the analysed Cournot model

We solve for equilibrium in this problem in the usual way: where the best-response functions intersect we have a Nash equilibrium. With this in mind, we plug in firm 2's best-response function to firm 1's,

$$\begin{aligned} q_1 &= \frac{1 - \frac{1-q_1}{4}}{4} \\ &= \frac{1}{4} - \frac{1}{16} + \frac{q_1}{16} \\ \implies \quad \frac{15}{16} q_1 &= \frac{3}{16} \\ \implies \quad q_1 &= \frac{1}{5} \end{aligned}$$

With this value in hand, solving for q_2 is simple.

$$\begin{aligned} q_2 &= \frac{1-q_1}{4} \\ &= \frac{1-\frac{1}{5}}{4} \\ &= \frac{\frac{4}{5}}{4} \\ q_2 &= \frac{1}{5} \end{aligned}$$

So in equilibrium, $q_1 = q_2 = \frac{1}{5}$. Total production $Q = q_1 + q_2 = \frac{2}{5}$, and therefore market price $P(Q) = 1 - Q = \frac{3}{5}$. Notice that the firms are producing the same quantities in equilibrium! Again, this is because of the symmetric nature of the problem: since both firms face the same cost curve and affect market price in identical ways, their solutions should be the same.

Now, how can we make this problem slightly more realistic and slightly more difficult? Let's assume that instead of 2 firms, we now have N firms in the market. To keep matters simple, we'll assume more or less the same structure that we had above: $P(Q) = 1 - Q$, $Q = \sum_{i=1}^{N} q_i$, $C_i(q_i) = q_i^2$. So the price-demand function has not changed in terms of how it responds to aggregate output, and each firm i faces the same cost function in terms of how much it produces. For the sake of notational simplicity, let $Q_{-i} = \sum_{j \neq i} q_j$ be the aggregate production of all other firms than firm i. In words: this market has N identical firms, whose aggregate production determines the price level in the economy.

We solve for the behaviour of a particular firm i in the usual way, by defining the firm's profits and then maximizing. Here,

$$\pi_i\left(q_i, Q_{-i}\right) = P\left(q_i + Q_{-i}\right) q_i - C(q_i) = \left(1 - q_i - Q_{-i}\right) q_i - q_i^2$$

We then setup the maximization problem and take first-order conditions.

$$\begin{aligned} \max_{q_i} \pi_i\left(q_i, Q_{-i}\right) &= \max_{q_i} \left(1 - q_i - Q_{-i}\right) q_i - q_i^2 \\ &= \max_{q_i} q_i - Q_{-i} q_i - 2q_i^2 \\ \frac{\partial}{\partial q_i} : \qquad 0 &= 1 - Q_{-i} - 4q_i \\ \implies \qquad q_i &= \frac{1 - Q_{-i}}{4} \end{aligned}$$

So the best-response function for firm i defines how it will behave given *aggregate* production of all other firms, Q_{-i}.

To compute explicit values for production choices, we have the option to compute this quantity for firm 1, firm 2, and so on up to firm N. However, this will become quite tedious and is not a good use of time. How then can we solve this problem? Recall from the two-firm setup that both firms had the same solution to their profit-maximization problem; this was due to the fact that their profit-maximization problems were, in essence, identical since they shared a common cost function. Here, all firms in the market again share a common cost function, so we guess that there is a symmetric solution to this problem and then check that it is valid.

In a symmetric equilibrium, all firms (or agents, depending on the problem setup) follow identical strategies; in the question at hand, this means that each firm will select the same production quantity. Let this quantity be q^*. Then $q_i = q^*$ for all firms i. If this is the case, then total production in the economy is $Q = Nq^*$ and the production of all firms other than i is given by $Q_{-i} = (N-1)q^*$.

What quantity defines equilibrium? We substitute q^* into the best-response function defined above.

$$\begin{aligned} && q_i &= \frac{1 - Q_{-i}}{4} \\ \implies && q^* &= \frac{1 - (N-1)q^*}{4} \\ \implies && 4q^* &= 1 - (N-1)q^* \\ \implies && q^* &= \frac{1}{N+3} \end{aligned}$$

This matches neatly with our answer to the two-firm problem, $q_1 = q_2 = \frac{1}{5} = \frac{1}{N+3}$, so we're probably on the right track. Notice that if we check this answer, each firm *is* best-responding to the actions of the others, so we are in equilibrium.

With N firms each producing $\frac{1}{N+3}$, aggregate production in this economy is $Q = \frac{N}{N+3}$ implying a market price level of $P = 1 - Q = \frac{3}{N+3}$. Then as the number of firms grows arbitrarily large, $Q \to 1$ and $P \to 0$; that is, as the number of firms in the economy grows to infinity, consumers are able to extract all of the available surplus in the market. This jibes nicely with our ideas that competition is good for consumers.

