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COURSE BIDDING AT BUSINESS SCHOOLS*

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Mechanisms that rely on course bidding are widely used at business schools in order to allocate seats at oversubscribed courses. Bids play two key roles under these mechanisms: to infer student preferences and to determine who have bigger claims on course seats. We show that these two roles may easily conflict, and preferences induced from bids may significantly differ from the true preferences. Therefore, these mechanisms, which are promoted as market mechanisms, do not necessarily yield market outcomes. We introduce a Pareto-dominant market mechanism that can be implemented by asking students for their preferences in addition to their bids over courses.

1. INTRODUCTION

Allocation of course seats to students is one of the major tasks of registrars' offices at most universities. Since demand exceeds supply for many courses, it is important to design mechanisms to allocate course seats equitably and efficiently. Many business and law schools rely on mechanisms based on *course bidding* to serve this purpose. The following statement is from Kellogg Course Bidding System Rules:[2] "The bidding is designed to achieve an equitable and efficient allocation of seats in classes when demand exceeds supply."

Although not all schools use the same version, the following simplest version captures the main features of a vast majority of these mechanisms:

(i) Each student is given a positive *bid endowment* to allocate across the courses he considers taking.
(ii) All bids for all courses and all students are ordered in a single list and processed one at a time starting with the highest bid. When it is the turn of a bid, it is *honored* if and only if the student has not filled his schedule and the course has not filled all its seats.

When the process terminates, a schedule is obtained for each student. Similarly, a market-clearing "price" is obtained for each course, which is simply the lowest honored bid unless the course has empty seats, and in that case the price is zero. The version we describe is closest to the version used by the University of Michigan Business School and thus we refer to it as *UMBS course-bidding mechanism*. Schools that rely on this mechanism and its variants include Columbia Business School, the Haas School of Business at UC Berkeley, the Kellogg Graduate School of Management at Northwestern, Princeton University, and the Yale School of Management.

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[2] http://www.kellogg.nwu.edu/script_html/CBSDEMO/cbs_demo.htm retrieved in 2003.

The UMBS course-bidding mechanism is inspired by the market mechanism, and schools that rely on this mechanism promote it as a market mechanism. Consider the following question and its answer borrowed from the University of Michigan Business School, Course Bidding Tips and Tricks:[3]

Q. How do I get into a course?
A. If you bid enough points to make market clear, a seat will be reserved for you in that section of the course, up to class capacity.

In this article, we show that the UMBS course-bidding mechanism does not necessarily yield a market outcome, and this is a potential source of efficiency loss, part of which can be avoided by an appropriate choice of a market mechanism. Although the UMBS course-bidding mechanism resembles the market mechanism, there is one major aspect in which they differ: Under the UMBS course-bidding mechanism, students do not provide direct information on their preferences, and, consequently, their schedules are determined under the implicit assumption that courses with higher bids are necessarily preferred to courses with lower bids. For example, consider the following statement from the guidelines for Allocation of Places in Oversubscribed Courses and Sections at the School of Law, University of Colorado at Boulder:[4] "The second rule is that places are allocated by the bidding system. Each student has 100 bidding points for each semester. You can put all your points in one course, section or seminar, or you can allocate points among several. By this means, you express the strength of your preferences."

The entire strategic aspect of course bidding is overseen under this interpretation of the role of the bids. Although the choice of bids is clearly affected by the preferences, it is not adequate to use them as a proxy for the strength of the preferences. For example, if a student believes that the "market clearing" price of a course will be low, it is suboptimal for him to bid highly for that course regardless of how much he desires to be assigned a seat at that course. Indeed, this point is often made by the registrars' offices. The following statement appears in the Bidding Instructions of both Columbia Business School and Haas School of Business at UC Berkeley:[5] "If you do not think a course will fill up, you may bid a token point or two, saving the rest for courses you think will be harder to get into."

Here is the crucial mistake made under the UMBS course-bidding mechanism: Bids play two important roles under this mechanism:

(i) Bids are used to infer student preferences
(ii) Bids are used to determine who has a bigger claim on each course seat, and, therefore, choice of a bid-vector is an important strategic tool.

These two roles may easily conflict: For example, a student may be declined a seat at one of his favorite courses, despite clearing the market, simply because he clears the market in "too many" other less favorite courses. Indeed, such bidding behavior is consistent with expected utility maximization, and thus it cannot be considered to be a mistake.

Once we understand what is wrong with the UMBS course-bidding mechanism, it is relatively easy to fix it: The key is separating the two roles of the bids and asking students to

1. submit their preferences, in addition to
2. allocating their bid endowment across the courses.

In this way, the registrar's office no longer needs to guess what student preferences are. Although there may be several market outcomes in the context of course bidding, choosing the

[3] http://webuser.bus.umich.edu/Departments/Admissions/AcademicServices/CurrentUpdates/BiddingTipsTricks.htm retrieved in 2003.

[4] http://www.colorado.edu/law/wait_list.html retrieved in 2003.

[5] http://www-1.gsb.columbia.edu/students/biddinginstructions.html and http://web.haas.berkeley.edu/Registrar retrieved in 2003.

"right" one is easy because there is a market outcome that Pareto-dominates any other market outcome. We show this by relating course bidding to *two-sided matching markets* (Gale and Shapley, 1962). The Pareto-dominant market outcome can be obtained via an extension of the celebrated *Gale–Shapley student-proposing deferred acceptance algorithm*.

The mechanism design approach has recently been very fruitful in similar real-life resource allocation problems. A pioneering example is the redesign of U.S. hospital-intern market (cf. Roth and Peranson, 1999; Roth, 2002). This approach had influenced policies on other important resource allocation problems as well. For example, Abdulkadiroğlu and Sönmez (2003) show how ideas in two-sided matching literature can be utilized to improve allocation of students to schools by school choice programs, and, consequently, Boston and New York City public schools started to use a version of one of their proposals (cf. Abdulkadiroğlu et al., 2005a, 2005b). Roth et al. (2004, 2005a) show how live kidney exchanges can be organized to increase the welfare of the patients. Consequently, two kidney exchange programs were established in the United States based on these proposals (cf. Roth et al., 2005b). The current article, to the best of our knowledge, is the first paper to approach course bidding from a mechanism design perspective.[6] We believe this approach may be helpful in improving course-bidding mechanisms in practice.

2. ASSIGNMENT OF COURSE SEATS TO STUDENTS

There are a number of students, each of whom should be assigned seats at a number of courses. Let $I = \{i_1, i_2, \ldots, i_n\}$ denote the set of students and $C = \{c_1, c_2, \ldots, c_m\}$ denote the set of courses. Each course has a maximum capacity, and, similarly, each student has a maximum number of courses that he can take. Without loss of generality, we assume that the maximum number of courses that each student can take is the same.[7] Let q_I denote the maximum number of courses that can be taken by each student, and let q_c denote the capacity of course c. We refer to any set of at most q_I courses as a *schedule*, any schedule with q_I courses as a *full schedule*, and any schedule with less than q_I courses as an *incomplete schedule*. Note that $\emptyset$ is also a schedule, and we refer to it as the *empty schedule*. Each student has strict preferences over all schedules including the empty schedule. We refer to a course c to be *desirable* if the singleton $\{c\}$ is preferred to the empty schedule. Let P_i denote the strict preferences of student i over all schedules and R_i denote the induced weak preference relation.

Assigning a schedule to each student is an important task faced by the registrar's office. A *matching* is an assignment of courses to students such that

1. no student is assigned more courses than q_I, and
2. no course is assigned to more students than its capacity.

Equivalently, a matching is an assignment of a schedule to each student such that no course is assigned to more students than its capacity. Given a matching μ, let μ_i denote the schedule of student i under μ and let μ_c denote the set of students enrolled in course c under μ. Different registrars' offices rely on different methods to assign course seats to students. However, methods based on course bidding are commonly used at business schools and law schools in order to assure that the assignment process is fair and course seats are assigned to students who value them most.

[6] Prior to our paper, Brams and Kilgour (2001) study allocation of course seats to students via a mechanism that does not rely on course bidding.

[7] It is straightforward to extend the model as well as the results (i) to the more general case where the maximum number of courses that can be taken by different students is possibly different, and (ii) to the case where each student can take a maximum number of credits.

3. COURSE BIDDING

At the beginning of each semester, each student is given a *bid endowment* $B > 0$. In order to keep the notation at a minimum, we assume that the bid endowment is the same for each student. Each student is asked to allocate his bid endowment across all courses. Let $b_i = (b_{ic_1}, b_{ic_2}, \ldots, b_{ic_m})$ denote the *bid vector* of student i where $b_{ic} \geq 0$ for each course c, and $\Sigma_{c\in C}\, b_{ic} = B$.

Course bidding is inspired by the market mechanism, and hence student bids are used

(i) to determine the market-clearing bid for each course, and
(ii) to determine a schedule for each student.

More specifically, consider the following mechanism, which can be used to determine market-clearing bids as well as student schedules:

1. Order *all* bids for *all* courses and *all* students from highest to smallest in a *single* list.
2. Consider one bid at a time, following the order in the list. When it is the turn of bid b_{ic}, the bid is *successful* if student i has unfilled slots in his schedule and course c has unfilled seats. If the bid is successful, then student i is assigned a seat at course c (i.e., the bid is honored) and the process proceeds with the next bid in the list. If the bid is unsuccessful, then proceed with the next bid in the list without an assignment.
3. When all bids are handled, no student is assigned more courses than q_I and no course is assigned to more students than its capacity. Hence, a matching is obtained. The *market clearing* bid of a course is the lowest successful bid in case the course is full, and zero otherwise.

Variants of this mechanism are used at many schools, including the University of Michigan Business School, Columbia Business School, the Haas School of Business at UC Berkeley, the Kellogg School of Management at Northwestern University, Princeton University, and Yale School of Management. The most basic version described above is closest to the version used at University of Michigan Business School and we refer to it as *UMBS course-bidding mechanism*. Although each of the above schools uses its own version, the points we make in this article carry over. We next give a detailed example illustrating the dynamics of the UMBS course-bidding mechanism.

