

ON THE EXISTENCE OF PURE AND MIXED STRATEGY NASH EQUILIBRIA IN DISCONTINUOUS GAMES

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A game is *better-reply secure* if for every nonequilibrium strategy x^* and every payoff vector limit u^* resulting from strategies approaching x^* , some player i has a strategy yielding a payoff strictly above u_i^* even if the others deviate slightly from x^* . If strategy spaces are compact and convex, payoffs are quasiconcave in the owner's strategy, and the game is better-reply secure, then a pure strategy Nash equilibrium exists. Better-reply security holds in many economic games. It also permits new results on the existence of symmetric and mixed strategy Nash equilibria.

KEYWORDS: Discontinuous games, better-reply security, payoff security, pure strategy, mixed strategy, Nash equilibrium, existence.

1. INTRODUCTION

IT IS OFTEN NATURAL TO MODEL strategic settings in economics as games with infinite strategy spaces. For example, models of price and spatial competition (Bertrand (1883), Hotelling (1929)), auctions (Milgrom and Weber (1982)), patent races (Fudenberg et al. (1983)), etc., typically allow participants to choose any action among a continuum. However, such games frequently exhibit discontinuities in the payoffs. Consequently, standard theorems such as those found in Debreu (1952) or Glicksberg (1952) cannot be applied to establish the existence of an equilibrium (pure or mixed). Now while in many of these games equilibria can be constructed, rendering the existence question moot, there are other instances in which this is not the case. Consider for example, the pay-your-bid multi-unit auction, a game that we shall consider in detail. While its rules are simple enough to state, rather little is known at present about this auction's equilibrium strategies. Indeed, up to now, even the existence of an equilibrium in this game has been at issue since payoffs are not continuous.²

The present paper offers a pure strategy Nash equilibrium existence result for a large class of discontinuous games. Indeed, our main result on the existence of *pure* strategy equilibrium strictly generalizes the *mixed* strategy equilibrium existence results of Nash (1950), Glicksberg (1952), Mas-Colell (1984), Dasgupta

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²Ties in winning bids must be broken according to some fixed rule. This inevitably leads to a discontinuity in a tying bidder's payoff since bidding slightly higher would have guaranteed winning the unit.

and Maskin (1986), Mertens (1986), Simon (1987), and Robson (1994), as well as the *pure* strategy existence result for two-person zero-sum games due to Sion (1958). Although our main result is formally unrelated to that of Baye et al. (1993), the present hypotheses are significantly easier to check in practical applications and as we indicate below have a natural interpretation, the latter property being important for theoretical applications. Also presented here are subsidiary results on the existence of pure and mixed strategy symmetric equilibria in games possessing enough symmetry.

Our results rely on a new condition called *better-reply security*. A game is better-reply secure if for every nonequilibrium strategy x^* and every payoff vector limit u^* resulting from strategies approaching x^* , some player i has a strategy yielding a payoff strictly above u_i^* even if the others deviate slightly from x^* .³

For example, Bertrand's price-competition game is better-reply secure because given any price vector p^* , each player can obtain a payoff very close to his supremum at a price that no one else charges. Hence this payoff can be secured since it will result even if the others deviate slightly from p^* . But if p^* is not an equilibrium, then because total industry profits are continuous, for any price vector near p^* someone's payoff must fall below that which he can secure.

Our main result is that games with compact, convex strategy spaces and payoffs that are quasiconcave in the owner's strategy possess a pure strategy Nash equilibrium if in addition they are better-reply secure.

Better-reply security combines and generalizes two conditions, *reciprocal upper semicontinuity* and *payoff security*. A game is reciprocally upper semicontinuous if, whenever some player's payoff jumps down, some other player's payoff jumps up. This condition was introduced by Simon (1987), and expresses a weak form of competition.⁴ A game is payoff secure if for every joint strategy, x , each player has a strategy that virtually guarantees the payoff he receives at x , even if the others play slightly differently than at x .⁵ Both conditions are satisfied in many economic games and are often quite simple to check. For example, Bertrand's game is trivially reciprocally u.s.c. since the sum of payoffs is continuous. Moreover, payoff security is easily checked by noting that each

³When $u^* = u(x^*)$, as results when the sequence of strategies approaching x^* is x^*, x^*, x^*, \dots , the strategy securing some player i a superior payoff is a better reply against x_{-i}^* than x_i^* . Note that because payoffs are not necessarily continuous, there can be many payoff vector limits, u^* , resulting from strategies approaching x^* .

⁴Simon used the term *complementary discontinuities*. The term used here instead emphasizes the condition's relation to upper semicontinuity of a real-valued function. Indeed, the role played by reciprocal upper semicontinuity in ensuring existence of equilibrium is similar to that played by upper semicontinuity of a function in guaranteeing the existence of a maximum of a real-valued function. Dasgupta and Maskin (1986) were the first to introduce a condition of this kind in a game setting. They require the sum of the players' payoffs to be upper semicontinuous. Reciprocal upper semicontinuity is a strict generalization.

⁵Payoff security was introduced in an earlier version of this paper (Reny (1995)) dealing solely with the existence of mixed strategy equilibria. We have abbreviated the term used there "local payoff security" to simply "payoff security."

firm can guarantee a payoff that is virtually no worse than the status quo (and perhaps significantly better) by decreasing its price slightly, so long as the others' prices do not change too much. The ease with which these two conditions can be checked provides sufficient reason to highlight them, despite the fact that better-reply security is more permissive.⁶

The method of proof employed to obtain our main result is new. Previous techniques fail for two reasons. First, the presence of discontinuities in payoffs may preclude the existence of best replies so that best reply correspondences need not be nonempty-valued, let alone upper hemicontinuous. Consequently, both lattice-theoretical techniques as well as the standard application of Kakutani's fixed point theorem to best reply correspondences fail. Second, to obtain the existence of a pure strategy equilibrium, one cannot approximate the original game by a sequence of finite games and expect that the sequence of *mixed* strategy equilibria of the finite games will converge to a *pure* strategy equilibrium of the limit game. Such an approximation technique has been employed by Dasgupta and Maskin (1986), Simon (1987), and Simon and Zame (1990) to obtain the existence of *mixed* strategy equilibria in discontinuous games. To circumvent these difficulties, we instead approximate the discontinuous *payoff functions*, not the strategy sets, by a sequence of continuous payoff functions. Because strategy spaces are compact, this yields an appropriate sequence of pure strategy profiles whose limit is a pure strategy equilibrium of the original game.

The remainder of the paper is organized as follows. Section 2 describes the environment and notation, and provides a number of preliminary definitions. Section 3 contains the new condition, *better-reply security*, as well as our main pure strategy equilibrium existence result and its proof. Examples illustrating the theorem are also given. Section 4 considers the existence of symmetric pure strategy equilibria. Once again better-reply security plays a role. However, owing to symmetry it need hold only *along the diagonal of payoffs*. Perhaps more significantly, it is observed (as in Baye et al. (1993)) that the quasiconcavity condition on a player's payoff in his own action can be weakened substantially, again, by requiring it to hold merely *along the diagonal of payoffs*. Examples of symmetric games without quasiconcave payoffs, yet covered by this result, are provided. Section 5 considers the existence of mixed strategy equilibria and provides a number of corollaries to our main results, each of which is obtained by considering the game's mixed extension. One of the examples we consider to illustrate the corollaries is the pay-your-bid multi-unit auction. It is shown that our *mixed* strategy equilibrium existence result can be used to establish the existence of a *pure* strategy equilibrium in that game. Section 6 discusses related work. It is shown there that the mixed strategy equilibrium existence theorems of Nash (1950), Glicksberg (1952), Dasgupta and Maskin (1986), Mertens (1986), and Simon (1987) are all special cases of the corollaries pertaining to mixed

⁶In Section 3 it is shown that reciprocal upper semicontinuity and payoff security together imply better-reply security.

strategy equilibria given in Section 5, and so also of our main result. Section 7 considers other approaches to the existence question, and Section 8 concludes with a brief discussion of issues relating to approximating infinite-action games by finite games.

2. PRELIMINARIES

There are N players. Each player $i = 1, 2, \dots, N$ has a pure strategy set, X_i , a nonempty compact subset of a topological vector space,⁷ and a bounded payoff function $u_i: X \rightarrow \mathbf{R}$, where $X = \times_{i=1}^N X_i$.⁸ Under these conditions, $G = (X_i, u_i)_{i=1}^N$ is called a *compact game*.

Throughout, the product of any number of sets is endowed with the product topology. The symbol $-i$ denotes "all players but i ." In particular, $X_{-i} = \times_{j \neq i} X_j$, and x_{-i} denotes an element of X_{-i} .

The vector of the players' payoff functions will be denoted by $u: X \rightarrow \mathbf{R}^N$ and is defined by $u(x) = (u_1(x), \dots, u_N(x))$ for every $x \in X$. The graph of the vector payoff function is the subset of $X \times \mathbf{R}^N$ given by $\{(x, u) \in X \times \mathbf{R}^N \mid u = u(x)\}$.