Stackelberg competition

The Cournot model can provide useful intuition about what happens in a small economy (for an exercise, compare P and Q in the two- or N-firm Cournot model to the P and Q which arise from the monopolist's problem), especially the way in which firm entry tends to leave consumers better off. One way we can make the model more realistic is to specify different cost functions for each firm (after all, Pepsi and Coke are probably paying slightly different amounts due to slightly different prodution technology); another is to notice that, in the real world, firms generally do not enter simultaneously: there is an incumbent firm and new entrants respond to this firm's existing market power (and, of course, there are many other ways of making the Cournot model more realistic, but let's not get out of control). Altering the Cournot model in this way to make it a sequential game leads us to the model of Stackelberg competition.

In the shortest case of Stackelberg competition, there are two firms: firm 1 first chooses which quantity to produce and then, having witnessed firm 1's production decision firm 2 chooses which quantity it would like to produce. A natural solution concept for this model is subgame perfection, where each firm best-responds at each point at which it chooses an action; here, this means that firm 2 will have to best-respond to firm 1's quantity choice *regardless of what that quantity choice is*[1].

Assume we have the same model as before: $P = 1 - Q$, $C_1(q_1) = q_1^2, C_2(q_2) = q_2^2$, with aggregate production $Q = q_1 + q_2$. To solve for the subgame perfect Nash equilibrium we follow the usual method of backwards induction and solve the last subgame first, then work our way backwards up the game tree.

Here, firm 2's problem is the last subgame, so we solve its profit maximization problem.

$$\begin{aligned} && \max_{q_2} \pi_2(q_2, q_1) &= \max_{q_2} (1 - q_1 - q_2)\, q_2 - q_2^2 \\ && &= \max_{q_2} q_2 - q_1 q_2 - 2q_2^2 \\ \frac{\partial}{\partial q_2}: && 0 &= 1 - q_1 - 4q_2 \\ \implies && q_2 &= \frac{1 - q_1}{4} \end{aligned}$$

[1]Why is this important? Without specifying a particular model, say that firm 2 claims to produce an extremely large amount unless firm 1 produces nothing at all. Then in Nash equilibrium, firm 1 is free to believe this (likely incredible) threat and will stay out of the market; firm 2 will then solve the monopolist's problem in its turn. By restricting market equilibrium to be subgame perfect, firm 2 cannot steal market power from firm 1 by idle threats alone.

This is firm 2's best-response function in the second stage. There isn't much that's terribly surprising about this result: here, firm 2 is best-responding to the production choice of firm 1 just as in the two-firm Cournot model. With the same cost functions and the same price-demand function it is logical that firm 2's best-response function is the same as it was in the two-firm Cournot model.

However, when we step back to the first stage of the game we will need to account for how firm 1's action will affect firm 2's action. This is standard in questions of subgame perfection (if slightly easier to see in models with discrete action spaces): firm 1 knows that by choosing q_1, firm 2 will choose $q_2(q_1)$; if firm 1 doesn't want this to happen, it will not choose quantity q_1. What this tells us is that firm 1 must account for how its production choice affects firm 2's production choice when it chooses its action; in economic parlance, firm 1 *endogenizes* its effect on the production of firm 2. Although this can seem like a mere bookkeeping problem, it turns out to have sizable implications for the nature of equilibrium.

With this in mind, firm 1's maximization problem is

$$\max_{q_1} \pi_1(q_1) = \max_{q_1} \left(1 - q_1 - q_2\left(q_1\right)\right) q_1 - q_1^2$$

Notice how firm 1's effect on firm 2's quantity decisions enters into the maximization problem! Substituting in for firm 2's best response function, we see the maximization problem as

$$\max_{q_1} \left(1 - q_1 - \frac{1 - q_1}{4}\right) q_1 - q_1^2 = \max_{q_1} \frac{3}{4}\left(1 - q_1\right) q_1 - q_1^2$$

Solving this problem is now merely an issue of taking first-order conditions:

$$\begin{aligned} & \max_{q_1} \frac{3}{4}\left(1 - q_1\right) q_1 - q_1^2 = \max_{q_1} \frac{3}{4} q_1 - \frac{7}{4} q_1^2 \\ \frac{\partial}{\partial q_1}: \quad & 0 = \frac{3}{4} - \frac{7}{2} q_1 \\ \Longrightarrow \quad & q_1 = \frac{3}{14} \end{aligned}$$

Plugging into firm 2's best-response function, we find

$$\begin{aligned} q_2 &= \frac{1 - q_1}{4} \\ &= \frac{1 - \frac{3}{14}}{4} \\ &= \frac{\frac{11}{14}}{4} \\ q_2 &= \frac{11}{56} \end{aligned}$$

So equilibrium in this model is $q_1 = \frac{3}{14}$, $q_2 = \frac{11}{56}$.