EXAMPLE 1. There are four students, i_1–i_4, each of whom should take two courses, and three courses, c_1–c_3, such that c_1 has three seats, c_2 has two seats, and c_3 has four seats. Each student has 100 bid points to allocate over courses c_1–c_3, and student bids are given in the following matrix:

b_{ic}	c_1	c_2	c_3
i_1	60	38	2
i_2	48	22	30
i_3	47	28	25
i_4	45	35	20

Positive bids are ordered from highest to smallest as follows:

$$b_{i_1c_1} - b_{i_2c_1} - b_{i_3c_1} - b_{i_4c_1} - b_{i_1c_2} - b_{i_4c_2} - b_{i_2c_3} - b_{i_3c_2} - b_{i_3c_3} - b_{i_2c_2} - b_{i_4c_3} - b_{i_1c_3}.$$

We next process each bid, one at a time, starting with the highest bid: $b_{i_1c_1} = 60$: i_1 is assigned c_1; $b_{i_2c_1} = 48$: i_2 is assigned c_1; $b_{i_3c_1} = 47$: i_3 is assigned c_1; $b_{i_4c_1} = 45$ is unsuccessful: c_1 has no seats left; $b_{i_1c_2} = 38$: i_1 is assigned c_2; $b_{i_4c_2} = 35$: i_4 is assigned c_2; $b_{i_2c_3} = 30$: i_2 is assigned c_3; $b_{i_3c_2} = 28$ is unsuccessful: c_2 has no seats left; $b_{i_3c_3} = 25$: i_3 is assigned c_3; $b_{i_2c_2} = 22$ is unsuccessful: i_2 has a full schedule, c_2 has no seats left; $b_{i_4c_3} = 20$: i_4 is assigned c_3; $b_{i_1c_3} = 2$ is unsuccessful: i_1 has a full schedule.

The outcome of the UMBS course-bidding mechanism is

$$\begin{pmatrix} i_1 & i_2 & i_3 & i_4 \\ c_1, c_2 & c_1, c_3 & c_1, c_3 & c_2, c_3 \end{pmatrix}$$

with a market-clearing price vector of (47,35,0).

Under the UMBS course-bidding mechanism, there can be two kinds of ties:

1. Bids of two or more students may be the same for a given course
2. A student may bid the same for two or more courses.

In practice, both kinds of ties are broken based on a previously determined lottery. We also assume that throughout the article.

4. MARKET EQUILIBRIUM

Schools that rely on the UMBS course-bidding mechanism promote it as a market mechanism. In this section, we will explore to what extent this is appropriate.

Most business and law schools provide data on market-clearing bids of previous years. Based on recent years' bid data and possibly some private information, students try to guess which market-clearing bids they will face. Strictly speaking, it is possible that a student can influence the market-clearing bids. However, since there are hundreds of students in most applications, this is rather unlikely. Throughout the article, we assume that students are *price takers under a belief system* and that they do not try to influence the market-clearing bids and do not take into consideration the influence of other students' behavior in formation of market-clearing bids. Each student, rather, forms a belief on market-clearing bids based on recent years' bid data and chooses an optimal bid.

What do we mean by price-taking behavior?

Let $p = (p_c)_{c\in C} \in \mathbb{R}^m_+$ be a *price vector* such that for each $c \in C$, p_c is the lowest bid required to clear a course. We assume that students have beliefs about the market-clearing bids of the courses. These beliefs are common to each student and are given through a joint probability distribution function $F(p)$ denoting the probability that the market-clearing prices will be less than or equal to p.

Each student $i \in I$ has a *utility function* $U_i : 2^C \to \mathbb{R}$ defined over the schedules of courses that represents his preferences P_i over the schedules.

Given a preference relation P_i over schedules and given a subset of courses $D \subseteq C$, let $Ch(D, P_i)$ denote the *best schedule from* D. Let $U = (U_i)_{i\in I}$ be a utility profile. Let P^U denote the *associated preference profile with* U. A student is a *price taker with respect to belief system* F if his objective is to maximize his expected utility with respect to the common belief system F that is exogenously formed and does not take into consideration the effect of his and other students' preferences in formation of F. Utility-maximizing behavior implies that, given a bid vector b_i, student i chooses the best schedule of the courses that are cleared by the bid vector b_i. Given a set of courses $D \subseteq C$, the student will choose $Ch(D, P_i^U)$, whenever D is the whole set of courses cleared by bid vector b_i, and the probability for that to happen is

$$\Pr(\{\mathbf{p}_c \le b_{ic}\}_{c\in D} \cup \{\mathbf{p}_c > b_{ic}\}_{c\in C\setminus D} \mid F) = \int_{\{0\le p_c\le b_{ic}\}_{c\in D}} \int_{\{p_c > b_{ic}\}_{c\in C\setminus D}} dF(p).$$

Therefore, for any $D \subseteq C$, the expected (indirect) utility of a student stating b_i under the price-taking behavior with respect to F, denoted by $\mathbf{U}_i(b_i)$, is given by

$$\mathbf{U}_i(b_i) = \sum_{D\subseteq C} \Pr(\{\mathbf{p}_c \le b_{ic}\}_{c\in D} \cup \{\mathbf{p}_c > b_{ic}\}_{c\in C\setminus D} \mid F)\, U_i\big(Ch\big(D, P_i^U\big)\big).$$

A *course bidding economy* is denoted by $(I, C, q_I, q_C, B, U, F)$. We fix I, C, q_I, q_C, and B throughout the article (except the Appendix) and denote an economy by its utility profile and belief system (U, F).[8]

A *market equilibrium* of economy (U, F) is given by a triple (μ, b, p), where

(i) μ is a matching and it is interpreted as a *market outcome*,
(ii) $b = [b_{ic}]_{i\in I, c\in C}$ is a bid matrix and interpreted as *equilibrium bid matrix,* and
(iii) $p = (p_c)_{c\in C} \in \mathbb{R}^m_+$ is a price vector and interpreted as the vector of realized *market-clearing prices*,

such that

1. (expected utility maximization with respect to price-taking behavior under F) for any student $i \in I$, there is a bid vector b_i^+ such that $b_i^+ = \arg\max_{b_i^*:\sum\limits_c b_{ic}^* \le B} \mathbf{U}_i(b_i^*)$.
2. (tie breaking) a positive tie-breaker lottery value λ_i is randomly generated for each student such that $\lambda_i \neq \lambda_j$ for all $\{i, j\} \subseteq I$, and $\max_{i\in I} \lambda_i < \min_{i,j\in I, c\in C: b_{ic}^+ > b_{jc}^+ > 0} b_{ic}^+ - b_{jc}^+$. Equilibrium bid matrix b is formed as follows using these tiebreakers: for each student i and course c,
$$b_{ic} = \begin{cases} b_{ic}^+ + \lambda_i & \text{if} b_{ic}^+ > 0 \\ 0 & \text{if } b_{ic}^+ = 0 \end{cases}.$$
3. (market clearing) for any student i and any schedule $s \neq \mu_i$, if $b_{ic} \geq p_c$ for all $c \in s$, then $U_i(\mu_i) > U_i(s)$,
4. (prices) for any course $c \in C$,
$$p_c = \begin{cases} \min_{i\in I}\{b_{ic} : c \in \mu_i\} & \text{if } |\mu_c| = q_c \\ 0 & \text{if } |\mu_c| < q_c \end{cases}.$$

Here (1) states that student *i* determines his bid vector b_i using expected utility maximization under the belief system F, (2) states that a tie-breaking lottery is generated to break ties among student bids for the same course and these values are added to the bid matrix, (3) states that his schedule μ_i is better than any other schedule he could *afford*, and (4) states that the market-clearing price of a course is the lowest successful bid if the course has no seats left and zero otherwise. Under the price-taking assumption with respect to a belief system, we do not force the belief system to be endogenously derived through the best response of the student's behavior to other students' behavior. Hence, our requirement eliminates the need of the students to have information or beliefs about other students' information and converts the students' problem to a decision-making problem. Since (2) is a nonstandard condition, we elaborate on it further. As beliefs need not to be consistent with the outcome, the tiebreaker plays the role of a rationing device to determine who will be assigned a course if multiple students clear the course and yet only a portion of these bids can be honored. If we did not use the tiebreaker, a market equilibrium may cease to exist.

Observe that an equilibrium bid matrix exists if F is continuous in the domain of the bids or the bids are only integer valued. Throughout the article, we assume that bids are only integer valued.

We refer to a mechanism as a *market mechanism* if it always selects a market outcome when students behave as expected utility maximizers under a belief system to choose the messages they send to the mechanism.

4.1. *Is the UMBS Course-Bidding Mechanism a Market Mechanism?* Given that the UMBS course-bidding mechanism is widely used in real-life implementation and given that it is

[8] We thank an anonymous referee for suggesting the use of exogenous beliefs.

promoted as a market mechanism, it is important to understand whether this mechanism indeed yields a market outcome. There is one major difficulty in this context: Although the market equilibrium depends on bids as well as student preferences under a given belief system, the UMBS course-bidding mechanism merely depends on bids. Business and law schools that use the UMBS course-bidding mechanism implicitly assume that bids carry sufficient information to infer the student preferences and thus it is not necessary to inquire about student preferences. Since higher bids are processed before lower bids, they implicitly assume that

1. for any student i and any pair of courses c, d, $b_{ic} > b_{id}$ if and only if $\{c\} P_i \{d\}$, and
2. a. for any student i, any course c, and any incomplete schedule s with $c \notin s$, $\{c\} P_i \emptyset$ if and only if $(s \cup \{c\}) P_i s$, and
   b. for any student i, any pair of courses c, d, and any incomplete schedule s with $c, d \notin s$, $\{c\} P_i \{d\}$ if and only if $(s \cup \{c\}) P_i (s \cup \{d\})$.

That is,

1. whenever a student bids higher for a course c than another course d, he necessarily prefers a seat at c to a seat at d, and
2. this preference ranking is independent of the rest of his schedule.

The first assumption relates induced bids to preferences over courses, and we refer to it as *bid monotonicity*. The second assumption relates preferences over schedules to preferences over courses and it is known as *responsiveness* (Roth, 1985) in the matching literature. So a key issue is whether it is appropriate to have induced bids that are monotonic and preferences are responsive.

4.2. *Are Bids Monotonic?* It turns out that bid monotonicity is not a realistic assumption under expected utility maximization. If a student believes that the market-clearing price of a course will be low, it is suboptimal for him to bid highly for that course regardless of how much he desires to be assigned a seat at this course. Indeed, this point is often made by the registrar's office. This not only violates bid monotonicity, but more importantly may result in a non market outcome as well as in efficiency loss. The following example is built on this simple intuition.

EXAMPLE 2. Consider a student i who shall register for up to $q_I = 5$ courses and suppose there are six courses. His utility for each individual course is given in the following table

Course	c_1	c_2	c_3	c_4	c_5	c_6
Utility	150	100	100	100	100	100

and his utility for a schedule is additively separable

$$U_i(s) = \Sigma_{c \in s} U_i(c).$$

Student i has a total of $B = 1001$ points to bid over courses $c_1 - c_6$ and the minimum acceptable bid is 1. Based on previous years' bid-data, student i has the following belief on the market-clearing bids:

(i) Market-clearing bid for course c_1 will be 0 with probability 1.
(ii) Market-clearing bids for the courses in c_2–c_6 have independent identical cumulative distribution functions, and for any of these courses c, the cdf F_c is strictly concave with $F_c(200) = 0.7$, $F_c(250) = 0.8$, and $F_c(1001) = 1$. That is, for each of the courses c_2–c_6, student i believes that the market-clearing bid will be no more than 200 with 70% probability and no more than 250 with 80% probability.