Finally, if each X_i is convex and for each i and every $x_{-i} \in X_{-i}$, $u_i(\cdot, x_{-i})$ is quasiconcave on X_i , we say that $G = (X_i, u_i)_{i=1}^N$ is *quasiconcave*.

3. PURE STRATEGY EQUILIBRIA

The following definitions play a central role in all of our results.

DEFINITION: Player i can *secure* a payoff of $\alpha \in \mathbf{R}$ at $x \in X$ if there exists $\bar{x}_i \in X_i$, such that $u_i(\bar{x}_i, x'_{-i}) \geq \alpha$ for all x'_{-i} in some open neighborhood of x_{-i} .

Thus, a payoff can be secured by i at x if i has a strategy that guarantees at least that payoff even if the other players deviate slightly from x .

The next condition employs the *closure* of the graph of the game's vector payoff function. While this set is perfectly well defined in our general topological space environment, it might supplement the reader's intuition to consider the metric space context. There, a pair (x^*, u^*) is in the closure of the graph of the vector payoff function if u^* is the limit of the vector of payoffs corresponding to some sequence of strategies converging to x^* .

⁷We follow Royden (1988) and say that a linear vector space V endowed with a topology is a *topological vector space* if addition is continuous from $V \times V$ into V , and multiplication by scalars is continuous from $\mathbf{R} \times V$ into V . In particular, unlike some other treatments, this definition does not require (although it of course permits) V to be Hausdorff.

⁸One can in fact allow payoffs to be unbounded, but this would require introducing the extended reals, since the limiting values of the payoff functions will come into play below. A more elementary yet equivalent treatment of unbounded payoff functions (indeed, even those taking on infinite values) results by first transforming the unbounded payoffs, v_i , into bounded ones, u_i , via $u_i = \exp v_i / (1 + \exp v_i)$, and then following our treatment below.

DEFINITION: A game $G = (X_i, u_i)_{i=1}^N$ is *better-reply secure* if whenever (x^*, u^*) is in the closure of the graph of its vector payoff function and x^* is not an equilibrium, some player i can secure a payoff strictly above u_i^* at x^* .

So, a game is better-reply secure if for every nonequilibrium strategy x^* and every payoff vector u^* resulting from strategies approaching x^* , some player i has a strategy yielding a payoff strictly above u_i^* even if the others deviate slightly from x^* .

Games with continuous payoff functions are better-reply secure, since any better reply will secure a payoff strictly above a player's inferior nonequilibrium payoff and those generated by nearby strategies. Many discontinuous economic games are better-reply secure. As already remarked in the introduction, Bertrand's price-competition game is one such example. As we shall see below, first-price auctions are also better-reply secure. Of course, auction-game payoffs need not be quasiconcave in one's own strategy, so Theorem 3.1 below may not directly apply. However, Corollary 5.2 to Theorem 3.1 guarantees the existence of a mixed strategy equilibrium in these games (see Section 5 below).

We now state our main result.

THEOREM 3.1: *If $G = (X_i, u_i)_{i=1}^N$ is compact, quasiconcave, and better-reply secure, then it possesses a pure strategy Nash equilibrium.*

A technical point of interest is that, as in Sion (1958), we do not require X to be Hausdorff or locally convex. This is in contrast to standard fixed point theorems (i.e., Glicksberg (1952)) upon which other equilibrium existence results are based.⁹

In Section 6 it is shown that the above result on the existence of *pure* strategy equilibria generalizes the *mixed* strategy equilibrium existence results of Nash (1950), Glicksberg (1952), Mas-Colell (1984), Dasgupta and Maskin (1986), Simon (1987), and Robson (1994). In addition (see Corollary 3.4 below), it extends to the N -person non-zero-sum case the pure strategy equilibrium existence result due to Sion (1958).¹⁰

While better-reply security is straightforward to verify, it is sometimes even simpler to verify other conditions leading to it. We now provide two rather useful conditions that together imply better-reply security.

DEFINITION: A game $G = (X_i, u_i)_{i=1}^N$ is *payoff secure* if for every $x \in X$ and every $\epsilon > 0$, each player i can secure a payoff of $u_i(x) - \epsilon$ at x .

⁹While we never require X to be locally convex, we shall later assume that X is Hausdorff when mixed strategies are considered.

¹⁰Sion (1958) shows that compact, quasiconcave, zero-sum games possess a value in pure strategies, whenever each player's payoff is u.s.c. in his own pure strategy and l.s.c. in the opponent's (i.e., whenever each player's payoff is, owing to the zero-sum property, l.s.c. in the opponent's pure strategy).

Payoff security requires that for every strategy x , each player has a strategy that virtually guarantees the payoff he receives at x , even if the others deviate slightly from x . Sion (1958), Mertens (1986), and Simon (1987) note the importance of the robustness of one's payoff to perturbations in the others' strategies, and our notion of payoff security is a natural continuation of their ideas.

DEFINITION: A game $G = (X_i, u_i)_{i=1}^N$ is *reciprocally upper semicontinuous* if, whenever (x, u) is in the closure of the graph of its vector payoff function and $u_i(x) \leq u_i$ for every player i , then $u_i(x) = u_i$ for every player i .

Reciprocal u.s.c. was first introduced by Simon (1987) under the name *complementary discontinuities*. It requires that some player's payoff jump up whenever some other player's payoff jumps down. It generalizes the condition introduced by Dasgupta and Maskin (1986) that the sum of the players' payoffs be u.s.c. Since the u.s.c.-sum condition is cardinal in nature, Simon's (1987) generalization is a distinct improvement.^{11,12} We now relate reciprocal u.s.c. and payoff security to better-reply security. The proof of the following result can be found in the Appendix.

PROPOSITION 3.2: *If $G = (X_i, u_i)_{i=1}^N$ is reciprocally u.s.c. and payoff secure, then it is better-reply secure.*

Bertrand's price competition game with zero costs provides a ready opportunity to apply Proposition 3.2. This game is reciprocally u.s.c. since the sum of the firms' payoffs is continuous. In addition it is payoff secure since each firm can secure a payoff that is at worst just below the status quo by lowering its price slightly. Consequently, as we have already argued directly, this game is better-reply secure.

On the other hand, first-price auction games, while payoff secure, fail to be reciprocally u.s.c. so that Proposition 3.2 does not apply.¹³ Nonetheless, as we shall show in Section 5, first-price auction games are better-reply secure. Consequently, better-reply security is strictly more permissive than the combination of reciprocal u.s.c. and payoff security. However, as mentioned before, Theorem 3.1 does not apply since first-price auction payoffs are not quasiconcave. First-price auctions and, more generally, multi-unit pay-your-bid auctions are considered in Example 5.2 in Section 5 below.

¹¹None of the three conditions, better-reply security, reciprocal upper semicontinuity, or payoff security is precisely an ordinal property. However, they are all virtually so in the following sense. If $f_i: \mathbf{R} \rightarrow \mathbf{R}$ is *continuous* and strictly increasing for every $i = 1, \dots, N$, then $(X_i, u_i)_{i=1}^N$ is (separately) better-reply secure, reciprocally u.s.c., and payoff secure if and only if $(X_i, f_i \circ u_i)_{i=1}^N$ is.

¹²I am grateful to Faruk Gul and Tim van Zandt for encouraging me to refrain from placing restrictions on the sum of the players' payoffs.

¹³Suppose there are just two bidders. Both of their (ex ante) payoffs can simultaneously jump down if each bidder suddenly ties (from below) the other bidder's high-profit winning bids.

Together, Theorem 3.1 and Proposition 3.2 immediately yield the following useful result.

COROLLARY 3.3: *If $G = (X_i, u_i)_{i=1}^N$ is compact, quasiconcave, reciprocally upper semicontinuous and payoff secure, then it possesses a pure strategy Nash equilibrium.*

A further implication is the following generalization of Sion's (1958) existence result for two-person zero-sum games.

COROLLARY 3.4: *Let $G = (X_i, u_i)_{i=1}^N$ be a compact, quasiconcave, reciprocally u.s.c. game. If each player's payoff is lower-semicontinuous in the pure strategies of the others, then G possesses a pure strategy Nash equilibrium.*

PROOF OF COROLLARY 3.4: By Corollary 3.3, it suffices to show that the game is payoff secure. So, choose $x \in X$ and $\epsilon > 0$. The lower semicontinuity of $u_i(x_i, \cdot)$ on X_{-i} implies that $\{x'_{-i} \in X_{-i} | u_i(x_i, x'_{-i}) > u_i(x) - \epsilon\}$ is an open neighborhood of x_{-i} . Consequently, the game is payoff secure. *Q.E.D.*

Before proceeding to the proof of Theorem 3.1, we describe the main ideas. Better-reply security is used to construct for each player i , a function, $\underline{u}_i(x)$, that is l.s.c. in the opponents' strategies yet capable of detecting the presence of profitable deviations in the sense that x^* is an equilibrium if and only if for every player i , $\sup_{x_i \in X_i} \underline{u}_i(x_i, x^*_{-i}) \leq u_i(x^*)$. The ability to detect u_i -profitable deviations via a function that is lower semicontinuous in the opponent's strategies helps provide enough continuity to reduce the general existence question to establishing that for every *finite* set of deviations, there is a strategy against which no \underline{u}_i can detect a u_i -profitable deviation among elements of the finite set. This is the import of Parts I and II of the proof. That such strategies do indeed exist for each finite deviation set is established in Part III.