Comparing this particular example to Cournot competition, we see that $\frac{3}{14} > \frac{1}{5}$ while $\frac{11}{56} < \frac{1}{5}$, so firm 1 is producing more while firm 2 is producing less. This jibes nicely with the intuition above: since there is an additively-inverse relationship between best-response q_1 and q_2, as one rises the other must fall (and vice-versa). That firm 1 produces more in Stackelberg than in Cournot reflects the fact that, as first-mover, it "sets the tone" in the market and determines the nature of the competitive equilibrium; in this case, this is to firm 1's advantage (later we will likely discuss games in which the second mover has an advantage).

Regarding aggregates, $Q = q_1 + q_2 = \frac{23}{56}$ while $P = 1 - Q = \frac{33}{56}$. Notice that $\frac{23}{56} > \frac{2}{5}$ while $\frac{33}{56} < \frac{3}{5}$. So in this Stackelberg model compared to the Cournot model above, aggregate production is higher while market price is lower: consumers must then be unambiguously better off (if you're interested, feel free to compute consumer surplus in both cases; the fractions make it a messy exercise).

Now, it may seem reasonable to ask why firm 1 looks at how its choice affects firm 2, while firm 2 does not look at how its choice affects firm 1. The short answer here is that since firm 1 moves *before* firm 2, there is no way that firm 2's behaviour can affect firm 1 without resorting threats or claims of future action; since subgame perfection rules out idle or incredible threats, firm 2 is left to merely best-respond to firm 1's choice without endogenizing how its strategy will affect firm 1.

A reasonable next step in Stackelberg equilibrium is to look at ways in which firm 1 may harvest more or less of its first-mover advantage. To this end, we introduce a fixed cost into the model; this means that firm 2 may no longer see a level of production which is profitable and may choose to stay out of the market. This, in turn, will affect firm 1's behaviour.

We retain the price-demand function $P = 1 - Q$ for now. Cost functions are now defined by

$$C_1(q_1) = \begin{cases} q_1^2 + K & \text{if } q_1 > 0 \\ 0 & \text{otherwise} \end{cases}, \qquad C_2(q_2) = \begin{cases} q_2^2 + K & \text{if } q_2 > 0 \\ 0 & \text{otherwise} \end{cases}$$

That is, each firm faces a quadratic cost curve plus a fixed cost if it chooses to enter the market, and pays nothing if it chooses not to enter.

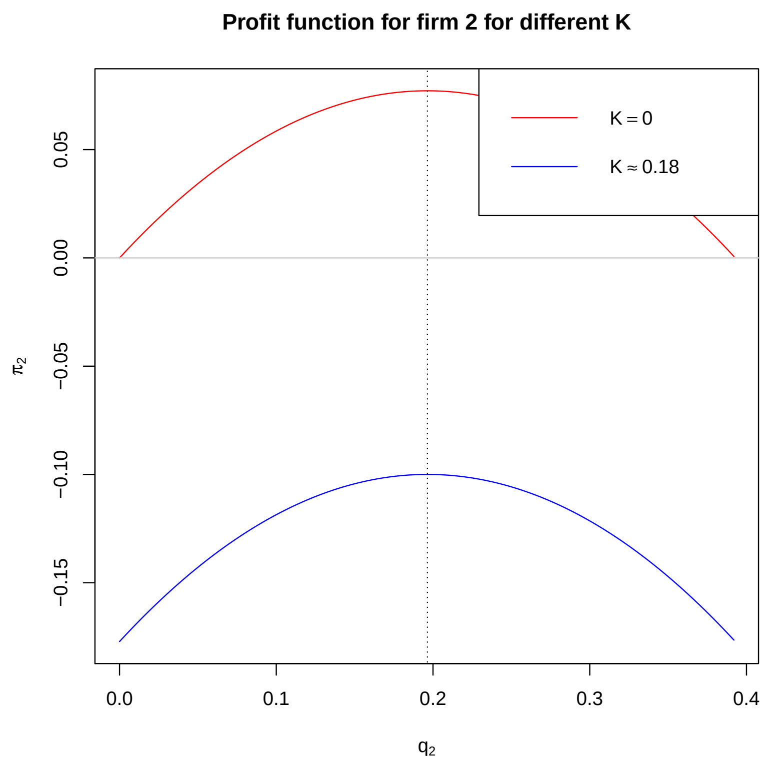

Figure 2: π_2 for two values of the fixed-cost K, assuming $q_1 = \frac{3}{14}$. The dashed line shows that both contingent profit functions attain their maxima at the same level of production, but with a sufficiently large fixed cost positive profits are not possible. In this case, the tested level of fixed cost is the cutoff level plus $\frac{1}{10}$.