Assuming that he is an expected utility maximizer, we next find the optimal bid vector for student i.

The expected utility of the student under the UMBS mechanism by submitting the bid vector b_i is given as follows:

$$\mathbf{U}_i^{UMBS}(b_i) = \sum_{D\subseteq C} \Pr(\{\mathbf{p}_c \leq b_{ic}\}_{c\in D} \cup \{\mathbf{p}_c > b_{ic}\}_{c\in C\setminus D} \mid F) U_i\big(Ch\big(D, P_i^{b_i}\big)\big),$$

where $P_i^{b_i}$ is the bid-monotonic preferences induced by the bid vector b_i. Observe that this differs from the expected utility under a market equilibrium.

By first-order necessary conditions and symmetry, student i shall bid 1 for course c_1, and the same value for each course $c \in \{c_2, c_3, c_4, c_5, c_6\}$ for which he devotes a positive bid.[9]

Therefore, the optimal bid vector is in the form $b_{ic_1} = 1,\ b_{ic} = 1000/k$ for any k of courses c_2–c_6. We next derive the expected utility of each such possibility.

Case 1: $b^1_{ic_1} = 1,\ b^1_{ic_2} = b^1_{ic_3} = b^1_{ic_4} = b^1_{ic_5} = b^1_{ic_6} = 200,$

$$\begin{aligned} u^1 = {} & \Pr\{p_{c_2} \leq 200, p_{c_3} \leq 200, p_{c_4} \leq 200, p_{c_5} \leq 200, p_{c_6} \leq 200\} U_i(\{c_2, c_3, c_4, c_5, c_6\}) \\ & + 5\Pr\{p_{c_2} > 200, p_{c_3} \leq 200, p_{c_4} \leq 200, p_{c_5} \leq 200, p_{c_6} \leq 200\} U_i(\{c_1, c_3, c_4, c_5, c_6\}) \\ & + 10\Pr\{p_{c_2} > 200, p_{c_3} > 200, p_{c_4} \leq 200, p_{c_5} \leq 200, p_{c_6} \leq 200\} U_i(\{c_1, c_4, c_5, c_6\}) \\ & + 10\Pr\{p_{c_2} > 200, p_{c_3} > 200, p_{c_4} > 200, p_{c_5} \leq 200, p_{c_6} \leq 200\} U_i(\{c_1, c_5, c_6\}) \\ & + 5\Pr\{p_{c_2} > 200, p_{c_3} > 200, p_{c_4} > 200, p_{c_5} > 200, p_{c_6} \leq 200\} U_i(\{c_1, c_6\}) \\ & + \Pr\{p_{c_2} > 200, p_{c_3} > 200, p_{c_4} > 200, p_{c_5} > 200, p_{c_6} > 200\} U_i(\{c_1\}) \\ = {} & 0.7^5 \times 500 + 5 \times 0.7^4(1-0.7)550 + 10 \times 0.7^3(1-0.7)^2 450 \\ & + 10 \times 0.7^2(1-0.7)^3 350 + 5 \times 0.7(1-0.7)^4 250 + (1-0.7)^5 150 = 474.79. \end{aligned}$$

Case 2: $b^2_{ic_1} = 1,\ b^2_{ic_2} = b^2_{ic_3} = b^2_{ic_4} = b^2_{ic_5} = 250,\ b^2_{ic_6} = 0.$

$$\begin{aligned} u^2 = {} & \Pr\{p_{c_2} \leq 250, p_{c_3} \leq 250, p_{c_4} \leq 250, p_{c_5} \leq 250\} U_i(\{c_1, c_2, c_3, c_4, c_5\}) \\ & + 4\Pr\{p_{c_2} > 250, p_{c_3} \leq 250, p_{c_4} \leq 250, p_{c_5} \leq 250\} U_i(\{c_1, c_3, c_4, c_5\}) \\ & + 6\Pr\{p_{c_2} > 250, p_{c_3} > 250, p_{c_4} \leq 250, p_{c_5} \leq 250\} U_i(\{c_1, c_4, c_5\}) \\ & + 4\Pr\{p_{c_2} > 250, p_{c_3} > 250, p_{c_4} > 250, p_{c_5} \leq 250\} U_i(\{c_1, c_5\}) \\ & + \Pr\{p_{c_2} > 250, p_{c_3} > 250, p_{c_4} > 250, p_{c_5} > 250\} U_i(\{c_1\}) \\ = {} & 0.8^4 \times 550 + 4 \times 0.8^3(1-0.8)450 + 6 \times 0.8^2(1-0.8)^2 350 \\ & + 4 \times 0.8(1-0.8)^3 250 + (1-0.8)^4 150 = 470.0. \end{aligned}$$

Since expected utility of bidding for three or less of courses c_2–c_6 can be no more than $150 + 3 \times 100 = 450$, the optimal bid vector for student i is b_i^1 with an expected utility of 474.79. There are two important observations we shall make. The first one is an obvious one: The optimal bid for the most deserved course c_1 is the smallest bid violating bid monotonicity. The second point is less obvious but more important: If the beliefs are consistent with the real bid distribution, under the optimal bid b_i^1, student i is assigned the schedule $s = \{c_2, c_3, c_4, c_5, c_6\}$ with probability $0.7^5 = 0.168$. So although the bid $b^1_{ic_1} = 1$ is high enough to claim a seat at

[9] For simplicity of exposition, we assume that beliefs regarding the market-clearing bids are independent across courses. We will use this simplifying assumption in our other examples, as well. Our examples can be generalized to a situation in which the beliefs about the market-clearing bids of various courses are dependent on each other.

course c_1, since it is the lowest bid, student i is not assigned a seat in an available course under the UMBS course-bidding mechanism.

Therefore, the outcome of the UMBS course-bidding mechanism cannot be supported as a market outcome, and this weakness is a direct source of efficiency loss. To summarize:

1. how much a student bids for a course under UMBS course-bidding mechanism is not necessarily a good indication of how much a student wants that course,
2. as an implication, the outcome of the UMBS course-bidding mechanism cannot always be supported as a market outcome, and
3. the UMBS course-bidding mechanism may result in unnecessary efficiency loss due to not seeking direct information on student preferences.

5. GALE–SHAPLEY PARETO-DOMINANT MARKET MECHANISM

Although the UMBS course-bidding mechanism is very intuitive, it makes one crucial mistake: Bids play two possibly conflicting roles under this mechanism:

1. Bids are used to determine who has a bigger claim on each course seat, and therefore choice of a bid vector is an important strategic tool.
2. Bids are used to infer student preferences.

As Example 2 clearly shows, these two roles can easily conflict. Fortunately, it is possible to fix this deficiency by utilizing the theory on two-sided matching markets developed by David Gale, Lloyd Shapley, Alvin Roth, and their followers. The key point is separating the two roles of the bids. Under the proposed two-sided matching approach, students are not only asked to allocate their bid endowment over courses but also to indicate their preferences over schedules. In order to simplify the exposition, we initially assume that preferences over schedules are responsive. Recall that under responsiveness students can simply reveal their preferences over individual courses and the empty schedule. Later on, we will show to what extent responsiveness can be relaxed.

We are now ready to adopt a highly influential mechanism in two-sided matching literature to course bidding.

Gale–Shapley Pareto-Dominant Market Mechanism:

1. Students are ordered with an even lottery to break ties.
2. Each student strictly ranks the courses in order to indicate his preferences. It is sufficient to rank only desirable courses.
3. Each student chooses a bid vector.
4. Based on stated preferences, bids, and the tie-breaking lottery, a matching is obtained in several steps via the following *student-proposing deferred acceptance algorithm*.

Step 1: Each student proposes to his top q_I courses based on his stated preferences. Each course c rejects all but the highest bidding q_c students among those who have proposed. Those who are not rejected are *kept on hold*. In case there is a tie, the tie-breaking lottery is used to determine who is rejected and who will be kept on hold.

In general, at

Step t: Each student who is rejected from $k > 0$ courses in Step $(t-1)$ proposes to his best remaining k courses based on his stated preferences. In case fewer than k courses remain, he proposes to all remaining courses. Each course c considers the new proposals together with the proposals on hold and rejects all but the highest bidding q_c students. Those who are not rejected are kept on hold. In case there is a tie, the tie-breaking lottery is used to determine who is rejected and who will be kept on hold.

The procedure terminates when no proposal is rejected, and at this stage course assignments are finalized.

Let μ^{GS} denote the outcome of the Gale–Shapley Pareto-dominant mechanism, and let a price vector p be determined as follows: For each course c with full capacity, p_c is the lowest successful bid, and for each course c with empty seats, $p_c = 0$.

Let $P = (P_i)_{i \in I}$ be the profile of (true) student preferences over schedules. Under responsiveness, for each student i, the preference relation P_i induces a strict ranking of all courses. We already assumed that students are price takers under a belief system, and thus they do not try to influence the market-clearing bids and do not respond necessarily in a best responding way to their fellow students. In the following lemma, we prove that, under this behavior, it is (part of) a weakly dominant strategy for the students to state their preferences truthfully under the Gale–Shapley Pareto-dominant market mechanism. Therefore, under price-taking behavior with respect to belief system F, the stated preferences of students over individual courses are their true preferences.

LEMMA 1. *Let (U, F) be an economy with responsive preferences over schedules. Then, revealing preferences truthfully is part of an optimal decision for each student for the Gale–Shapley Pareto-dominant market mechanism under price-taking behavior with respect to F.*

The intuition behind this lemma is simple. Since in each round of the deferred acceptance algorithm each student proposes to his best schedule that has not rejected him yet, for a given belief system he maximizes his ex ante chances of being placed in the best possible schedule by revealing his preferences truthfully.

Based on this lemma, from now on, we will assume that students reveal their preferences truthfully to the Gale–Shapley Pareto-dominant market mechanism.

We are now ready to show that the Gale–Shapley Pareto-dominant market mechanism functions as a market mechanism when students behave as expected utility maximizers under price-taking behavior under any common belief system.