Part III then needs only to establish the existence of a strategy with equilibrium-like properties when strategy spaces are restricted, yet must include some fixed, finite set of potential deviations. As we shall explain momentarily, the restricted strategy spaces cannot be finite so that the presence of discontinuities remains an issue. Fortunately, the lower semicontinuity of $\underline{u}_i(x)$ in x_{-i} , allows one to approximate it from below by a sequence of functions that are continuous in x_{-i} . Consequently, standard equilibrium existence results guarantee a mixed strategy equilibrium to games with the restricted strategy spaces whose payoffs are the continuous approximations. An essential component of the approximating games is that while each player is allowed to employ mixed strategies, each views the others as employing only *pure* strategies. This is accomplished by assessing each player's payoff at the average (pure) strategy determined by the others' mixtures,¹⁴ and permits the quasiconcavity of $u_i(x)$ in

¹⁴It is this construction that requires the restricted strategy spaces to be infinite.

x_i to be exploited to construct an appropriate purification of the sequence of mixed equilibria that converges to a pure strategy “equilibrium” of the game with the restricted strategy spaces and original payoffs. A key to establishing that no \underline{u}_i can detect a u_i -profitable deviation from the limit strategy among elements of the finite set is the property that the sequence of approximating game payoffs *underestimates* the value of u_i along the sequence of purified strategies, while it *overestimates* the \underline{u}_i -value of a deviation at the limit. The former property follows because each \underline{u}_i (and so u_i) is approximated from below, while the latter property requires the approximation from below to be “as high as possible.”

The following lemma provides the technical result that is needed to obtain the continuous payoff approximation described above. Its proof can be found in the Appendix.

LEMMA 3.5: *Suppose that Y is compact, metric, and $f: Y \rightarrow \mathbf{R}$ is lower semicontinuous. Then there exists a sequence $\{f_n\}$ of real-valued continuous functions on Y such that $\forall y \in Y$:*

- (i) $f_n(y) \leq f(y) \forall n$, and
- (ii) $\forall y_n \rightarrow y, \liminf_n f_n(y_n) \geq f(y)$.

Part (i) of Lemma 3.5 ensures that payoffs can be approximated continuously from below, while part (ii) ensures that the approximations are “high enough.”

PROOF OF THEOREM 3.1: For each player i , and every $x \in X$, let

$$\underline{u}_i(x) = \sup_{U \ni x_{-i}} \inf_{x'_{-i} \in U} u_i(x_i, x'_{-i}),$$

where the sup is taken over all open neighborhoods, U , of x_{-i} . So defined, for each i and every $x_i \in X_i$, $\underline{u}_i(x_i, \cdot)$ is both real-valued (since u_i is bounded), and lower semicontinuous on X_{-i} . To see lower semicontinuity, observe that for fixed $x_i \in X_i, \underline{u}_i(x_i, x_{-i}) = \sup_{\text{open } U \subseteq X_{-i}} f_U(x_{-i})$, where $f_U(x_{-i}) = \inf_{x'_{-i} \in U} u_i(x_i, x'_{-i})$, if $x_{-i} \in U$, and $f_U(x_{-i}) = -\infty$ otherwise. Therefore, for each $x_i \in X_i, \underline{u}_i(x_i, \cdot)$ is the pointwise supremum of a collection of lower semicontinuous functions on X_{-i} , and hence lower semicontinuous itself.

Note that player i can secure a payoff strictly greater than $\alpha \in \mathbf{R}$ at $x^* \in X$ if and only if

$$(3.1) \quad \sup_{x_i \in X_i} \underline{u}_i(x_i, x^*_{-i}) > \alpha.$$

Let Γ denote the *closure* of the graph of the game’s vector payoff function, $u: X \rightarrow \mathbf{R}^N$. The remainder of the proof will be broken into three parts.

Part I (Detecting Nash Equilibria): By (3.1) and the better-reply security property, we have the following:

If $(x^*, u^*) \in \Gamma$, and for all i

$$(3.2) \quad \sup_{x_i \in X_i} \underline{u}_i(x_i, x_{-i}^*) \leq u_i^*,$$

then x^* is a Nash equilibrium.

Part II (Sufficiency of Finite Deviation Sets): For every $x, y \in X$, let $\underline{u}(x, y) = (\underline{u}_1(x_1, y_{-1}), \dots, \underline{u}_N(x_N, y_{-N}))$, and for each $x \in X$, define $E(x) = \{(y, u) \in \Gamma \mid \underline{u}(x, y) \leq u\}$, where \leq denotes componentwise weak order in \mathbf{R}^N . Letting $E = \bigcap_{x \in X} E(x)$, Part I implies that $x^* \in X$ is a Nash equilibrium if $(x^*, u^*) \in E$ for some $u^* \in \mathbf{R}^N$. Thus, it suffices to show that E is nonempty.

Now, because each $u_i(\cdot)$ is bounded, Γ is a compact subset of $X \times \mathbf{R}^N$. Hence, each $E(x)$ is also a compact subset of $X \times \mathbf{R}^N$ since each component of $\underline{u}(x, y)$ is l.s.c. in y for every x . Thus, it suffices to show that the collection $\{E(x)\}_{x \in X}$ possesses the finite intersection property. We shall establish this next.

Part III (Constructing Strategies Immune to Finite Deviation Sets): Parts I and II imply that it now suffices to demonstrate the following:

For every finite subset F of X , there exists $(\bar{x}, \bar{u}) \in \Gamma$ such that

$$(3.3) \quad \underline{u}(x, \bar{x}) \leq \bar{u}, \quad \text{for every } x \in F.$$

To prove (3.3), let $F = \{x^1, \dots, x^m\}$ be a finite subset of X , and for $i = 1, 2, \dots, N$, let $X_i^0 = \{x_i^1, \dots, x_i^m\}$. Since each X_i^0 is finite, we may endow its convex hull, $\text{co } X_i^0$, with the Euclidean metric. Let the symbol " \rightarrow_E " denote convergence in the Euclidean metric.

Now because each X_i is a topological vector space, the topology on $\text{co } X_i^0 \subseteq X_i$ induced by the Euclidean metric is at least as fine as the original one.¹⁵ Consequently, each $\underline{u}_i(x_i, \cdot)$, being l.s.c. on $\times_{j \neq i} X_j$ in the original product topology, is also l.s.c. on $\times_{j \neq i} \text{co } X_j^0$ in the Euclidean metric. Moreover, the Euclidean metric renders the sets $\times_{j \neq i} \text{co } X_j^0$ compact.

Hence, according to Lemma 3.5, for each $i = 1, 2, \dots, N$, and every $x_i \in X_i^0$, there is a sequence of functions, $\{u_i^n(x_i, \cdot)\}$, each continuous in the Euclidean metric on $\times_{j \neq i} \text{co } X_j^0$, such that for all $x_{-i} \in \times_{j \neq i} \text{co } X_j^0$,

- (i) $u_i^n(x_i, x_{-i}) \leq \underline{u}_i(x_i, x_{-i})$, for all n , and
- (ii) $\forall x_{-i} \rightarrow_E x_{-i}, \liminf_n u_i^n(x_i, x_{-i}^n) \geq \underline{u}_i(x_i, x_{-i})$.

For every n , construct the N -person game, G^n , as follows. Player i 's set of pure strategies is $\Delta(X_i^0)$, the set of probability measures on X_i^0 . For $\mu \in$

¹⁵The two topologies are identical when X_i is Hausdorff. See footnote 7.

$\times_{i=1}^N \Delta(X_i^0)$, player i 's payoff is

$$v_i^n(\mu) = \int_{X_i^0} u_i^n(x_i, \bar{x}_{-i}) d\mu_i;$$

where $\bar{x}_j = \int_{X_j^0} x_j d\mu_j \in \text{co } X_j^0$, for $j = 1, 2, \dots, N$.