To solve this problem we apply the usual method of subgame perfection. We solve firm 2's second-stage optimization problem in terms of firm 1's action in the first stage, then solve for firm 1's optimal first-stage action. In the second stage, firm 2's maximization problem is

$$\begin{aligned} \max_{q_2} \pi_2(q_2, q_1) &= \max_{q_2}(1 - q_1 - q_2)q_2 - \left(q_2^2 + K\right) \\ &= \max_{q_2} q_2 - q_1 q_2 - 2q_2^2 - K \\ \frac{\partial}{\partial q_2} : \quad 0 &= 1 - q_1 - 4q_2 \\ \implies \quad q_2^* &= \frac{1 - q_1}{4} \end{aligned}$$

This is the solution to firm 2's first-order conditions. But as for all problems with a fixed cost, we need to make sure that firm 2 would actually want to produce at this level; that is, if firm 2 is making negative profits at the optimum, it would certainly not want to enter the market (see figure 2). So we check,

$$\begin{aligned} \pi_2(q_2^*, q_1) &= (1 - q_1 - q_2)q_2 - \left(q_2^2 + K\right) \\ &= \left(1 - q_1 - \frac{1 - q_1}{4}\right)\frac{1 - q_1}{4} - \left(\frac{1 - q_1}{4}\right)^2 - K \\ &= \frac{3}{16}\left(1 - q_1\right)^2 - \left(\frac{1 - q_1}{4}\right)^2 - K \\ \pi_2(q_2^*, q_1) &= 2\left(\frac{1 - q_1}{4}\right)^2 - K \end{aligned}$$

From this, we see that if $K > 2(\frac{1-q_1}{4})^2$ firm 2 can earn only negative profits and will choose not to enter the market. Firm 2's production function (in this case, its best-response function) is then

$$q_2 = \begin{cases} \frac{1-q_1}{4} & \text{if } K < 2\left(\frac{1-q_1}{4}\right)^2 \\ 0 & \text{otherwise} \end{cases}$$

It is important to note that we have not completely specified firm 2's production function until we know when it will enter the market and when it will not! A correct answer to a question like this *always* requires making sure that the firm's response is sensible. Notice that in this case, 0 profit was the cutoff as this was the firm's outside option; this will usually be the case, but is not guaranteed.

Now, solving firm 1's optimization problem is going to prove a little more difficult. That is, since firm 1 now knows that it can affect whether or not firm 2 will enter the market (we refer to this as the *extensive margin*, but this is not important) we will have to account for a few things:

- If firm 2 enters, what is firm 1's strategy?
- If firm 2 does not enter, what is firm 1's strategy?
- With these strategies in hand, does firm 1 want to allow firm 2 to enter or not?
- With this in mind, does firm 1 want to enter the market?

Let's abstract a little bit from the algebraic solution to firm 2's problem, so that we're not lugging notation around unnecessarily. It is intuitive that if firm 1 produces a sufficiently large quantity, firm 2 will see no benefit from entering due to the fixed cost in the model. So we claim that there is some threshhold quantity $\underline{q}$ such that if $q_1 \geq \underline{q}$ firm 2 does not enter the market, while if $q_1 < \underline{q}$ firm 2 does enter the market. For the moment, we will not consider what this $\underline{q}$ actually is, but just keep in mind that there is *some* threshhold production level which will keep firm 2 out of the market.

Firm 1's problem can now be phrased as follows: if $q_1 < \underline{q}$, profits should be optimized with respect to firm 2's subsequent entry; if $q_1 \geq \underline{q}$, profits should be optimized with respect to the fact that firm 2 will not enter the market. Let's look at the second problem. If firm 2 does not enter, firm 1 is effectively a monopolist. So if firm 1 can keep firm 2 out of the market by producing more than $\underline{q}$ its problem is

$$\max_{q_1} (1-q_1)q_1 - \left(q_1^2 + K\right) \qquad \text{s.t. } q_1 \geq \underline{q}$$

Here, "s.t." stands for "such that." There are a bevvy of purely-mathematical approaches to what we refer to as constrained optimization, but since this is not a class in mathematics we're going to take a more intuitive approach to the issue. In this monopolist's problem, the first-order conditions are

$$\frac{\partial}{\partial q_1}: \qquad 1 - 4q_1 = 0$$

So we can see that $q_1 = \frac{1}{4}$ is the optimal production level. But what if $q_1 = \frac{1}{4} < \underline{q}$? Then at firm 1's optimum, firm 2 would still enter the market, according to our intuitive definition of $\underline{q}$. Since we are looking for a solution that keeps firm 2 out of the market, this will not do.