PROPOSITION 1. *Let (U, F) be an economy. Let P be a responsive preference profile represented by U. Suppose that students reveal their preferences over courses and bid matrix b to the Gale–Shapley Pareto-dominant market mechanism as expected utility maximizers under price-taking behavior with respect to F and consistent with Lemma 1. Under (b, P), let μ^{GS} be the outcome of the Gale–Shapley Pareto-dominant market mechanism as a result, and p be the induced price vector. The triple (μ^{GS}, b, p) is a market equilibrium of the economy (U, F).*

As a corollary, we state that, given an economy (U, F), the expected utility of a student under the Gale–Shapley Pareto-dominant market mechanism when the student states truthful preferences and a bid vector, b_i, is identical to the expected utility of the student in a market setting:

$$\mathbf{U}_i^{GS}(b_i, P_i^U) = \sum_{D \subseteq C} \Pr(\{\mathbf{p}_c \leq b_{ic}\}_{c \in D} \cup \{\mathbf{p}_c > b_{ic}\}_{c \in C \setminus D} | F) U_i(Ch(D, P_i^U)) = \mathbf{U}_i(b_i).$$

It is easy to show that, in general, there can be several market outcomes induced by the same equilibrium bid matrix. Consider the following example:

EXAMPLE 3. There are three students, i_1, i_2, and i_3, each of whom should take one course, and three courses, c_1, c_2, and c_3, each of which has one seat. The bid endowment of each student is 101 and student utility profiles are given as follows:

U	c_1	c_2	c_3
i_1	100	$100 - \varepsilon$	0
i_2	0	100	$100 - \varepsilon$
i_3	$100 - \varepsilon$	0	100

where ε is positive and sufficiently small. Let P be the list of induced preferences by U. The beliefs about the market-clearing prices are independent and given as follows:

	$F_c(1)$	$F_c(100)$	$F_c(101)$
c_1	0.01	0.7	0.7
c_2	0.01	0.8	0.8
c_3	0.01	0.9	0.9

Suppose that $F_c(h)$ is strictly convex between bids 1 and 100 for each course c.

Next, we determine the equilibrium bid matrix when students behave as expected utility maximizers under price-taking behavior with respect to F. By strict positivity and strict convexity of the cumulative distribution functions between 1 and 100 and their constancy between 100 and 101, students will bid 100 for one of their two desirable courses and 1 for the other one. The ε value is chosen sufficiently small such that they will bid 100 for the desirable course that has the highest probability of clearance at bid 100. Hence, they generate the following bid matrix:

b^*	c_1	c_2	c_3
i_1	1	100	0
i_2	0	1	100
i_3	1	0	100

Suppose that to break the ties a uniform draw between 0 and 1 is determined for each student. Let the following tie-breaking draw be added to positive bids of the students:

	i_1	i_2	i_3
tie-breaking lottery	0.1	0.3	0.2

;

therefore the resulting equilibrium bid matrix is given by:

b	c_1	c_2	c_3
i_1	1.1	100.1	0
i_2	0	1.3	100.3
i_3	1.2	0	100.2

We will show that there are two market outcomes supported by this bid matrix. First, we can find the first market outcome using the Gale–Shapley Pareto-dominant market mechanism under (P, b):

$$\mu = \begin{pmatrix} i_1 & i_2 & i_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$$

is a market outcome with price vector $p = (1.1, 1.3, 100.2)$.

Let (ν, b, r) be a market equilibrium with $\nu \neq \mu$. Suppose $\nu(i_1) \neq \mu(i_1) = c_1$. Since c_1 is i_1's first choice course, we need $r_1 > 1.1 = b_{i_1c_1}$. In this case $r_1 = 1.2 = b_{i_3c_1}$, and $\nu(i_3) = c_1$, since if r_1 were any higher, nobody could afford this course and it would have an empty slot, contradicting $r_1 > 0$. Thus, $r_3 > 100.2 = b_{i_3c_3}$, since otherwise i_3 can afford c_3 and he prefers it to $\nu(i_3) = c_1$, contradicting that ν is a market outcome. Hence, $r_3 = 100.3 = b_{i_2c_3}$ and $\nu(i_2) = c_3$. If r_3 were any higher, nobody could afford it, and it would have an empty slot, contradicting $r_3 > 0$. Thus, $r_2 > 1.3 = b_{i_2c_2}$, since otherwise i_2 can afford c_2 and he prefers it to $\nu(i_2) = c_3$, contradicting that ν is a market outcome. Hence, $r_2 = 100.1 = b_{i_1c_1}$. If r_2 were any higher, nobody could afford it, and it would have an empty slot, contradicting $r_2 > 0$. Therefore, matching

$$\nu = \begin{pmatrix} i_1 & i_2 & i_3 \\ c_2 & c_3 & c_1 \end{pmatrix}$$

together with price vector $r = (1.2, 100.1, 100.3)$ and bid matrix b is another market equilibrium. Observe that if we started the construction of market outcome ν with any other student than i_1, then we would still end up with the same matching ν. Therefore, matchings μ and ν are the only market outcomes under equilibrium matrix b. Observe that the outcome of the Gale–Shapley Pareto-dominant market mechanism, μ, Pareto dominates the other market outcome ν.

The conclusion of this example can be generalized. That is, the outcome of the Gale–Shapley Pareto-dominant market mechanism is the right one: Thanks to its direct relation with two-sided matching markets, the outcome of this mechanism Pareto dominates any other market outcome.

PROPOSITION 2. *Let (U, F) be an economy such that the represented student preference profile, P, by U is responsive. Let bid matrix b denote an equilibrium bid matrix for (U, F), and μ^{GS} be the outcome of the Gale–Shapley Pareto-dominant market mechanism for b and P. Matching μ^{GS} Pareto dominates any other matching μ that is market outcome of economy (U, F) when b is the equilibrium bid matrix.*

5.1. *The Gale–Shapley Pareto-Dominant Market Mechanism and Efficiency.* Replacing the UMBS course-bidding mechanism with the Gale–Shapley Pareto-dominant market mechanism eliminates inefficiencies that result from registrars' offices using bids as a proxy of the strength of the preferences.

Although the Gale–Shapley Pareto-dominant market mechanism Pareto dominates any other market mechanism, there may be situations where all market outcomes are Pareto inefficient for the same equilibrium bid matrix. The following example, which is inspired by a similar example in Roth (1982), makes this point.[10]

EXAMPLE 4. There are four students, i_1, i_2, i_3, and i_4, each of whom should take one course, and four courses, c_1, c_2, c_3, and c_4, each of which has one seat. The bid endowment of each student is 101 and student utility profiles are given as follows:

U	c_1	c_2	c_3	c_4
i_1	100	$100 - \varepsilon$	0	0
i_2	0	100	$100 - \varepsilon$	0
i_3	$100 - \varepsilon$	0	100	0
i_4	100	0	0	$100 - \varepsilon$

where ε is positive and sufficiently small. Let P be the list of preferences induced by U. The beliefs about the market-clearing prices are independent and given as follows:

	$F_c(1)$	$F_c(100)$	$F_c(101)$
c_1	0.01	0.8	0.8
c_2	0.01	0.9	0.9
c_3	0.01	0.7	0.7
c_4	0.01	0.9	0.9

[10] See also Balinski and Sönmez (1999), Ergin (2002), and Abdulkadiroğlu and Sönmez (2003) for similar examples in the context of school–student matching.

Suppose that $F_c(h)$ is strictly convex between bids 1 and 100 for each course c. Hence, each student will bid 100 for one of the two desirable courses she has and 1 for the other one. In particular, ε is chosen sufficiently small such that she will bid 100 for the course with the highest probability of clearance at 100 (as in Example 3). This behavior generates the following bid matrix:

b^*	c_1	c_2	c_3	c_4
i_1	1	100	0	0
i_2	0	100	1	0
i_3	100	0	1	0
i_4	1	0	0	100

Suppose that to break the ties, a uniform draw between 0 and 1 is determined for each student. Let the following tie-breaking draw be added to positive bids of the students:

	i_1	i_2	i_3	i_4
tie-breaking lottery	0.3	0.2	0.1	0.4

;

therefore, the resulting equilibrium bid matrix is given by:

b	c_1	c_2	c_3	c_4
i_1	1.3	100.3	0	0
i_2	0	100.2	1.2	0
i_3	100.1	0	1.1	0
i_4	1.4	0	0	100.4

We can find the outcome of the Gale–Shapley Pareto-dominant market mechanism as follows for (b, P):

$$\mu = \begin{pmatrix} i_1 & i_2 & i_3 & i_4 \\ c_2 & c_3 & c_1 & c_4 \end{pmatrix}.$$

However, the following matching Pareto-dominates μ under P :

$$\nu = \begin{pmatrix} i_1 & i_2 & i_3 & i_4 \\ c_1 & c_2 & c_3 & c_4 \end{pmatrix}.$$

Even though the UMBS mechanism does not result with a market outcome in many cases, can it be more efficient than the Gale–Shapley Pareto-dominant market mechanism? Under certain conditions, the answer is no:

PROPOSITION 3. *Let (U, F) be an economy. Let P denote the list of responsive student preferences over schedules represented by U, bid matrix b denote an equilibrium bid matrix for (U,F), and μ^{GS} be the outcome of the Gale–Shapley Pareto-dominant market mechanism for (b,P). Suppose that in the economy (U,F) under the UMBS mechanism, when the students maximize their expected utility with respect to F, they also generate bid matrix b. Let μ^{UMBS} be the outcome of the UMBS mechanism for b. Then μ^{UMBS} cannot Pareto dominate μ^{GS}.*

On the other hand, as the following example shows, the best market outcome (i.e., the outcome of the Gale–Shapley Pareto-dominant mechanism) may Pareto dominate the outcome of the UMBS mechanism under the same bid matrix consistent with expected utility maximization:

EXAMPLE 3 CONTINUED. If we use the UMBS mechanism here for the same economy given by (U, F) in Example 3, the students will submit the same bid matrix b^* as in Example 3. If the tie-breaking lottery is given as in Example 3, then b of Example 3 will be the bid matrix. We repeat b as follows:

b	c_1	c_2	c_3
i_1	1.1	100.1	0
i_2	0	1.3	100.3
i_3	1.2	0	100.2

In this case, the UMBS outcome is given by

$$\nu = \begin{pmatrix} i_1 & i_2 & i_3 \\ c_2 & c_3 & c_1 \end{pmatrix},$$

which is Pareto dominated by the Gale–Shapley Pareto-dominant market mechanism outcome

$$\mu = \begin{pmatrix} i_1 & i_2 & i_3 \\ c_1 & c_2 & c_3 \end{pmatrix}.$$

It is worthwhile to note that if different bid matrices are generated as a result of expected utility maximization with respect to a belief system under the two mechanisms, the UMBS mechanism outcome can ex post Pareto dominate the best market outcome. On the other hand, ex ante, the GS outcome will always Pareto dominate the UMBS outcome, since a student can always mimic the UMBS mechanism's expected utility under the GS mechanism by submitting bid-monotonic preferences and the UMBS optimal bid vector.