Consequently, each v_i^n is continuous on $\times_{i=1}^N \Delta(X_i^0)$ in the Euclidean metric and quasiconcave in μ_i for every μ_{-i} . By standard results for continuous games (e.g., Fudenberg and Tirole (1991), Theorem 1.2), G^n possesses an equilibrium μ^n for every n . For every player i , we must then have for every x_i given positive weight by μ_i^n , and every $x'_i \in X_i^0$

$$u_i^n(x'_i, \bar{x}_{-i}^n) \leq v_i^n(\mu^n) = u_i^n(x_i, \bar{x}_{-i}^n) \leq \underline{u}_i(x_i, \bar{x}_{-i}^n) \leq u_i(x_i, \bar{x}_{-i}^n),$$

where the first inequality and the equality hold since μ^n is an equilibrium of G^n , the second inequality holds by condition (i) of the constructed approximating sequences u_i^n , and the last equality holds by the definition of \underline{u}_i .

Therefore, for each x_i given positive weight by μ_i^n , $u_i(x_i, \bar{x}_{-i}^n)$ is at least as large as i 's equilibrium payoff $v_i^n(\mu^n)$. By the quasiconcavity of $u_i(\cdot, \bar{x}_{-i}^n)$ on X_i , we must then have

$$(3.4) \quad u_i^n(x_i, \bar{x}_{-i}^n) \leq v_i^n(\mu^n) \leq u_i(\bar{x}^n), \quad \text{for all } x_i \in X_i^0.$$

Since for every n , \bar{x}^n is an element of the compact metric space $\times_{i=1}^N \text{co } X_i^0$, and each u_i is bounded, we may assume without loss of generality that $\bar{x}^n \rightarrow_E \bar{x} \in \times_{i=1}^N \text{co } X_i^0$, and that $\lim_n u_i(\bar{x}^n) = \bar{u}_i$ exists for every i . The inequality (3.4) then yields for all $x \in \times_{i=1}^N X_i^0$, and for all i ,

$$\underline{u}_i(x_i, \bar{x}_{-i}) \leq \liminf_{n \rightarrow \infty} u_i^n(x_i, \bar{x}_{-i}^n) \leq \bar{u}_i$$

where the first inequality follows by condition (ii) of the approximating sequences u_i^n .

Since $F \subseteq \times_{i=1}^N X_i^0$ is an arbitrary finite subset of X , and $\bar{x}^n \rightarrow_E \bar{x}$ implies $\bar{x}^n \rightarrow \bar{x}$, so that $(\bar{x}, \bar{u}) \in \Gamma$, this proves (3.3). *Q.E.D.*

REMARK 3.1: Better-reply security ensures that the set of Nash equilibria is closed and that limits of ϵ -equilibria are equilibria. Both of these follow from Part I of the proof. To see the first, note that if x^* is a Nash equilibrium, then $(x^*, u(x^*))$ satisfies (3.2). Hence, the closed set $\{x^* | (x^*, u^*) \in \Gamma \text{ satisfies (3.2) for some } u^*\}$ coincides with the set of Nash equilibria. To see the second, note that if x^ϵ is a sequence of ϵ -equilibria converging to x^* as ϵ tends to zero, then for every player i , every $x_i \in X_i$, and every ϵ , $\underline{u}_i(x_i, x_{-i}^\epsilon) \leq u_i(x_i, x_{-i}^\epsilon) \leq u_i(x_i, x_{-i}^\epsilon) + \epsilon$. Hence, the lower semicontinuity of each $\underline{u}_i(x_i, \cdot)$ in the strategies of the others implies that $(x^*, u^*) \in \Gamma$ satisfies (3.2), where $u^* = \lim_\epsilon u(x^\epsilon)$.¹⁶

¹⁶In a metric space context, the result that limits of ϵ -equilibria are equilibria implies that the set of Nash equilibria is closed, since one can choose $\epsilon = 0$ along the sequence. However, in a topological space the two results must be considered separately since the behavior of sequences and their limits is not sufficient to establish closedness of a set.

Two examples illustrate the above results.

EXAMPLE 3.1: Consider a class of two-person, non-zero-sum, noisy games of timing on the unit square.¹⁷ Related games have been used to model behavior in duels as well as in R&D and patent races. The payoff to player i is given by

$$u_i(x_i, x_{-i}) = \begin{cases} l_i(x_i), & \text{if } x_i < x_{-i}, \\ \phi_i(x_i), & \text{if } x_i = x_{-i}, \\ m_i(x_{-i}), & \text{if } x_i > x_{-i}, \end{cases}$$

where $l_i(\cdot)$ and $m_i(\cdot)$ are continuous on $[0, 1]$, and $l_i(\cdot)$ is nondecreasing on $[0, 1]$. We have the following:

If for $i = 1, 2$ and every $x \in [0, 1]$, (i) $\phi_i(x) \in \text{co}\{l_i(x), m_i(x)\}$ and (ii) $\text{sgn}(l_i(x) - \phi_i(x)) = \text{sgn}(\phi_{-i}(x) - m_{-i}(x))$, then the above game possesses a pure strategy Nash equilibrium.

To see this, note first that this game is clearly compact, while condition (ii) together with the continuity of l_i and m_i imply that the vector of payoffs is reciprocally u.s.c. Moreover, condition (i) and l_i nondecreasing imply that the game is quasiconcave; and condition (i) combined with the continuity of l_i and m_i imply that the game is payoff secure. Thus, Corollary 3.3 applies.

The next example shows how Theorem 3.1 can be employed as an alternative to Debreu's (1952) social equilibrium existence result to establish the existence of a Walrasian equilibrium. Consumers' choice sets are rendered independent of the price vector chosen by the auctioneer by defining their payoffs to be (discontinuously) unattractive at unaffordable choices.

EXAMPLE 3.2: Consider an n -consumer m -commodity exchange economy in which each consumer i has a continuous, locally nonsatiated, quasiconcave utility function $u_i: \mathbf{R}_+^m \rightarrow \mathbf{R}_+$, and a strictly positive endowment vector e_i . The following game is played by the n consumers and an auctioneer. Each consumer i chooses a nonnegative bundle $x_i \leq e + \mathbf{1}$, where e is the aggregate endowment vector. The auctioneer chooses a price vector p from the unit simplex in \mathbf{R}_+^m . Consequently, strategy spaces are compact and convex. Payoffs in the game are as follows. Given the joint strategy $(x_1, x_2, \dots, x_n, p)$, consumer i 's payoff is $u_i(x_i)$ if $p \cdot x_i \leq p \cdot e_i$ and is -1 otherwise, while the auctioneer's payoff is $p \cdot \sum_i (x_i - e_i)$, the value of excess demand. So defined, each player's payoff is quasiconcave in his own strategy and upper semicontinuous in all strategies. Hence the game is compact, quasiconcave, and reciprocally u.s.c. Moreover, at every joint strategy each consumer can secure his payoff through a small

¹⁷A game of timing, like a duel, is "noisy" when each player can hear the shot fired by the other. See Karlin (1959) for an extensive study of two-person zero-sum games of timing, and Pitchik (1982) for necessary and sufficient conditions for the existence of *mixed* strategy equilibria in non-zero-sum noisy games of timing.

proportional reduction in his consumption of every commodity.¹⁸ Together with the continuity of the auctioneer's payoff, this implies that the game is payoff secure. So, we may apply Corollary 3.3 to conclude that this game possesses a pure strategy Nash equilibrium, which by standard arguments constitutes a Walrasian equilibrium of the exchange economy.

Theorem 3.1 can also be applied to obtain the existence of a symmetric equilibrium in nondecreasing bidding functions in symmetric m th-price auctions for $m \geq 3$. However, in order to apply the theorem, ties in bids must be broken in favor of high-value bidders. This unusual tie-breaking rule, together with the fact that ties can occur in equilibrium, suggests that equilibria in these auctions may fail to exist under the standard tie-breaking rule. See Reny (1998).

4. PURE STRATEGY SYMMETRIC EQUILIBRIA

The result of the previous section can be improved upon when the game in question possesses enough symmetry. Throughout this section, we shall assume that for all players i, j , $X_i = X_j$. So, in contrast to the previous section, we here let $X = X_1 = \dots = X_N$. If in addition, $u_1(x, y, \dots, y) = u_2(y, x, y, \dots, y) = \dots = u_N(y, \dots, y, x)$ for all $x, y \in X$, we say that $G = (X_i, u_i)_{i=1}^N$ is a *quasi-symmetric* game. Note that quasi-symmetry is weaker than symmetry for games with three or more players.

We shall also maintain the following convention throughout this section. For each player i , and for all $x, y \in X$, $u_i(y, \dots, x, \dots, y)$ denotes the function u_i evaluated at the strategy in which player i chooses x and all others choose y .

Define a quasi-symmetric game's *diagonal payoff function* $v: X \rightarrow \mathbf{R}$ by $v(x) = u_1(x, \dots, x) = \dots = u_N(x, \dots, x)$ for every $x \in X$.

Recall that a Nash equilibrium, (x_1^*, \dots, x_N^*) , is symmetric if $x_1^* = \dots = x_N^*$.

DEFINITION: Player i can *secure a payoff* of $\alpha \in \mathbf{R}$ *along the diagonal* at $(x, \dots, x) \in X^N$, if there exists $\bar{x} \in X$, such that $u_i(x', \dots, \bar{x}, \dots, x') \geq \alpha$ for all $x' \in X$ in some open neighborhood of $x \in X$.