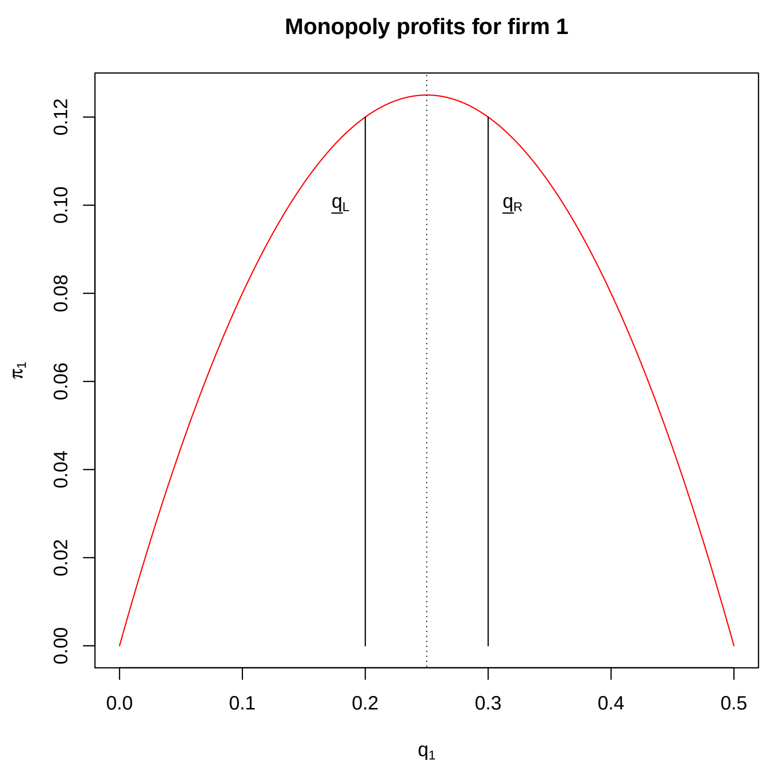

Figure 3: monopoly profits for firm 1, assuming $K = 0$ for simplicity. $\underline{q}_1$ and $\underline{q}_2$ are two possible threshholds necessary to keep firm 2 out of the market in the second stage.

There are two ways of thinking about this issue (although they are in a sense one in the same): graphically, or thorugh derivatives. Graphically, we appeal to figure 3. In the case where the threshhold is $\underline{q}_1 < \frac{1}{4}$ (or any threshhold less than $\frac{1}{4}$) the optimum production level $q_1 = \frac{1}{4}$ is consistent with producing a sufficient quantity to keep firm 2 out of the market. However, in the case where the treshhold is $\underline{q}_2 > \frac{1}{4}$ (or any threshhold greater than $\frac{1}{4}$) the optimum production level $q_1 = \frac{1}{4}$ is *not* consistent with producing a sufficient quantity to keep firm 2 out of the market in the second stage. What then will firm 1 do to keep firm 2 out of the market? Since profits are evidently falling for all quantities above $\frac{1}{4}$, the firm will want to produce as little as possible while still producing $q_1 \geq \underline{q}_2$. Since the lowest level of production which is no greater than $\underline{q}_2$ is specifically $\underline{q}_2$, firm 1's optimum choice in this context is $q_1 = \underline{q}_2$.

Using calculus, we can see this pretty succinctly: notice that $\frac{\partial \pi_1}{\partial q_1} = 1 - 4q_1$. So if $q_1 > \frac{1}{4}$ the first derivative of profits is negative. Then the firm, looking to maximize profits, will keep q_1 as close to $\frac{1}{4}$ as possible while still ensuring $q_1 \geq \underline{q}$. Evidently this has firm 1 choosing $q_1 = \underline{q}$.

So firm 1 is left with two options: choose optimal production $q_1 = \frac{1}{4}$ if this level is above $\underline{q}$, or choose production $q_1 = \underline{q}$ if $\underline{q} > \frac{1}{4}$. Since $\underline{q}$ is the threshhold for q_1 above which firm 2 will opt to not enter the market, this optimization is consistent with our assumptions that firm 2 does not enter. For now, we will ignore the impact of the fixed cost; this will come back later (intuitively: since the fixed cost enters the decision regardless of whether or not firm 2 enters, we can look at firm 1's decision between allowing firm 2 to enter or not without regard to the fixed cost; when we look at whether or not this decision leaves firm 1 wanting to enter, we will need to reincorporate the fixed cost). Then firm 1's production function here looks like $q_1 = \max\{\frac{1}{4}, \underline{q}\}$.

We now look at the second half of the problem: what if firm 1 does not want to keep firm 2 out of the market? This is the case where $q_1 < \underline{q}$, firm 1's optimum production is insufficient to keep firm 2 from entering. We setup firm 1's maximization problem as in the original Stackelberg case, respecting firm 2's best response to firm 1's action.

$$
\begin{aligned}
\max_{q_1} \pi_1(q_1) &= \max_{q_1} \left(1 - q_1 - \frac{1-q_1}{4}\right) q_1 - \left(q_1^2 + K\right) \\
&= \max_{q_1} \frac{3}{4}q_1 - \frac{7}{4}q_1^2 - K \\
\frac{\partial}{\partial q_1}: \quad 0 &= \frac{3}{4} - \frac{7}{4}q_1 \\
\implies \quad q_1 &= \frac{3}{14}
\end{aligned}
$$

This result isn't surprising: the case in which firm 1's action allows firm 2 to enter is exactly the first case of Stackelberg competition we analysed; so it is sensible that firm 1's action is identical.