The following example shows that bid vectors can differ under the GS and UMBS mechanisms:

EXAMPLE 5. Consider a student whose preferences over two courses are given by $U(c_1) = 200$, $U(c_2) = 175$. Suppose that 0 is the utility he gets from being unmatched. The student is given 1001 points to bid and he has to enroll in one course. He has the following beliefs about the market-clearing prices for the two courses:

$$F_{c_1}(1) = 0,\ F_{c_1}(300) = 0.7,\ F_{c_1}(500) = 0.8,\ F_{c_1}(1001) = 0.9,$$
$$F_{c_2}(1) = 0,\ F_{c_2}(500) = 0.6,\ F_{c_2}(700) = 0.9,\ F_{c_2}(1001) = 0.95.$$

Suppose that for other bids b, the probabilities are given by the probability for the highest bid reported in the above table that is less than b, i.e., $F_{c_1}(499) = F_{c_1}(300) = 0.7$, whereas $F_{c_1}(501) = F_{c_1}(500) = 0.8$.

Under the UMBS mechanism, the course for highest bid is considered first, and if the bid cannot clear the market then the second bid is considered (in case of a tie, bids are uniformly randomly ordered, and the above principle is applied). Thus, the student maximizes his expected utility with respect to (b_1, b_2). We have:

(i) If $b_1 > b_2$, then $\mathbf{U}_i^{UMBS}(b_i) = u^1(b_1, b_2) = U(c_1)F_{c_1}(b_1) + U(c_2)(1 - F_{c_1}(b_1))F_{c_2}(b_2)$.
(ii) If $b_2 > b_1$, then $\mathbf{U}_i^{UMBS}(b_i) = u^2(b_1, b_2) = U(c_1)(1 - F_{c_2}(b_2))F_{c_1}(b_1) + U(c_2)F_{c_2}(b_2)$.
(iii) If $b_2 = b_1$, then $\mathbf{U}_i^{UMBS}(b_i) = \frac{1}{2}u^1(b_1, b_2) + \frac{1}{2}u^2(b_1, b_2)$.

On the other hand, under the GS mechanism, with truthful revelation of preferences, the highest ranked course is processed first, and if the student cannot clear the market price then his second bid is considered. Thus, he maximizes his expected utility for bidding (b_1, b_2), which is given by

$$\mathbf{U}_i^{GS}(b_i, P_i^U) = u^1(b_1, b_2) = U(c_1)F_{c_1}(b_1) + U(c_2)(1 - F_{c_1}(b_1))F_{c_2}(b_2).$$

It is easy to show that under the UMBS assignment mechanism, the student will bid (501,500) and that under the GS one he will bid either (300,701) or (301,700), with the two latter bids being equivalent since they both imply the same probability distribution over outcomes.

5.2. *To What Extent Can the Responsiveness Assumption Be Relaxed?* Responsiveness is a very convenient assumption because it simplifies the task of indicating preferences over schedules to the much simpler task of indicating preferences over courses. However, in practice it may be violated for many reasons. For instance:

1. A student may wish to bid for different sections of the same course. More generally a student may bid for two courses he considers to be "substitutes" and may wish to take one or the other but not both.
2. There can be additional difficulties due to timing of courses. A student may bid for two courses meeting at the same time and hence it may not be possible to assign him seats in both courses due to scheduling conflicts.

Therefore, it is important to understand to what extent the responsiveness assumption can be relaxed so that the Gale–Shapley Pareto-dominant market mechanism is still well defined. We need further notation in order to answer this question.

A preference relation P_i is *substitutable* (Kelso and Crawford, 1982) if, for any set of courses $D \subseteq C$ and any pair of courses $c, d \in D$,

$$c, d \in Ch(D, P_i) \text{ implies } c \in Ch(D \setminus \{d\}, P_i).$$

The substitutability condition simply states that if two courses are both in the best schedule from a set of available courses and if one of the courses becomes unavailable, then the other one is still in the best schedule from the smaller set of available courses. Substitutability is a milder assumption on schedules than responsiveness, and complications due to bidding for several alternate courses or courses with conflicting schedules are easily handled under substitutability. That is because one can easily extend the Gale–Shapley Pareto-dominant market mechanism when preferences are substitutable.

Gale–Shapley Pareto-Dominant Market Mechanism under Substitutable Preferences:

1. Students are ordered with an even lottery to break ties.
2. Each student strictly ranks the schedules in order to indicate his substitutable preferences.[11]
3. Each student chooses a bid vector.
4. Based on stated preferences, bids, and the tie-breaking lottery, a matching is obtained in several steps via the following student-proposing deferred acceptance algorithm.

Step 1: Each student proposes to courses in his best schedule out of all courses. Each course c rejects all but the highest bidding q_c students among those who have proposed. Those who are not rejected are kept on hold. In case there is a tie, the tie-breaking lottery is used to determine who is rejected and who will be kept on hold.

In general, at

Step t: Each student who is rejected from one or more courses in Step $(t-1)$ proposes to courses in his best schedule out of those courses which have not rejected him. By substitutability, this will include all courses for which he is on hold. Each course c considers the new proposals together with the proposals on hold and rejects all but the highest bidding q_c students. Those

[11] If only violation of responsiveness is due to conflicting schedules or bidding for alternate courses, simply indicating preferences over courses and indicating the constraints is sufficient.

who are not rejected are kept on hold. In case there is a tie, the tie-breaking lottery is used to determine who is rejected and who will be kept on hold.

The procedure terminates when no proposal is rejected, and at this stage course assignments are finalized.

Lemma 1 and Propositions 1, 2, and 3 immediately extend: Under substitutable preferences, when students behave as expected utility maximizers under a belief system, the best decision of each student is revealing her preferences truthfully to the Gale–Shapley Pareto-dominant market mechanism; the outcome of the Gale–Shapley Pareto-dominant market mechanism is a market outcome; it Pareto dominates any other market outcome; and if the induced bid matrices are the same, the UMBS mechanism cannot Pareto dominate the Gale–Shapley Pareto-dominant market mechanism. In Appendix A, we prove these results for this more general case with substitutable preferences.

What if preferences are not substitutable? For instance, what happens if there are complementarities and a student wishes to take two courses together but does not wish to take either one in case he cannot take the other? Recent literature on related models with indivisibilities such as Gul and Stacchetti (2000), Milgrom (2000), Hatfield and Milgrom (2005), and Hatfield and Kojima (2008) suggest that such complementarities might be bad news. Our next result shows that the course-bidding approach for individual courses collapses unless preferences are substitutable. More specifically, we show that a market equilibrium may not exist unless preferences are substitutable.[12]

PROPOSITION 4. *Let C be the set of courses and suppose there is a student i whose preferences P_i over schedules is not substitutable. If the number of courses in C and the bid endowment B are high enough, there exists a belief system F for market-clearing prices, a set of students J with responsive preferences, and a utility profile U representing preferences such that there is no market equilibrium for the problem (U, F).*

6. CONCLUSION

Mechanisms that rely on course bidding are widely used at business schools and law schools in order to allocate seats in oversubscribed courses. Bids play two important roles under these mechanisms:

1. Bids are used to infer student preferences over schedules, and
2. bids are used to determine who has a bigger claim on each seat.

We have shown that these two roles may easily conflict, and the preferences induced from bids may significantly differ from the true preferences. Therefore, although these mechanisms are promoted as market mechanisms, they are not truly market mechanisms. The two conflicting roles of the bids may easily result in efficiency loss due to inadequately using bids as a proxy for the strength of the preferences. We have shown that under a "true" market mechanism the two roles of the bids shall be separated and students should state their preferences in addition to bidding over courses. In this way, registrars' offices no longer need to guess student preferences and they can directly use the stated preferences. This will also give registrars' offices a more reliable measure of underdemanded courses, and, in case this measure is used in policy decisions, more solid decisions can be given.[13]

[12] Intuitively bidding for individual courses is not appropriate when preferences have complementarities, and instead one may consider course allocation mechanisms that rely on bidding for schedules (instead of courses). The University of Chicago Business School uses one such mechanism. Analysis of schedule-bidding mechanisms is very important but it is beyond the scope of our paper.

[13] For example, the following statement from the Bidding Instructions at the Haas School of Business, UC Berkeley shows that low bids may result in cancellation of courses:

> Bidding serves three functions. First, it allows us to allocate seats fairly in oversubscribed classes. Second, it allows us to identify and cancel courses with insufficient demand. Third, ….

One possible appeal of inferring preferences from bids is that there is a unique market outcome of the induced economy. Conversely, once students directly submit their preferences in addition to allocating their bids, there may be several market outcomes. Fortunately, there exists a market outcome that Pareto dominates any other market outcome, and therefore multiplicity of market outcomes is not a serious drawback for our proposal. It is important to emphasize that, although relying on the Pareto-dominant market mechanism eliminates inefficiencies based on "miscalculation" of student preferences, it does not eliminate all inefficiencies. There is a potential conflict between Pareto efficiency and market equilibria in the context of course bidding and even the Pareto-dominant market equilibria cannot escape from "market failure." Furthermore, if student preferences do not satisfy a condition known as substitutability, then course bidding loses much of its appeal, as a market equilibrium may cease to exist.

In theory, the reason for ex post inefficiencies, which cannot be eliminated by the Gale–Shapley Pareto-dominant market mechanism, is that under our market equilibrium definition, some bids of the students are wasted, i.e., they spend too many points on a course that they could have bought for less. In order to eliminate such an efficiency loss, our future research agenda involves investigation of dynamic matching mechanisms that clear in multiple rounds.

APPENDIX

A. *Proofs of Results*

A.1. *Course bidding and two-sided matching markets.* We first relate course bidding to two-sided matching markets in order to prove Propositions 1 and 2.

Let I be the set of students, C be the set of courses, q_I be the maximum number of courses each student can take, $q_C = (q_c)_{c \in C}$ be the list of course capacities, and $b = [b_{ic}]_{i \in I c \in C}$ be a bid matrix. Let $P_I = (P_i)_{i \in I}$ be the list of student preferences over schedules and suppose preferences are substitutable. We simply refer to each six-tuple (I, C, q_I, q_C, P_I, b) as an *ex post problem*.

Given an ex post problem, construct a *two-sided matching market* as follows: In addition to students who have preferences over schedules (i.e., sets of courses of size at most q_I), pretend that each course c is also an agent who has strict preferences P_c over groups of students of size at most q_c. Furthermore, suppose that preferences of courses are responsive and based on student bids. That is, for each course c,

1. for any pair of students i, j, $\{i\} P_c \{j\}$ if and only if $b_{ic} > b_{jc}$,
2. for any student i, and any group of students J with $|J| < q_c$, $i \notin J$,

$$(J \cup \{i\}) P_c J,$$

3. for any pair of students i, j, and any group of students J with $i, j \notin J$ as well as $|J| < q_c$,

$$(J \cup \{i\}) P_c (J \cup \{j\}) \text{ if and only if } \{i\} P_c \{j\}.$$

Let $P_C = (P_c)_{c \in C}$ be the list of course preferences. Given an ex post problem (I, C, q_I, q_C, P_I, b) we refer to the six-tuple $(I, C, q_I, q_C, P_I, P_C)$ as an *induced two-sided matching market*.