DEFINITION: The game $G = (X_i, u_i)_{i=1}^N$ is *diagonally better-reply secure* if whenever $(x^*, u^*) \in X \times \mathbf{R}$ is in the closure of the graph of its diagonal payoff function and (x^*, \dots, x^*) is not an equilibrium, some player i can secure a payoff strictly above u^* along the diagonal at (x^*, \dots, x^*) .

DEFINITION: The game $G = (X_i, u_i)_{i=1}^N$ is *diagonally quasiconcave* if X is convex, and for every player i , all $x^1, \dots, x^m \in X$ and all $\bar{x} \in \text{co}\{x^1, \dots, x^m\}$

$$u_i(\bar{x}, \dots, \bar{x}) \geq \min_{1 \leq k \leq m} u_i(\bar{x}, \dots, x^k, \dots, \bar{x}).$$

¹⁸ Proportional reductions ensure that bundles remain nonnegative.

Letting $w_i(x, y) = u_i(y, \dots, x, \dots, y)$ for each player i , and every $x, y \in X$, G is diagonally quasiconcave if and only if each w_i is diagonally quasiconcave in the sense of Zhou and Chen (1988). See also Baye et al. (1993). It is worthwhile to note that diagonal quasiconcavity is strictly weaker than quasiconcavity, even for two-player games. Our main result for quasi-symmetric games is as follows.

THEOREM 4.1: *If $G = (X_i, u_i)_{i=1}^N$ is quasi-symmetric, compact, diagonally quasiconcave, and diagonally better-reply secure, then it possesses a symmetric pure strategy Nash equilibrium.*

As with better-reply security, diagonal better-reply security can sometimes be verified by checking simpler, yet more restrictive, conditions.

DEFINITION: The game $G = (X_i, u_i)_{i=1}^N$ is *diagonally payoff secure* if for every $x \in X$ and every $\epsilon > 0$, each player i can secure a payoff of $u_i(x, \dots, x) - \epsilon$ along the diagonal at (x, \dots, x) .

For games with three or more players, diagonal payoff security is strictly weaker than payoff security. For example, Hotelling's location game is not payoff secure when X is taken to be a firm's set of mixed strategies there, although it is diagonally payoff secure. The proof of the following proposition is omitted as it follows the same lines as the proof of Proposition 3.2, which can be found in the Appendix.

PROPOSITION 4.2: *If the quasi-symmetric game $G = (X_i, u_i)_{i=1}^N$ is diagonally payoff secure and each $u_i(x, \dots, x)$ is upper semicontinuous as a function of x on X , then G is diagonally better-reply secure.*

Since quasi-symmetry implies that $u_1(x, \dots, x) = \dots = u_N(x, \dots, x)$ for all $x \in X$, requiring $u_i(x, \dots, x)$ to be u.s.c. in x on X in a quasi-symmetric game is weaker than requiring the vector of the player's payoffs to be reciprocally u.s.c. on X^N . Consequently, for quasi-symmetric games, the hypotheses of Proposition 4.2 are strictly weaker than those of Proposition 3.2.

An immediate consequence of Theorem 4.1 and Proposition 4.2 is the following result, which tends to be sufficient for most applications.

COROLLARY 4.3: *If $G = (X_i, u_i)_{i=1}^N$ is quasi-symmetric, compact, diagonally quasiconcave, diagonally payoff secure, and each $u_i(x, \dots, x)$ is upper semicontinuous as a function of x on X , then it possesses a symmetric pure strategy Nash equilibrium.*

We now provide the proof of Theorem 4.1.

PROOF OF THEOREM 4.1: For every $x, y \in X$, let

$$\underline{u}_1(x, y) = \sup_{U \ni y} \inf_{y' \in U} u_1(x, y', \dots, y'),$$

where the sup is taken over all open neighborhoods $U \subseteq X$ of y . So defined, $\underline{u}_1(x, y)$ is, for every $x \in X$, real-valued (since u_1 is bounded) and l.s.c. as a function of y on X . Lower semicontinuity follows as in the proof of Theorem 3.1.

Consequently, player 1 can secure a payoff strictly above $\alpha \in \mathbf{R}$ along the diagonal at $(y, \dots, y) \in X^N$ if and only if

$$(4.1) \quad \sup_{x \in X} \underline{u}_1(x, x) > \alpha.$$

Let Γ denote the closure of the graph of the game's diagonal payoff function. Because G is quasi-symmetric and diagonally better-reply secure, (4.1) implies that $(x^*, \dots, x^*) \in X^N$ is a symmetric Nash equilibrium of G if for some $u^* \in \mathbf{R}, (x^*, u^*) \in \Gamma$ and

$$(4.2) \quad \underline{u}_1(x, x^*) \leq u^*, \quad \forall x \in X.$$

For each $x \in X$, let

$$E(x) = \{y \in X: \underline{u}_1(x, y) \leq \bar{u}_1(y)\},$$

where $\bar{u}_1(y) = \inf_{U \ni y} \sup_{y' \in U} u_1(y', \dots, y')$, and where the infimum is taken over all open neighborhoods of y . Because payoffs are bounded, $\bar{u}_1(\cdot)$ is a well-defined real-valued function on X . Moreover, it is upper semicontinuous on X (the proof being analogous to that demonstrating the lower semicontinuity of \underline{u}_1).

Now, suppose that $x^* \in E(x)$ for every $x \in X$. Then, $\underline{u}_1(x, x^*) \leq \bar{u}_1(x^*)$ for every $x \in X$. However, because $(x^*, \bar{u}_1(x^*)) \in \Gamma$, this implies that (4.2) is satisfied with $u^* = \bar{u}_1(x^*)$, so that (x^*, \dots, x^*) is a symmetric Nash equilibrium. Thus, it suffices to show that $\bigcap_{x \in X} E(x)$ is nonempty.

Because $\bar{u}_1(\cdot)$ is upper semicontinuous on X , and for every $x \in X, \underline{u}_1(x, y)$ is l.s.c. in $y, E(x)$ is a closed (hence compact) subset of the compact set X ; for each $x \in X$. Therefore, it suffices to show that the collection of compact sets $\{E(x)\}_{x \in X}$ possesses the finite intersection property, which by the KKM Theorem, would be the case if for every finite number of points $x^1, \dots, x^m \in X, \text{co}\{x^1, \dots, x^m\} \subset E(x^1) \cup \dots \cup E(x^m)$.

So, suppose by way of contradiction that $\bar{x} \in \text{co}\{x^1, \dots, x^m\}$, and $\bar{x} \notin E(x^1) \cup \dots \cup E(x^m)$. This means that $u_1(x^k, \bar{x}, \dots, \bar{x}) \geq \underline{u}_1(x^k, \bar{x}) > \bar{u}_1(\bar{x}) \geq u_1(\bar{x}, \dots, \bar{x})$ for each $k = 1, \dots, m$, where the first and third inequalities follow from the definitions of \underline{u}_1 and \bar{u}_1 , respectively. But this contradicts the diagonal quasiconcavity of G . Q.E.D.

The following class of examples illustrates Theorem 4.1 through Corollary 4.3. In particular, it should be noted that payoffs are not required to be quasiconcave in the owner's strategy or jointly continuous in all strategies. Particular games among the class include Bertrand duopoly with continuous, but otherwise unrestricted, demand; as well as models of brand loyalty (see, e.g., Baye et al. (1993)).

EXAMPLE 4.1: Consider a compact, quasi-symmetric, diagonally payoff secure game $G = (X_i, u_i)_{i=1}^N$ with $X_i = [0, 1]$, and such that $u_i(x, \dots, x)$ is u.s.c. in x on $[0, 1]$. If for each $\bar{x} \in [0, 1]$, $u_i(\bar{x}, \dots, x, \dots, \bar{x})$ is either:

- (i) nondecreasing in x on $[0, \bar{x}]$, or
- (ii) nonincreasing in x on $[\bar{x}, 1]$,

then G possesses a symmetric pure strategy Nash equilibrium.¹⁹

This follows directly from Corollary 4.3 since conditions (i) and (ii) imply that G is diagonally quasiconcave.

5. MIXED STRATEGY EQUILIBRIA

In this section, we present a number of mixed strategy corollaries to the pure strategy existence results derived in the previous sections. Out of the need to calculate expected payoffs, we shall assume throughout this section that each u_i is both bounded and measurable. In addition, we shall assume that each X_i is a compact Hausdorff space, and we'll then call $G = (X_i, u_i)_{i=1}^N$ a compact, Hausdorff game. Consequently, if M_i denotes the set of (regular, countably additive) probability measures on the Borel subsets of X_i , M_i is compact in the weak* topology.²⁰ Extend each u_i to $M = \times_{i=1}^N M_i$ by defining $u_i(\mu) = \int_X u_i(x) d\mu$ for all $\mu \in M$, and let $\bar{G} = (M_i, u_i)_{i=1}^N$ denote the *mixed extension* of G .