Of course, we must apply the same consistency check as above; if $\underline{q} < \frac{3}{14}$, firm 1 forces firm 2 out of the market by behaving optimally to allow firm 2 entry. This is less of a concern in this case — since any firm is better off under monopoly than under competition, if firm 1 kicks firm 2 out when behaving optimally to keep firm 2 in, firm 1 is happy with this outcome — but for mathematical consistency we need to check this constraint. The logic for firm 1's choice in this case is roughly the same as in the case in which firm 1 seeks to keep firm 2 out; here, if $\underline{q} < \frac{3}{14}$, firm 1 will choose $q_1 = \underline{q}$ instead. So in this case firm 1's production function is $q_1 = \min\{\underline{q}, \frac{3}{14}\}$.

Having solved for firm 1's optimal behaviour in both the case in which it allows firm 2 to enter and the case in which it locks firm 2 out, we look to see which strategy firm 1 will prefer. To do this, we need to compute $\underline{q}$. Looking back to firm 2's optimal strategy, we see that firm 2 will enter and produce $\frac{1-q_1}{4}$ if and only if $K \leq 2(\frac{1-q_1}{4})^2$. Rearranging algebraically, this is equivalent to firm 2 producing if and only if $q_1 \leq 1 - \sqrt{8K}$. That is to say, firm 2 will enter the market if and only if $q_1 \leq 1 - \sqrt{8K}$, so the threshhold in this particular problem is $\underline{q} = 1 - \sqrt{8K}$.

To categorize firm 1's behaviour we look at its resultant profits in various situations.

- *Monopolistic production,* $q_1 = \frac{1}{4}$.

$$
\pi_1\left(\frac{1}{4}\right) = \left(1 - \frac{1}{4}\right)\frac{1}{4} - \left(\frac{1}{4}\right)^2 - K = \frac{1}{8} - K
$$

- *Monopolistic production,* $q_1 = 1 - \sqrt{8K}$.

$$\begin{aligned}\pi_1\left(1-\sqrt{8K}\right) &= \left(\sqrt{8K}\right)\left(1-\sqrt{8K}\right) - \left(1-\sqrt{8K}\right)^2 - K \\ &= \sqrt{8K} - 8K - 1 + 2\sqrt{8K} - 8K - K \\ &= 3\sqrt{8K} - 17K - 1\end{aligned}$$

- *Competitive production,* $q_1 = \frac{3}{14}$.

$$\begin{aligned}\pi_1\left(\frac{3}{14}\right) &= \left(1 - \frac{3}{14} - \frac{1-\frac{3}{14}}{4}\right)\frac{3}{14} - \left(\frac{3}{14}\right)^2 - K \\ &= \left(\frac{11}{14} - \frac{11}{56}\right)\frac{3}{14} - \frac{9}{196} - K \\ &= \left(\frac{33}{56}\right)\frac{3}{14} - \frac{9}{196} - K \\ &= \frac{63}{784} - K \\ &= \frac{9}{112} - K\end{aligned}$$

- *Competitive production,* $q_1 = 1 - \sqrt{8K}$.

$$\begin{aligned}\pi_1\left(1-\sqrt{8K}\right) &= \left(\sqrt{8K} - \frac{\sqrt{8K}}{4}\right)\left(1-\sqrt{8K}\right) - \left(1-\sqrt{8K}\right)^2 - K \\ &= \left(\frac{3}{4}\right)\sqrt{8K} - \left(\frac{3}{4}\right)8K - 1 + 2\sqrt{8K} - 8K - K \\ &= \left(\frac{11}{4}\right)\sqrt{8K} - 15K - 1\end{aligned}$$

Immediately, we see that since $\frac{1}{8} - K > \frac{9}{112} - K$ firm 1 will prefer to keep firm 2 out of the market so long as the monopolistic level of production — $q_1 = \frac{1}{4}$ — is sufficient to keep firm 2 from entering. Further, since $\frac{1}{8} - K > \frac{11}{4}\sqrt{K} - 15K - 1$ everywhere if given the choice firm 1 would rather produce the monopoly quantity and capture the market than to brush up against the entry threshhold for firm 2 and allow it to enter. With these facts in place, we know that for K sufficiently small the firm will produce competitively according to the usual Stackelberg model; for K sufficiently large the firm will prefer producing at the monopolistic level to producing at the competitive level. We are then left to find the point at which firm 1 switches from competitive production to monopolistic production.