For a problem, the central concept is a market equilibrium. For a two-sided matching market, the central concept is *pairwise stability*: A matching μ is pairwise stable if there is no unmatched student-course pair (i, c) such that

1. a. student i has an incomplete schedule and $(\mu_i \cup \{c\}) P_i \mu_i$ or
   b. student i has a course d in his schedule such that $[(\mu_i \setminus \{d\}) \cup \{c\}] P_i \mu_i$ and
2. a. course c has an empty slot under μ or
   b. course c has a student j in its class such that $[(\mu_c \setminus \{j\}) \cup \{i\}] P_c \mu_c$.

The following well-known result is due to Blair (1988).

PROPOSITION 5. *Suppose both students and courses have substitutable preferences over other side of the market. Then*

1. *a student-proposing deferred acceptance algorithm yields a pairwise stable matching, and*
2. *this pairwise stable matching is at least as good as any pairwise stable matching for any student.*

We next state the proof of Lemma 1 for substitutable preferences.

PROOF OF LEMMA 1. Let F be a belief system for market-clearing bids. For any student i, let b_i be a bid vector, P_i be the substitutable preferences of student i over schedules, and $\tilde{P}_i \neq P_i$ be any other preference relation. We will prove that for every utility function U_i that represents P_i, the expected utility of revealing $(b_i, \tilde{P}_i)$ cannot exceed the expected utility of revealing (b_i, P_i) for the Gale–Shapley Pareto-dominant market mechanism under price-taking behavior with respect to F.

Consider any realization of price vector p as a draw from the distribution function F. Consider the instances under which the student reveals $(b_i, \tilde{P}_i)$ and (b_i, P_i). The student believes that he will be placed in a course c if and only if $b_{ic} \geq p_c$ as long as he makes an offer under the Gale–Shapley Pareto-dominant market mechanism. Whenever a student is rejected by a course under (b_i, P_i), it means that the quota of the course will be full and $p_c > b_{ic}$. Therefore, the student will also be rejected by the same course c under $(b_i, \tilde{P}_i)$. Since the mechanism's algorithm allows the student to make an offer to the best schedule of courses among the ones that had not yet rejected the student in each round, the student believes that revealing (b_i, P_i) will bring at least as much utility as $(b_i, \tilde{P}_i)$, which includes his incorrect preferences. Since the above observation is true for each draw of the price vector, this observation will also hold under expected utility maximization under price-taking behavior. ■

Proposition 5 together with Lemma 1 and the following lemma will be key to proving Propositions 1 and 2.

LEMMA 2. *Let $(I, C, q_I, q_C, B, U, F)$ be a course-bidding economy. Let bid matrix b satisfy condition 1 of the market equilibrium for $(I, C, q_I, q_C, B, U, F)$, that is, it is the market equilibrium bid matrix. Let (I, C, q_I, q_C, P_I, b) be the induced ex post problem and $(I, C, q_I, q_C, P_I, P_C)$ be any of its induced two-sided matching markets. A matching μ is a market outcome of the economy $(I, C, q_I, q_C, B, U, F)$ if and only if it is a pairwise stable matching of the two-sided matching market $(I, C, q_I, q_C, P_I, P_C)$.*

PROOF OF LEMMA 2. Let (μ, b, p) be a market equilibrium of the economy $(I, C, q_I, q_C, B, U, F)$ and suppose μ is not pairwise stable for the induced two-sided matching market $(I, C, q_I, q_C, P_I, P_C)$. There are four possibilities.

Case 1: There exists an unmatched student-course pair (i, c) such that

(a) student i has an incomplete schedule and $(\mu_i \cup \{c\}) P_i \mu_i$, and
(b) course c has an empty slot.

Since c has an empty slot, $p_c = 0$. But then whenever a student affords schedule μ_i he can afford schedule $s = \mu_i \cup \{c\}$ as well and hence $s P_i \mu_i$ for an affordable schedule s contradicting (μ, b, p) is a market equilibrium.

Case 2: There exists an unmatched student–course pair (i, c) such that

(a) student i has a course d in his schedule such that $[(\mu_i \setminus \{d\}) \cup \{c\}] P_i \mu_i$, and
(b) course c has an empty slot.

Since student i can afford schedule μ_i, he can afford schedule $s = \mu_i \setminus \{d\}$ as well. Moreover, since c has an empty slot, $p_c = 0$ and hence he can also afford schedule $s' = s \cup \{c\} = [(\mu_i \setminus \{d\}) \cup \{c\}$. Therefore, $s' P_i \mu_i$ for an affordable schedule s' contradicting (μ, b, p) is a market equilibrium.

Case 3: There exists an unmatched student–course pair (i, c) such that

(a) student i has an incomplete schedule and $(\mu_i \cup \{c\}) P_i \mu_i$, and
(b) course c has a student j in its class such that $[(\mu_c \setminus \{j\}) \cup \{i\}] P_c \mu_c$.

Since $|\mu_i| < q_I$, we have $|(\mu_i \cup \{c\})| \leq q_I$ and therefore $s = \mu_i \cup \{c\}$ is a schedule. Moreover (μ, b, p) being a market equilibrium with $c \in \mu_j$ and $[(\mu_c \setminus \{j\}) \cup \{i\}] P_c \mu_c$ imply $b_{ic} \geq b_{jc} \geq p_c$, and therefore since student i can afford μ_i, he can afford $s = \mu_i \cup \{c\}$ as well. Hence, $s P_i \mu_i$ for an affordable schedule s contradicting (μ, b, p) is a market equilibrium.

Case 4: There exists an unmatched student–course pair (i, c) such that

(a) student i has a course d in his schedule such that $[(\mu_i \setminus \{d\}) \cup \{c\}] P_i \mu_i$, and
(b) course c has a student j in its class such that $[(\mu_c \setminus \{j\}) \cup \{i\}] P_c \mu_c$.

Since (μ, p) is a market outcome with $c \in \mu_j$, $[(\mu_c \setminus \{j\}) \cup \{i\}] P_c \mu_c$ implies $b_{ic} \geq b_{jc} \geq p_c$, and therefore student i can afford a seat at course c. Moreover since he can afford schedule μ_i, he can afford schedule $s = \mu_i \setminus \{d\}$ as well. Therefore, he can also afford schedule $s' = s \cup \{c\} = [(\mu_i \setminus \{d\}) \cup \{c\}$, and hence $s' P_i \mu_i$ for an affordable schedule s' contradicting (μ, b, p) is a market equilibrium.

These four cases exhaust all possibilities and hence μ shall be pairwise stable for the two-sided matching market $(I, C, q_I, q_C, P_I, P_C)$.

Conversely, let μ be a pairwise stable matching for the two-sided matching market $(I, C, q_I, q_C, P_I, P_C)$. Construct the price vector $p = (p_c)_{c \in C}$ as follows:

1. If c has a full class, then $p_c = b_{ic}$ where student i is the least desirable student who is assigned a seat at course c under μ.
2. If c has an empty slot, then $p_c = 0$.

We will show that (μ, b, p) is a market equilibrium of the problem (I, C, q_I, q_C, P_I, b):

1. By construction, $b_{ic} \geq p_c$ for any student i and any course $c \in \mu_i$.
2. Again by construction, if $|\mu_c| < q_c$, then $p_c = 0$.
3. Finally suppose there exists a student i and a schedule $s \neq \mu_i$ that he could afford such that $s R_i \mu_i$. Since preferences are strict, $s P_i \mu_i$, and therefore there is a course c student i could afford such that $c \in s$, $c \notin \mu_i$, and either
   (a) student i has an incomplete schedule μ_i with $(\mu_i \cup \{c\}) P_i \mu_i$, or
   (b) there is a course $d \in \mu_i$ such that $[(\mu_i \setminus \{d\}) \cup \{c\}] P_i \mu_i$.

Moreover since student i can afford a seat at course c either

(a) course c has an empty seat under μ or
(b) there exists a student $j \in \mu_c$ such that $[(\mu_c \setminus \{j\}) \cup \{i\}] P_c \mu_c$.

Existence of the pair (i, c) contradicts pairwise stability of matching μ, and therefore for any schedule $s \neq \mu_i$ student i can afford, $\mu_i P_i s$.

Hence (μ, b, p) is a market equilibrium. ■

PROOF OF PROPOSITION 1 AND PROPOSITION 2. We prove the stronger versions of the propositions for substitutable student preferences. Let I be the set of students, C be the set of courses, q_I be the maximum number of courses each student can take, $q_C = (q_c)_{c \in C}$ be the list of course capacities, $b = [b_{ic}]_{i \in I c \in C}$ be the bid matrix, and $P_I = (P_i)_{i \in I}$ be the list of substitutable student preferences represented by utility profile U. Let F be a belief system for the market-clearing bids and B be the bid endowment of students. By Lemma 1, every student i reveals P_i to the Gale–Shapley Pareto-dominant market mechanism. Moreover, let b be a bid matrix obtained by each student maximizing his expected utility under price-taking behavior with respect to F. Given that each student reveals his preferences truthfully under the Gale–Shapley Pareto-dominant market mechanism, maximizing expected utility is identical to maximizing $\mathbf{U}_i(\tilde{b}_i)$ with respect to $\tilde{b}_i$. Therefore, b is an equilibrium bid matrix. Let μ^{GS} be the outcome of Gale–Shapley Pareto-dominant market mechanism under (P, b). Given the ex post problem (I, C, q_I, q_C, P_I, b), let $(I, C, q_I, q_C, P_I, P_C)$ be an induced two-sided matching market. By Proposition 6, μ^{GS} is a pairwise stable matching for the two-sided matching market $(I, C, q_I, q_C, P_I, P_C)$, and it is at least as good as any pairwise stable matching for any student. Therefore, by Lemma 2, μ^{GS} is a market outcome for the problem $(I, C, q_I, q_C, B, U, F)$ and it Pareto dominates any other market outcome. ■