The definitions of better-reply security, reciprocal upper semicontinuity, payoff security, etc. given in Sections 2–4 apply in the obvious ways to the mixed extension \bar{G} .²¹ However, it should be noted that although reciprocal upper semicontinuity of \bar{G} implies that of G , better-reply security (resp., payoff security) of \bar{G} neither implies nor is implied by better-reply security (resp., payoff security) of G .²²

¹⁹Note that the conditions allow $u_i(\bar{x}, \dots, x, \dots, \bar{x})$ to be nondecreasing in x on $[0, \bar{x}]$ for some values of \bar{x} , and nonincreasing in x on $[\bar{x}, 1]$ for other values of \bar{x} .

²⁰This follows from the Riesz representation theorem and Alaoglu's theorem. See, for example, Dunford and Schwartz (1988).

²¹Simply replace X_i by M_i everywhere in each definition.

²²When one moves to mixed strategies, securing any particular payoff becomes both easier and more difficult. It becomes easier because one can now employ mixed strategies to attempt to secure a payoff and this can increase the payoff one can secure (e.g., as in matching pennies settings). On the other hand, one's payoff must be secure against perturbations from the (larger) set of mixed strategies of the others, which can reduce the payoff one can secure.

The following result often provides a convenient means for checking reciprocal upper semicontinuity of \bar{G} by reducing the task to checking that the sum of the players' payoffs is u.s.c. on the set of pure strategies, $X = \times_{i=1}^N X_i$. A proof can be found in the Appendix.²³

PROPOSITION 5.1: *If $\sum_{i=1}^N u_i(x)$ is upper semicontinuous in x on X , then $\sum_{i=1}^N \int_X u_i(x) d\mu$ is upper semicontinuous in μ on M . Consequently, the mixed extension of $G = (X_i, u_i)_{i=1}^N$ is reciprocally upper semicontinuous.*

We now present the mixed strategy implications of Theorem 3.1 and Proposition 3.2.

COROLLARY 5.2 (TO THEOREM 3.1): *Suppose that $G = (X_i, u_i)_{i=1}^N$ is a compact, Hausdorff game. Then G possesses a mixed strategy Nash equilibrium if its mixed extension, \bar{G} , is better-reply secure. Moreover, \bar{G} is better-reply secure if it is both reciprocally upper semicontinuous and payoff secure.*

It is the above corollary that directly generalizes the mixed strategy equilibrium existence results of Nash (1950), Glicksberg (1952), Mas-Colell (1984), Dasgupta and Maskin (1986), Robson (1994), and Simon (1987). See subsections 6.1 and 6.2 below.

Corollary 5.2 can be applied to prove the existence of mixed strategy equilibria in many standard economic games: Bertrand price competition with arbitrary continuous cost and demand functions, Cournot quantity competition with fixed costs of production, models of adverse selection, etc. The example to follow illustrates a simple application.

EXAMPLE 5.1: Consider a zero-sum concession game between two players (see Hendricks and Wilson (1983)). The players must choose a time $t_1, t_2 \in [0, 1]$ to quit the game. The player who quits last wins, although conditional on winning, quitting earlier is preferred. If both players quit at the same time, the unit prize is divided evenly between them. Then payoffs are

$$u_i(t_i, t_{-i}) = \begin{cases} -t_i, & \text{if } t_i < t_{-i}, \\ \frac{1}{2} - t_i & \text{if } t_i = t_{-i}, \\ 1 - t_i & \text{if } t_i > t_{-i}. \end{cases}$$

This game does not possess a pure strategy equilibrium. Indeed, the pay not quasiconcave (nor are they quasiconcave along the diagonal of square). However, this game is compact, Hausdorff (indeed, metric), and zero-sum (and so its mixed extension is clearly reciprocally u.s.c.). Moreover, it is easy

²³This result is well known when X is a compact metric space (see, for example, Billingsley (1968)), but we have been unable to find a reference when X is merely a compact, Hausdorff topological space.

to see that the mixed extension is payoff secure: increasing slightly the time you choose to quit at worst only slightly reduces your payoff so long as the other player's mixed strategy does not change too much. Thus, Corollary 5.2 applies, ensuring that this game possesses a mixed strategy Nash equilibrium. It is perhaps notable that Dasgupta and Maskin's (1986) weak lower semicontinuity requirement fails in this game at the point (1,1), so that their main existence theorem cannot be applied here.²⁴

We now apply Corollary 5.2 on the existence of *mixed* strategy Nash equilibrium to show that the multi-unit pay-your-bid auction possesses a *pure* strategy equilibrium in which the bidders employ nondecreasing bidding functions.

EXAMPLE 5.2: There are N risk-neutral bidders competing for K units of a homogeneous good. Bidder i receives a multi-dimensional signal $x_i \in [0, 1]^m$ according to the distribution function F_i , having continuous and positive density, f_i on $[0, 1]^m$. The bidders' signals are independent. The signal x_i determines bidder i 's marginal valuation, $v_k^i(x_i)$ for the k th unit of the good. Each $v_k^i(\cdot)$ is assumed to be continuous and strictly increasing on $[0, 1]^m$, with $v_1^i(\cdot) \geq v_2^i(\cdot) \geq \dots \geq v_K^i(\cdot)$, and $v_1^i(0) = \dots = v_K^i(0) = 0$.²⁵ Knowing only their own signals, each bidder i submits K nonnegative bids $b_1^i \geq b_2^i \geq \dots \geq b_K^i$. The highest K bids among the KN bids submitted are winning bids, and winning bidders pay the seller their winning bid for each unit won. Relevant ties are broken equi-probably.²⁶ We shall establish the following.

The asymmetric multi-unit pay-your-bid auction above possesses a pure strategy equilibrium in which the bidders employ nondecreasing bidding functions.

We first wish to show that the auction possesses a mixed strategy Nash equilibrium in which bidders mix over nondecreasing bidding functions (although they can choose arbitrary measurable bidding functions). To begin, restrict each bidder's pure strategies to the set of ordered K -tuples of nondecreasing bidding functions (with the first majorizing the second, ..., majorizing the K th) each from $[0, 1]^m$ into $[0, \bar{v}]$, where \bar{v} is an upper bound on each bidder's marginal valuations, such that no bidder bids above his value on any unit. Consequently, the pure strategy sets are compact metric spaces in the topology of (almost everywhere) pointwise convergence.²⁷

²⁴This has been pointed out by Simon (1987).

²⁵By strictly increasing, we mean that each marginal valuation is nondecreasing and strictly increases when *all* components of the signal strictly increase. We allow the possibility that marginal valuations remain constant when only some, but not all, signals strictly increase.

²⁶A tie is *relevant* if it must be broken in order to determine the allocation. In single-unit auctions all high-bid ties are relevant, but this is not always so in multi-unit auctions.

²⁷As in L_p spaces, two nondecreasing bidding functions here are deemed identical if they are equal almost everywhere with respect to Lebesgue measure.

We now argue that this game's mixed extension is better-reply secure. So, suppose that m^* is not an equilibrium and let (m^*, u^*) be an element of the closure of the graph of the mixed extension's vector (ex ante) payoff function. By definition, $\lim u(m^n) = u^*$ for some sequence of mixed strategies, $\{m^n\}$, converging to m^* . Given m_{-i}^* and $\varepsilon > 0$, bidder i can achieve a payoff within ε of his supremum (when restricted to nondecreasing bid functions) by employing a K -tuple of strictly increasing bid functions, b_i^ε , say. Moreover, his payoff is continuous in the others' mixed strategies at $(b_i^\varepsilon, m_{-i}^*)$. Now, if relevant ties occur with probability zero given m^* , then payoffs are continuous at the limit, i.e., $u^* = u(m^*)$. So, because m^* is not an equilibrium there is a bidder i and $\varepsilon > 0$ small enough such that $u_i(b_i^\varepsilon, m_{-i}^*) > u_i(m^*) = u_i^*$. Hence, the continuity of $u_i(b_i^\varepsilon, \cdot)$ at m_{-i}^* implies that bidder i can secure a payoff strictly above u_i^* at m^* . On the other hand, if relevant ties occur with positive probability given m^* , then all those bidders tying with positive probability would strictly prefer to win with probability one (recall that bids are never above one's value on any unit). Moreover, along the sequence, m^n , one of those bidders who ties at the limit loses with probability bounded away from zero. This bidder i can then achieve a strictly higher payoff than the limiting payoff of $u_i^* = \lim u_i(m^n)$ by employing b_i^ε for $\varepsilon > 0$ small enough. That is, $u_i(b_i^\varepsilon, m_{-i}^*)$ is above and bounded away from $u_i(m^n)$ for large n . So once again, the continuity of $u_i(b_i^\varepsilon, \cdot)$ at m_{-i}^* implies that $u_i(b_i^\varepsilon, m_{-i}^*) > u_i^*$ and that bidder i can secure a payoff strictly above u_i^* at m^* . Consequently, better-reply security holds. We may therefore apply Corollary 5.2 to conclude that the game possesses a mixed strategy equilibrium.