To find this level of K, we equate profits under open competitive production with those under monopolistic production in which firm 1 acts suboptimally solely to keep firm 2 out of the market.

$$\begin{aligned} & 3\sqrt{8K} - 17K - 1 = \frac{9}{112} - K \\ \iff \quad & (6\sqrt{2})\sqrt{K} - 16K - \frac{121}{112} = 0 \\ \iff \quad & \sqrt{K} = \frac{6\sqrt{2} \pm \sqrt{72 - \frac{64(121)}{112}}}{32} \\ \iff \quad & \sqrt{K} \approx 0.2562 \pm 0.0528 \\ \iff \quad & K \in \{0.0451, 0.1011\}\end{aligned}$$

Notice that if $K \approx 0.1011$, firm 1 cannot make positive profits no matter its strategy and it will prefer to not enter. Then the relevant inflection point in this question is $K \approx 0.0451$; when K is below this value, the firm cannot optimally obtain enough market share to force firm 2 out of the market, and when K is above this value the firm wants to produce at the monopolistic level.

Lastly, notice that when $K > \frac{1}{8}$ firm 1 cannot obtain positive profits anywhere. So when $K > \frac{1}{8}$ firm 1's optimal strategy is not to enter the market. With this in mind, the final equilibrium of this problem is

$$q_1 = \begin{cases} \frac{3}{14} & \text{if } K < 0.0451 \\ 1 - \sqrt{8K} & \text{if } K \in \left[0.0451, \frac{9}{128}\right) \\ \frac{1}{4} & \text{if } K \in \left[\frac{9}{128}, \frac{1}{8}\right] \\ 0 & \text{otherwise} \end{cases}$$

$$q_2 = \begin{cases} \frac{1-q_1}{4} & \text{if } K < 2\left(\frac{1-q_1}{4}\right)^2 \\ 0 & \text{otherwise} \end{cases}$$

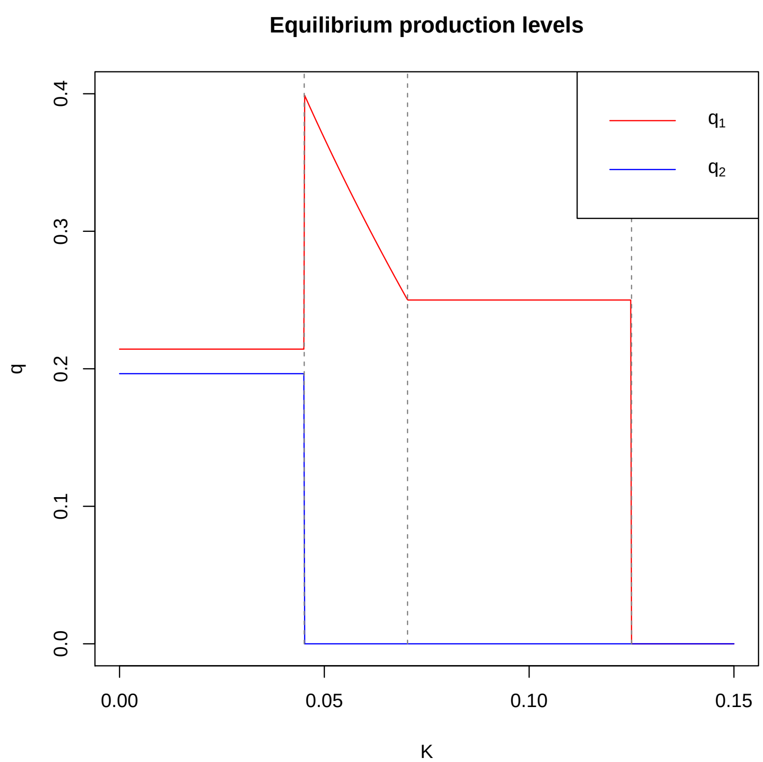

Figure 4: best-response quantities under Stackelberg competition with fixed costs.

Figures 4 and 5 show the relationship between the two best-response functions (and their associated profits) in equilibrium, for various values of K. The regions of the graph are subdivided to indicate ranges over which various strategies are pursued by the firms.