Proof of Proposition 3. Let b be a bid matrix and P be a list of substitutable student preferences. Let μ^{UMBS} be the outcome of the UMBS mechanism for b and μ^{GS} be the outcome of the Gale–Shapley Pareto-dominant mechanism for bid matrix b and preference profile P. Suppose that on the contrary, μ^{UMBS} Pareto dominates μ^{GS} under P. There exists some student i^1 with $\mu^{UMBS}(i^1) P_{i^1} \mu^{GS}(i^1)$. Hence, there exists some $c^1 \in \mu^{UMBS}(i^1) \backslash \mu^{GS}(i^1)$ such that $c^1 \in Ch(\mu^{UMBS}(i^1) \cup \mu^{GS}(i^1), P_{i^1})$. There exists some $i^2 \in I \backslash \{i^1\}$ such that $c^1 \in \mu^{GS}(i^2) \backslash \mu^{UMBS}(i^2)$ and $b_{i^1c^1} < b_{i^2c^1}$. The last part of the previous statement holds, as otherwise student i^1 would have enrolled in the preferred course c^1 under μ^{GS}, which is a market outcome. We have $\mu^{UMBS}(i^2) P_{i^2} \mu^{GS}(i^2)$ and there exists some $c^2 \in \mu^{UMBS}(i^2) \backslash \mu^{GS}(i^2)$ such that $c^2 \in Ch(\mu^{UMBS}(i^2) \cup \mu^{GS}(i^2), P_{i^2})$ and $b_{i^2c^1} < b_{i^2c^2}$. The last part of the previous statement holds because $c^1 \notin \mu^{UMBS}(i^2)$ whereas $c^1 \in \mu^{UMBS}(i^1)$ despite the fact that $b_{i^1c^1} < b_{i^2c^1}$, it should be the case that the bid of i^2 for c^1 was not valid under the UMBS mechanism although course c^1 had a seat when it was this bid's turn. Similarly, there exists $i^3 \in I \backslash \{i^2\}$ such that $c^2 \in \mu^{GS}(i^3) \backslash \mu^{UMBS}(i^3)$ such that $b_{i^2c^2} < b_{i^3c^2}$. We continue iteratively, and this construction results with a sequence of courses $\{c^k\}$ and a sequence of students $\{i^k\}$ such that $b_{i^1c^1} < b_{i^2c^1} < b_{i^2c^2} < \cdots < b_{i^kc^k} < b_{i^{k+1}c^k} < b_{i^{k+1}c^{k+1}} < \cdots$ However, this contradicts the fact that there are finitely many student–course pairs. Hence, μ^{UMBS} cannot Pareto dominate μ^{GS}, completing the proof. ■

Proof of Proposition 4. Let $C = \{c_1, \ldots, c_m\}$ be the set of courses, $q_C = (q_{c_1}, q_{c_2}, \ldots, q_{c_m})$ be the vector of course capacities, and q_I be the maximum number of courses each student can take. Suppose there is a student i whose preferences are not substitutable. Relabel the students so that i_1 is this student. Since P_{i_1} is not substitutable, for some $C' \subseteq C$ there are two distinct courses—without loss of generality—$c_1, c_2 \in Ch(C', P_{i_1})$ such that $c_2 \notin Ch(C' \backslash \{c_1\}, P_{i_1})$. We will construct a set of students J, a bid vector b, and a list of responsive preferences $P_J = (P_i)_{i \in J}$ such that the resulting economy has no market equilibrium.

Let $I = J \cup \{i_1\}$ denote the set of all students. For each course $c \in C$ and bid matrix b define

$$J(c, b) = \{i \in I \backslash \{i_1, i_2\} : b_{ic} > \max\{b_{i_1c}, b_{i_2c}\}\} \quad \text{and}$$
$$K(c, b) = \{i \in I \backslash \{i_1, i_2\} : c \in Ch(C, P_i)\}.$$

That is, $J(c, b)$ is the set of students, each of whom bids more than students i_1, i_2 for course c, and $K(c, b)$ is the set of students other than i_1, i_2, each of whom has course c in his best schedule. Also define

$$C^* = Ch(C', P_{i_1}) \cup Ch(C' \backslash \{c_1\}, P_{i_1}).$$

Note that

$$c_1, c_2 \in C^* \quad \text{and} \quad Ch(C^*, P_{i_1}) = Ch(C', P_{i_1}).$$

Let C'' be the set of courses such that

$$s\ P_{i_1}\ \emptyset \Longrightarrow s \subseteq C''.$$

Relabel courses so that

$$C'' \cap \{c_3, c_4, \dots, c_{q_I+1}\} = \emptyset.$$

This can be done, provided that the number of courses is high enough. Construct the set of students J, the list of responsive preferences $P_J = (P_i)_{i\in J}$, the utility profile $U = (U_i)_{i\in J\cup\{i_1\}}$ representing P, and the belief system F, as

1. $P_J = (P_i)_{i\in J}$ and $U = (U_i)_{i\in J\cup\{i_1\}}$ satisfy the following:
   (a) For student i_1, any desirable schedule s that is at least as good as $Ch(C^*, P_{i_1})$ brings a utility of $u - \varepsilon_{1,s}$, and any desirable schedule s worse than $Ch(C^*, P_{i_1})$ brings a utility of $\frac{u}{2} - \varepsilon_{1,s}$ such that $u > 0$ and all $\varepsilon_{1,s}$ are positive and arbitrarily close to 0. Remaining unmatched brings utility 0, and any undesirable schedule brings a negative utility.
   (b) For student i_2, courses $c_1, c_2, \dots, c_{q_I+1}$ are the only desirable courses with

   $$\{c_2\} P_{i_2} \{c_3\} P_{i_2} \{c_4\} P_{i_2} \dots P_{i_2} \{c_{q_I+1}\} P_{i_2} \{c_1\}, \quad \text{and}$$

   such that P_{i_2} is a responsive preference relation over schedules with quota q_I. His utility schedule over the schedules is such that any schedule s with q_I desirable courses brings student i_2 utility $u - \varepsilon_{2,s}$, and any schedule s with less than q_I desirable courses and no undesirable courses brings $\frac{u}{2} - \varepsilon_{2,s}$ utility such that all $\varepsilon_{2,s}$ values are positive and arbitrarily close to 0. Remaining unmatched brings utility 0, and any other schedule brings a negative utility.
   (c) There are sufficiently many students in $I\backslash\{i_1, i_2\}$ such that each of such students desires a single course in C and finds any other course undesirable with the conditions that (1) the number of students in $I\backslash\{i_1, i_2\}$ that desire a course $c \in C^* \cup \{c_1, c_2, c_3, \dots, c_{q_I+1}\}$ is $q_c - 1$ and (2) the number of students in $I\backslash\{i_1, i_2\}$ that desire a course $c \notin C^* \cup \{c_1, c_2, c_3, \dots, c_{q_I+1}\}$ is q_c. The utility of these students from their desirable course is arbitrary but higher than the option value of remaining unmatched, which is in turn larger than the utility of any other course.
2. F is such that beliefs are independent for each course and satisfy for some ε positive but arbitrarily small:
   a. For course c_1, any bid between 1 and $\frac{2B}{3}$ succeeds with probability 0.5, any higher bid less than B succeeds with probability $1 - \varepsilon$, and bid B succeeds with probability 1.
   b. For course c_2, any bid between 1 and $\frac{B}{2}$ succeeds with probability 0.1, any higher bid less than B succeeds with probability 0.5, and bid B succeeds with probability 1.
   c. For any course $c \in \{c_3, \dots., c_{q_{I+1}}\} \cup C^*$, any bid between 1 and $B - 1$ succeeds with probability $1 - \varepsilon$, and bid B succeeds with probability 1.
   d. For any course $c \notin \{c_1, c_2, c_3, \dots, c_{q_I+1}\} \cup C^*$, any bid between 1 and $B - 1$ succeeds with probability ε, and bid B succeeds with probability 0.1, where ε is given above.

We next prove the following claim:

CLAIM 1. Given that B is sufficiently large and ε is sufficiently small, for any equilibrium bid matrix b of the economy (U, F), we have

1. $b_{i_2c} < b_{i_1c} < B$ for all $c \in C^*\backslash\{c_1\}$,
2. $b_{i_1c} < b_{i_2c} < B$ for all $c \in \{c_1, c_3, c_4, \dots, c_{q_I+1}\}$,
3. for all $i \in I\backslash\{i_1, i_2\}$, $b_{ic} \in \{0, B\}$ for all $c \in C$.
4. $J(c, b) = K(c, b)$ for all $c \in C$,
5. $|J(c, b)| = |K(c, b)| = q_c - 1$ for all $c \in \{c_1, c_2, c_3, \dots, c_{q_I+1}\} \cup C^*$, and
6. $|J(c, b)| = |K(c, b)| = q_c$ for all $c \notin \{c_1, c_2, c_3, \dots, c_{q_I+1}\} \cup C^*$.

PROOF OF CLAIM 1. We prove the claim by deriving the equilibrium bid matrix for the students:

(a) Consider any student $i \in I\backslash\{i_1, i_2\}$. Given that he only finds some course $c \in \{c_1, c_2, c_3, \dots, c_{q_I+1}\} \cup C^*$ desirable and that he believes that his bid will clear with the highest possible probability, if and only if he bids B points, his optimal bid vector satisfies $b_{ic} = B$ and $b_{ic'} = 0$ for all $c' \neq c$.

(b) Consider student i_1. We consider three cases for his bidding behavior:

  (i) If he bids more than $\frac{2B}{3}$ for c_1, 1 for each remaining course in C'' (in particular less than $\frac{B}{2} + 1$ for course c_2): First note that this can be feasible for sufficiently large B. Observe that he will clear course c_2 with probability 0.1 and clear c_1 with probability 1, and clear all other courses in C^* almost surely, and he will not clear any other course almost surely. Thus, whenever he clears c_2, which happens with 0.1 probability, he will almost surely clear $Ch(C^*, P_{i_1})$ as his best schedule and otherwise almost surely he will receive a less desirable schedule. Thus, his expected utility in this case will be $U^* = 0.1u + 0.9\frac{u}{2} + o^*(\varepsilon, \{\varepsilon_{1,s}\})$.

  (ii) If he bids more than $\frac{B}{2}$ for course c_2, 1 for each remaining course in C'' (in particular less than $\frac{2B}{3} + 1$ for course c_1): Observe that he will clear course c_2 with probability 0.5 and clear c_1 with probability 0.5 and clear all other courses in C^* almost surely, although he will not clear any other course almost surely. Thus, whenever he clears both c_1 and c_2, which happens with 0.25 probability, he will almost surely clear $Ch(C^*, P_{i_1})$ as his best schedule and otherwise he will receive a less desirable schedule. Thus, his expected utility will be $U^{**} = 0.25u + 0.75\frac{u}{2} + o^{**}(\varepsilon, \{\varepsilon_{1,s}\})$.

  (iii) Observe that whenever he bids B for a course in $C''\backslash C^*$ he can only clear it with probability 0.1, and his maximum payoff will be $U^{***} = 0.1u + o^{***}(\varepsilon, \{\varepsilon_{1,s}\})$.

  For sufficiently small but positive ε and $\{\varepsilon_{1,s}\}$ values, $o^*(\varepsilon, \{\varepsilon_{1,s}\}), o^{**}(\varepsilon, \{\varepsilon_{1,s}\}), o^{***}(\varepsilon, \{\varepsilon_{1,s}\})$ will be sufficiently close to 0, implying $U^{**} > U^* > U^{***}$. Note that he will bid at least 1 point for each course in C^*.