So, up to this point, we have demonstrated that the pay-your-bid auction possesses a mixed strategy equilibrium when *restricting attention to pure strategies that are nondecreasing bidding functions*. Assume now that all bidders' marginal valuation functions are strongly increasing. That is, they are nondecreasing, and strictly increase whenever at least one signal strictly increases and no signal decreases. As we show in the Appendix every mixed strategy equilibrium as above is then a full equilibrium in the sense that there are no profitable deviations among measurable bid functions. Moreover, it is shown there that all mixed equilibria are in fact pure. That is, in equilibrium each bidder's mixed strategy places probability one on a single nondecreasing bid function. Finally, it is shown in the Appendix that as regards existence, the restriction to strongly increasing marginal valuation functions is unnecessary. This completes the argument.

Note that because the multi-unit pay-your-bid auction fails to be reciprocally u.s.c. (see footnote 13), the results of Dasgupta and Maskin (1986) and Simon (1987) cannot be directly applied.

The model of the multi-unit pay-your-bid auction presented here generalizes that of Engelbrecht-Wiggans and Khan (1995) in a number of ways. For example, their formulation does not allow the distribution of a single bidder's K marginal values to lie along a lower-dimensional surface. Thus, they rule out cases in which the number of objects exceeds the dimensionality of the signal

space. In addition, they assume that the bidders are symmetric in terms of the distribution of their values. Finally, they assume, but do not prove, the existence of a symmetric nondecreasing equilibrium. We examine the symmetric case below in Example 5.3.

Since the analysis above allows $K = 1$, it applies to single-unit auctions. This then furnishes a proof of the existence of a pure strategy equilibrium in nondecreasing bid functions for asymmetric first-price auctions with risk neutral bidders and multi-dimensional signals in the independent private values case.

We now provide the mixed strategy consequences of Theorem 4.1 and Proposition 4.2 for quasi-symmetric games. For the following result only, let M denote the common set of mixed strategies for each player i .

COROLLARY 5.3 (TO THEOREM 4.1): *Suppose that $G = (X_i, u_i)_{i=1}^N$ is a quasi-symmetric, compact, Hausdorff game. Then G possesses a symmetric mixed strategy Nash equilibrium if its mixed extension, \bar{G} , is better-reply secure along the diagonal. Moreover, \bar{G} is better-reply secure along the diagonal if it is diagonally payoff secure and each $u_i(\mu, \dots, \mu)$ is upper semicontinuous as a function of μ on M .*

EXAMPLE 5.3: Engelbrecht-Wiggans and Khan (1995) provide a number of results on the character of symmetric equilibria in nondecreasing bidding functions in the multi-unit pay-your-bid auction when the bidders are symmetric. However, they do not establish conditions under which such an equilibrium exists. We do so here. Consider the model of this auction introduced above. In addition, assume that each bidder has the same K marginal valuation functions and the same signal distribution function. An analysis similar to that above, but now employing Corollary 5.3, establishes that this symmetric K -unit pay-your-bid auction possesses a symmetric pure strategy equilibrium in which the bidders employ a common vector of nondecreasing bid functions.

Corollary 5.3 can also be applied to provide a rather permissive symmetric mixed strategy equilibrium existence result for Hotelling's location game. However, we shall not present this result here.²⁸

PROOFS OF COROLLARIES 5.2 AND 5.3: Both corollaries follow by simply verifying that the conditions of the associated theorem and proposition from Sections 3 and 4 are satisfied for the game's mixed extension and by noting that a pure strategy equilibrium of the mixed extension constitutes a mixed strategy equilibrium of the original game. *Q.E.D.*

²⁸ Dasgupta and Maskin (1986) and Simon (1987) provide conditions under which a mixed strategy equilibrium exists in Hotelling's location game. In particular, they require that the set of locations be convex, or have smooth boundaries. By applying Corollary 5.3, both of these conditions can be dispensed with and a symmetric mixed strategy equilibrium is nonetheless guaranteed to exist.

6. RELATED WORK ON MIXED STRATEGY EQUILIBRIA

We now indicate how Corollary 5.2 yields the results of Glicksberg (1952) (and hence of Nash (1950)), Dasgupta and Maskin (1986), Mertens (1986), Robson (1994), and Simon (1987).²⁹

6.1. *Glicksberg (1952), Mertens (1986), and Robson (1994)*

Glicksberg (1952) shows that compact, Hausdorff games with continuous payoff functions possess mixed strategy Nash equilibria. Robson (1994), in the course of studying the informational robustness of equilibria, proves that in a compact game with metric strategy spaces, if each player's payoff is u.s.c. in all players' strategies, and continuous in the other players' strategies, then the game possesses a mixed strategy Nash equilibrium. Mertens (1986) shows that two-person zero-sum compact games possess a value whenever player I's payoff is u.s.c. in his own strategy and l.s.c. in player II's strategy.³⁰

All of these results follow directly from Corollary 5.2. To see this note that by Proposition 5.1, lower-semicontinuity of one's payoff in the opponents' pure strategies implies lower-semicontinuity in the opponents' mixed strategies. But lower-semicontinuity in the opponents' mixed strategies for each of a player's pure strategies implies payoff security on the set of mixed strategies. Consequently, if every player's payoff is lower-semicontinuous in the others' pure strategies, as in each case above, then the game's mixed extension is payoff secure. Since in each case above the sum of payoffs is upper semicontinuous, the mixed extension is (by Proposition 5.1) reciprocally u.s.c. Hence, in each case Corollary 5.2 applies.

6.2. *Dasgupta and Maskin (1986) and Simon (1987)*

Dasgupta and Maskin (1986) impose the following conditions in order to guarantee the existence of a mixed strategy Nash equilibrium: (a) $G = (X_i, u_i)_{i=1}^N$ is a compact game whose sum of payoffs is u.s.c. on X ; (b) each X_i is a convex subset of \mathbf{R}^m ; (c) the discontinuities of each u_i lie along a finite number of "diagonal" sets; (d) each u_i is "weakly" lower-semicontinuous in i 's strategy choice.

Simon (1987) strictly generalizes the result of Dasgupta and Maskin (1986) by making the following assumptions: (i) G is a compact game with metric strategy spaces and a reciprocally u.s.c. mixed extension; (ii) there exists a sequence

²⁹The result of Mas-Colell (1984) is a variant of Dasgupta and Maskin's (1986), and it too can be shown to follow from Corollary 5.2

³⁰Actually, this u.s.c.-l.s.c. mixed strategy Nash equilibrium existence result for two-person zero-sum games follows from Sion's (1958) theorem. Mertens' (1986) main interest is in showing that in addition, both players have ϵ -optimal strategies with finite support. Our results *do not* generalize this latter result of Mertens'.

$\{\times_i X_i^n\}_{n \geq 1}$ of finite subsets of the original pure strategy sets satisfying, where M_i^n denotes i 's mixed strategies on X_i^n :

For each player i , for every $(\bar{x}_i, \bar{\mu}_{-i}) \in X_i \times M_{-i}$, there exists a sequence of pairs (μ_i^n, Y^n) , with $\mu_i^n \in M_i^n$, and $Y^n \subseteq X_{-i}$, $n = 1, 2, \dots$, such that for all n : $\bar{\mu}_{-i}(Y^n) > 1 - 1/n$, and

$$\sum_{x_i \in X_i^n} \mu_i^n(x_i) \left[\liminf_{x'_{-i} \rightarrow x_{-i}} u_i(x_i, x'_{-i}) \right] > u_i(\bar{x}_i, x_{-i}) - 1/n,$$

for all $x_{-i} \in Y^n$.

Simon's (1987), and hence also Dasgupta and Maskin's (1986), conditions imply the following. Since by (i) u_i is bounded, for every $\epsilon > 0$, there is an n large enough such that³¹

$$\begin{aligned} & \liminf_{\mu'_{-i} \rightarrow \bar{\mu}_{-i}} u_i(\mu_i^n, \mu'_{-i}) \\ & \geq \int_{X_{-i}} \left[\liminf_{x'_{-i} \rightarrow x_{-i}} u_i(\mu_i^n, x'_{-i}) \right] d\bar{\mu}_{-i}(x_{-i}) \\ & \geq \int_{X_{-i}} \left\{ \sum_{x_i \in X_i^n} \mu_i^n(x_i) \left[\liminf_{x'_{-i} \rightarrow x_{-i}} u_i(x_i, x'_{-i}) \right] \right\} d\bar{\mu}_{-i}(x_{-i}) \\ & > u_i(\bar{x}_i, \bar{\mu}_{-i}) - \epsilon. \end{aligned}$$

Thus, the game's mixed extension is payoff secure. Hence, Simon's (1987) conditions (i) and (ii), and so also Dasgupta and Maskin's (1986) conditions (a)–(d), imply the hypotheses of Corollary 5.2.