Let's take the question one step further. Suppose that in the interest of consumer welfare, the government would like to encourage firm 2 to enter by providing a subsidy to cover the cost of entry; what is the lowest subsidy the government could provide to firm 2 to prevent firm 1 from having monopoly power? Naïvely, we might think that this amount is equal to K, the fixed cost to the firm. However, the government can do

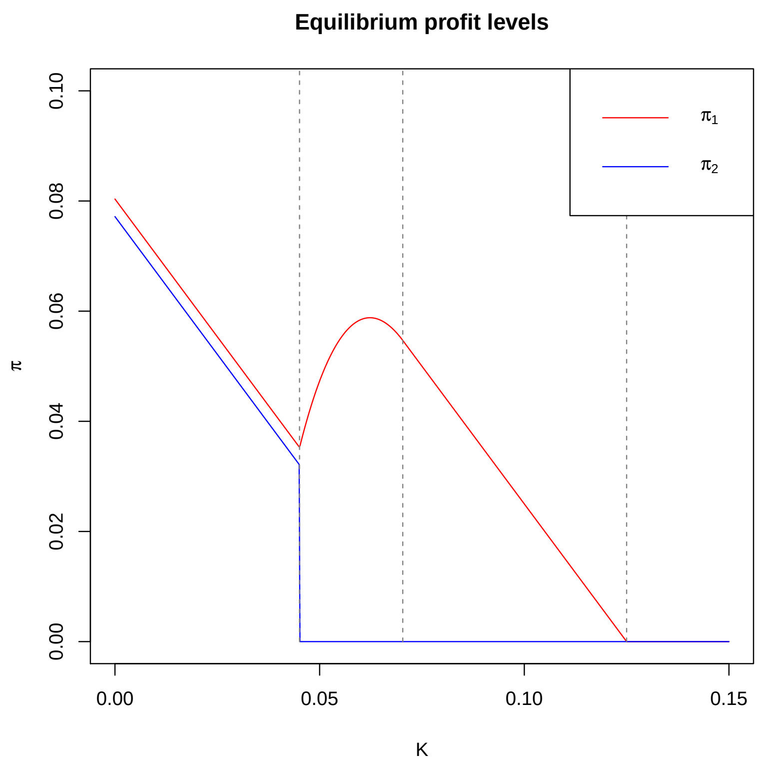

Figure 5: profits under Stackelberg competition with fixed costs.

better than this if it takes into account that, at some point, it is no longer cost-effective for firm 1 to keep firm 2 out of the market.

First, notice that at some point (i.e., for some K) the government can no longer accomplish this goal. Since firm 1's profits under Stackelberg competition are $\frac{9}{112} - K$, when $K > \frac{9}{112}$ firm 1 will not want to enter a competitive market. Then knowing that the government will subsidize firm 2's entry, firm 1 will not enter (we have not set this up as an explicit game, but the intuition should seem passable) if K is above $\frac{9}{112}$. The promise of a government subsidy will then create a monopoly for firm 2! This is a particular instance of second-mover advantage. Still, to retain "good" features of this game we will for now assume that $K \leq \frac{9}{112}$.

Denote this government subsidy by B. Returning to firm 2's problem, we see that firm 2 is now willing to enter the market so long as

$$K \leq 2\left(\frac{1-q_1}{4}\right)^2 + B$$

From this definition, we rearrange things algebraically to see that firm 2 will enter if and only if

$$q_1 \leq 1 - \sqrt{8K - 8B}$$

Since it is fair to assume that the government would like to minimize cost, the government will select the subsidy B such that $q_1 = 1 - \sqrt{8K - 8B}$. This gives us

$$B = K - \frac{1-q_1}{8}$$

Firm 1 now has a problem: since the government is issuing a subsidy to firm 2, there is no action it can take to keep firm 2 out of the market! We have the option to show this algebraically in terms of the game form, but given the mess gone through above to solve the problem without subsidies it's almost certainly better to justify this idea with intuition. As firm 2 will be subsidized to enter regardless of the actions of firm 1 (here we assume the government has no other objective than multi-firm competition), firm 1 will *always* find itself receiving the profits of Stackelberg competition. If this is the case, we already know firm 1's best response: to play as if firm 2 is entering in the second period.

From above, we know that firm 1's best response under this qualification is to choose $q_1 = \frac{3}{14}$. Then the government's desired subsidy — the minimum possible to commit firm 2 to entry — is

$$B = K - \frac{1 - \frac{3}{14}}{8} = K - \frac{11}{112}$$

Here, we should add the qualification that if $K < \frac{11}{112}$ the government will not issue any subsidy (that is, the subsidy is bounded below by 0). However, the mere threat of being able to provide the subsidy — and the government's strict committal to firm 2's entry — causes firm 1 to allow firm 2 to enter *even though no subsidy is formally issued.* This is slightly different from an idle threat: the government in this case is more than willing to provide the subsidy if firm 1 "tests" its commitment; this ability causes firm 1 to behave amicably towards firm 2 and entry is allowed.

But! Notice from above that if $K > \frac{9}{112}$ firm 1 will not want to enter a competitive market. Then in all relevant cases where competition is possible — that is, where $K \leq \frac{9}{112}$ — the government does not issue a subsidy but firm 2 enters anyway, because firm 1 acts obligingly due to fear of the government issuing a subsidy. That is, the mere threat of government intervention in this case causes firm 1 to behave. I'd consider this a pretty cool result.