(c) Consider student i_2. We consider two cases for his bidding behavior:

  (i) If he bids more than $\frac{2B}{3}$ for c_1, 1 for each course in $\{c_2, \dots, c_{q_I+1}\}$ (in particular less than $\frac{B}{2} + 1$ for course c_2): Observe that he will clear course c_2 with probability 0.1 and clear all other courses in $\{c_1, c_3, \dots, c_{q_I+1}\}$ with probability 1. Thus, since his preferences are responsive, he will clear a schedule with q_I desirable courses. Thus, his expected utility in this case will be $U^* = u + o^*(\varepsilon, \{\varepsilon_{2,s}\})$.

  (ii) If he bids more than $\frac{B}{2}$ for course c_2, 1 for each course in $\{c_1, c_3, \dots, c_{q_I+1}\}$ (in particular less than $\frac{2B}{3} + 1$ for course c_1): Observe that he will clear course c_2 with probability 0.5 and clear c_1 with probability 0.5 and clear all other courses in $\{c_2, \dots, c_{q_I+1}\}$. Thus, whenever he cannot clear both c_1 and c_2, which happens with 0.25 probability, he will have to get a schedule of $q_I - 1$ desirable courses with a utility close to $\frac{u}{2}$, and otherwise he will receive a schedule with q_I desirable courses with a utility close to u. Thus, his expected utility can be expressed as $U^{**} = 0.75u + 0.25\frac{u}{2} + o^{**}(\varepsilon, \{\varepsilon_{2,s}\})$.

For sufficiently small but positive ε and $\{\varepsilon_{2,s}\}$ values, $o^*(\varepsilon, \{\varepsilon_{2,s}\})$ and $o^{**}(\varepsilon, \{\varepsilon_{2,s}\})$ will be sufficiently close to 0, implying $U^* > U^{**}$.

Hence,

$$0 < b_{i_1c_1} \leq \frac{B}{2} < \frac{2B}{3} < b_{i_2c_1} < B \qquad \text{and} \qquad 0 < b_{i_2c_2} \leq \frac{B}{3} < \frac{B}{2} < b_{i_1c_2} < B.$$

Since $C^* \subseteq C''$ and $C'' \cap \{c_3, \ldots, c_{q_I+1}\} = \emptyset$, for all $c \in \{c_3, \ldots, c_{q_I+1}\}$, $b_{i_1c} = 0 < b_{i_2c} < B$. Moreover, for all $c \in C^*\backslash\{c_1, c_2\}$, $b_{i_2c} = 0 < b_{i_1c} < B$. We also have for all $i \in I\backslash\{i_1, i_2\}$, $b_{ic} \in \{0, B\}$ for all $c \in C$.

Since neither i_1 nor i_2 will bid B for any course $c \in C^* \cup \{c_1, c_2, c_3, \ldots, c_{q_I+1}\}$, although there are $q_c - 1$ students bidding B for c and preferring $\{c\}$ as their most desirable schedule, we have $J(c, b) = K(c, b)$ and $|J(c, b)| = |K(c, b)| = q_c - 1$.

Since neither i_1 nor i_2 will bid B for any course $c \notin C^* \cup \{c_1, c_2, c_3, \ldots, c_{q_I+1}\}$ although there are q_c students bidding B for c and preferring $\{c\}$ as their most desirable schedule, we have $J(c, b) = K(c, b)$ and $|J(c, b)| = |K(c, b)| = q_c$. ■

We will show that there is no market equilibrium of the resulting economy (U, F). On the contrary, suppose (μ, b, p) is a market equilibrium such that b satisfies the conditions in Claim 1.

CLAIM 2. For all $c \in \{c_1, c_2, c_3, \ldots, c_{q_I+1}\} \cup C^*$ and for all $i \in J(c, b)$, we have $c \in \mu_i$.

PROOF OF CLAIM 2. Suppose that there is a student $i \in J(c, b)$ such that $c \in \{c_1, c_2, c_3, \ldots, c_{q_I+1}\} \cup C^*$ and yet $c \notin \mu_i$. By Condition (4) of Claim 1, $i \in K(c, b)$. By Condition (3) of Claim 1 and the assumption that $i \in J(c, b)$, $p_c \leq b_{ic} = B$. By the construction of i's preferences, $\{c\} = Ch(C, P_i)$. Moreover, student i can afford the schedule $\{c\}$ and therefore $\{c\} P_i \mu_i$ contradicting (μ, b, p) is a market equilibrium. ■

CLAIM 3. $\{c_3, c_4, \ldots, c_{q_I+1}\} \subseteq \mu_{i_2}$.

PROOF OF CLAIM 3. Suppose that there is a course $c \in \{c_3, c_4, \ldots, c_{q_I+1}\}$ such that $c \notin \mu_{i_2}$. By responsiveness, $c \in Ch(\mu_{i_2} \cup \{c\}, P_{i_2})$. Therefore, since (μ, b, p) is a market equilibrium, $p_c > b_{i_2c}$. But then the definition of $J(c, b)$ together with Conditions (2) and (5) of Claim 1 imply only $q_c - 2$ students can afford a seat at course c, and therefore course c has an empty seat, contradicting $p_c > b_{i_2c}$. ■

CLAIM 4. $\mu_{i_1} \subseteq C^*$.

PROOF OF CLAIM 4. Suppose that there is a course $c \in \mu_{i_1}$ such that $c \in (C\backslash C^*)$. There are two possible cases:

Case 1: $c \in \{c_3, c_4, \ldots, c_{q_I+1}\}$: By assumption, $c \in \mu_{i_1}$ and by Claim 3, $c \in \mu_{i_2}$. By Conditions (4) and (5) of Claim 1, there is a student $j \in J(c, b) = K(c, b)$ such that $c \notin \mu_j$.

Case 2: $c \notin \{c_3, c_4, \ldots, c_{q_I+1}\}$: By Condition (4) and Condition (6) of Claim 1, there is a student $j \in J(c) = K(c)$ such that $c \notin \mu_j$.

In either case, (μ, b, p) being a market equilibrium together with $j \in J(c, b)$ implies $b_{jc} > b_{i_1c} \geq p_c$, and this together with $j \in K(c, b)$ and construction of P_j implies $\{c\} = Ch(C, P_j)$ contradicting (μ, b, p) is a market equilibrium. ■

We now have the machinery to execute the final part of the proof. Since only courses $c_1, c_2, c_3, \ldots, c_{q_I+1}$ are desirable for student i_2, Claim 3 leaves us with three possibilities: $\mu_{i_2} = \{c_3, c_4, \ldots, c_{q_{_I+1}}\}$ or $\mu_{i_2} = \{c_1, c_3, c_4, \ldots, c_{q_{_I+1}}\}$ or $\mu_{i_2} = \{c_2, c_3, c_4, \ldots, c_{q_{_I+1}}\}$. We will show that none of the three can be the case at a market equilibrium.

Case 1: $\mu_{i_2} = \{c_3, c_4, \ldots, c_{q_{_I+1}}\}$: Since (μ, b, p) is a market equilibrium and since $(\mu_{i_2} \cup \{c_1\}) P_{i_2} \mu_{i_2}$ by responsiveness, we have $p_{c_1} > b_{i_2c_1}$. However by Conditions (2), (3), and (5) of

Claim 1, there are only $q_{c_1} - 1$ students whose bids for course c_1 are higher than the bid of student i_2. Therefore, course c_1 has an empty seat under μ, contradicting $p_{c_1} > b_{i_2 c_1}$.

Case 2: $\mu_{i_2} = \{c_1, c_3, c_4, \ldots, c_{q_I+1}\}$: By assumption, $c_1 \in \mu_{i_2}$ and by Claim 2, each one of the $q_{c_1} - 1$ students in $J(c_1, b)$ is assigned a seat at course c_1; therefore

$$c_1 \notin \mu_{i_1}.$$

By Conditions (1), (3), and (5) of Claim 1, there are exactly q_{c_2} students, including student i_1, whose bids for course c_2 are higher than the bid of student i_2. Therefore, since $[(\mu_{i_2} \setminus \{c_1\}) \cup \{c_2\}] P_{i_2} \mu_{i_2}$ by responsiveness, each one of these students should be assigned a seat at course c_2 for otherwise $p_{c_2} = 0$ and student i_2 affords the better schedule $[(\mu_{i_2} \setminus \{c_1\}) \cup \{c_2\}$. Hence,

$$c_2 \in \mu_{i_1}.$$

By Conditions (1) and (5) of Claim 1, exactly $q_c - 1$ students bid more than student i_1 for each course $c \in Ch(C' \setminus \{c_1\}, P_{i_1}) \subseteq C^* \setminus \{c_1\}$ and since (μ, b, p) is a market equilibrium, student i_1 can afford the schedule $Ch(C' \setminus \{c_1\}, P_{i_1})$. Moreover, by Claim 4, $\mu_{i_1} \subseteq C^* \subseteq C'$, and we have already shown that $c_1 \notin \mu_{i_1}$. Therefore $\mu_{i_1} = Ch(C' \setminus \{c_1\}, P_{i_1})$. However the preferences of student i_1 are not substitutable and, in particular, $c_2 \notin Ch(C' \setminus \{c_1\}, P_{i_1})$, and therefore $c_2 \notin \mu_{i_1}$, directly contradicting $c_2 \in \mu_{i_1}$.

Case 3: $\mu_{i_2} = \{c_2, c_3, c_4, \ldots, c_{q_I+1}\}$: By assumption, $c_2 \in \mu_{i_2}$ and by Claim 2, each one of the $q_{c_2} - 1$ students in $J(c_2, b)$ is assigned a seat at course c_2; therefore, $c_2 \notin \mu_{i_1}$. Since $c_2 \in Ch(C', P_{i_1})$,

$$\mu_{i_1} \neq Ch(C', P_{i_1}).$$

Consider course c_1. Although $b_{i_2 c_1} > b_{i_1 c_1}$, by assumption $c_1 \notin \mu_{i_2}$ and by Conditions (4) and (5) of Claim 1, exactly $q_{c_1} - 1$ other students bid higher than student i_1 for course c_1. Therefore, since (μ, b, p) is a market equilibrium, student i_1 can afford a seat at course c_1. Next, consider any course $c \in C^* \setminus \{c_1\}$. By Conditions (1) and (5) of Claim 1, $q_c - 1$ students bid higher than student i_1 for each such course c. Therefore, student i_1 can afford each course in C^*. Moreover, by Claim 3, $\mu_{i_1} \subseteq C^*$ and therefore $\mu_{i_1} = Ch(C^*, P_{i_1}) = Ch(C', P_{i_1})$, directly contradicting $\mu_{i_1} \neq Ch(C', P_{i_1})$ and completing the proof. ■

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