The failure of reciprocal u.s.c. in the better-reply secure auction game of Example 5.2 shows that Corollary 5.2 is a strict generalization of all the results discussed above.

7. OTHER APPROACHES

Like Sion (1958), Mas-Colell (1984), Dasgupta and Maskin (1986), Mertens (1986), Simon (1987), Baye et al. (1993), and Robson (1994), we have followed here what might be called the topological approach to existence of equilibrium. An alternative approach is based upon lattice-theoretical concepts, and at its heart lies Tarski's (1955) fixed point theorem. Perhaps the best examples of this method are due to Topkis (1979), Vives (1990), and Milgrom and Roberts (1990) although its flavor can also be found in Roberts and Sonnenschein (1976), and Nishimura and Friedman (1981). In each of these papers, payoffs need not be quasiconcave, and in some needn't be continuous. The key property is that

³¹The first inequality follows since the function of x_{-i} in square brackets is l.s.c. and less than or equal to $u_i(\mu_i^n, x_{-i})$ for every x_{-i} ; and the third by (ii) together with the facts that n is large enough and u_i is bounded (by (i)).

best-replies are increasing in the opponents' strategies. This method typically yields the existence of pure strategy equilibria. Now, while lattice-theoretic methods do not require payoffs to be continuous, enough continuity must be assumed in order to guarantee the existence of best replies. So, typically payoffs are required to be u.s.c. in one's own strategy, an assumption that fails to hold in virtually all auctions, as well as in the classic games of Bertrand and Hotelling. Consequently, most practical applications of lattice-theoretical techniques tend to be confined to continuous games.

A third approach, although still fundamentally topological in nature, has been introduced by Simon and Zame (1990). They show that if one is willing to modify the (vector of) payoffs at points of discontinuity so that they correspond to points in the convex hull of limits of nearby payoffs, then one can ensure a mixed strategy equilibrium of such a suitably modified game. As an example of the usefulness of their approach, Simon and Zame note that the discontinuities in Hotelling's (1929) location model arise only when two firms choose the same location, and in this case, consumers are indifferent between which firm they patronize. Consequently, rather than insist that the two firms split the market evenly, Simon and Zame suggest that one ought to be content with any division of consumers among the two firms. Simon and Zame's result ensures that there is a "sharing rule" specifying how consumers are divided among firms when indifference arises such that the resulting game possesses a mixed strategy equilibrium. While in some settings involving discontinuities this approach is remarkably helpful (i.e., Hotelling's location game), in others it is less so. For example, in a mechanism design environment where discontinuities are sometimes deliberately introduced (auction design, for example), the participants must be presented with a game that fully describes the strategies and payoffs. One cannot leave some of the payoffs unspecified, to somehow be endogenously determined. In addition, this method is only useful in establishing the existence of a mixed, as opposed to pure, strategy equilibrium.

8. APPROXIMATION BY FINITE GAMES

For those who are more comfortable with the finite than with the infinite, a legitimate concern might be whether some infinite game under consideration is well approximated by a finite game. To be sure, the ability to approximate the infinite-action game by a finite-action game lends a degree of robustness to the analysis of the infinite-action game: the presence of literally infinitely many actions and the discontinuities that may accompany them are not essential. While space does not permit a full discussion of these issues we briefly mention some relevant approximation results that follow from Reny (1996).

Call one game a finite approximation of another if the strategy spaces of the one are finite subsets of the other, and the payoff functions of the one are the other's payoff functions restricted to the finite sets. Suppose that a compact, metric game's mixed extension is better-reply secure and that for every fixed strategy of the opponents, each player's payoff is discontinuous at the resulting

joint strategy for at most countably many of his own strategies.³² Then there is a sequence of finite approximations with the property that the limit of any sequence of mixed strategy equilibria of the approximating games is an equilibrium of the original game. Suppose, in addition, that every player's value function is continuous in the mixed strategies of the others.³³ Then, every equilibrium of the original game is the limit of ϵ -equilibria of a sequence of finite approximations in which ϵ tends to zero along the sequence.³⁴

In short, a reasonably large class of infinite-action games that are better-reply secure are well approximated by and robust to finite-game approximations. Consequently, the analytic convenience of the infinite-action setting is not in conflict with a finite-action viewpoint.

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APPENDIX

PROOF OF PROPOSITION 3.2: Suppose that (x^*, u^*) is in the closure of the graph of the game's vector payoff function, and that x^* is not an equilibrium. By reciprocal u.s.c, either $u_i(x^*) > u_i^*$ for some i , or $u_i(x^*) = u_i^*$ for all i . In the latter case, because x^* is not an equilibrium, some player i has a deviation, \hat{x}_i , such that $u_i(\hat{x}_i, x_{-i}^*) > u_i(x^*) = u_i^*$. Consequently, in either case there is a player i and a strategy \hat{x}_i (equal to x_i^* in the former case) such that $u_i(\hat{x}_i, x_{-i}^*) > u_i^*$. Fix now this player i .

Choose $\epsilon > 0$ so that $u_i(\hat{x}_i, x_{-i}^*) > u_i^* + \epsilon$. Because the game is payoff secure player i has a strategy \bar{x}_i such that $u_i(\bar{x}_i, x_{-i}^*) \geq u_i(\hat{x}_i, x_{-i}^*) - \epsilon > u_i^*$ for all x_{-i}^* in some open neighborhood of x_{-i}^* . Consequently, player i can secure a payoff strictly above u_i^* at x^* . Thus, the game is better-reply secure. *Q.E.D.*

PROOF OF LEMMA 3.5: Let $F = \{g \in C(Y) : g(y) \leq f(y) \forall y \in Y\}$, where, because Y is metric, $C(Y)$ denotes the metric space of bounded, continuous functions on Y endowed with the maximum metric. Since F is closed and $C(Y)$ is separable, F is separable. Let $\{g_1, g_2, \dots\}$ denote a countable dense subset of F , and define $f_n \in C(Y)$ by $f_n(y) = \max\{g_1(y), \dots, g_n(y)\}$ for all $y \in Y$. So defined, it is clear that $\{f_n\}$ satisfies (i). It remains to show that it satisfies (ii). To do so, we need a preliminary result:

- (*) If $f(y^0) > \alpha$, then there exists $h \in C(Y)$ and a neighborhood U of y^0 such that
 - (a) $h(y) \leq f(y), \forall y \in Y$,
 - (b) $h(y) > \alpha, \forall y \in U$.

To see that (*) holds, suppose that $f(y^0) > \alpha$. Choose $\alpha' \in (\alpha, f(y^0))$ and note that because f is lower semicontinuous, the set $\{y \in Y : f(y) > \alpha'\}$ is open. Consequently, there are open balls B_1 and B_2 around y^0 with radii $r_1 > r_2 > 0$, respectively, such that $f(y) > \alpha'$ for all $y \in B_1 \supset B_2$.

³²This can be weakened significantly.

³³A player's value function is the supremum of his payoff as a function of the mixed strategy of the opponents. Most auctions have continuous value functions as do Bertrand's and Hotelling's price competition games.

³⁴I am grateful to the editor for encouraging me to obtain this result.

Define $h(y)$ to be constant and equal to α' on the closure of B_2 , and to be constant and equal to $\min_{y \in Y} f(y)$ on the complement of B_1 . By the Tietze extension theorem, h can be extended to a continuous function on all of Y so that $h(y) \leq f(y)$ for all $y \in B_1$. Consequently, setting $U = B_2$, both (a) and (b) above are satisfied, and this proves (*).

To show now that $\{f_n\}$ satisfies (ii), suppose, to the contrary, that $y_n \rightarrow y^0$ and that $f_n(y_n) \rightarrow \alpha < f(y^0)$.³⁵ Then $f(y^0) > \alpha + \epsilon$, for some $\epsilon > 0$. By (*), there exists $h \in F$ and a neighborhood U of y^0 such that $h(y) > \alpha + \epsilon$ for all $y \in U$. Since $\{g_1, g_2, \dots\}$ is dense in F , we may choose k so that g_k is within $\epsilon/2$ of h . Consequently, for all $n \geq k$ such that $y_n \in U$, we have

$$f_n(y_n) \geq g_k(y_n) \geq h(y_n) - \epsilon/2 > \alpha + \epsilon/2.$$

But this contradicts $f_n(y_n) \rightarrow \alpha$.

Q.E.D.

PROOF OF PROPOSITION 5.1: Clearly, it suffices to show that $\sum_{i=1}^N u_i(\mu)$ is upper semicontinuous on M . Thus, it suffices to show that if $f: X \rightarrow \mathbf{R}$ is upper semicontinuous on X , then $F(\mu) = \int_X f(x) d\mu$ is upper semicontinuous on M . (Recall that X is compact, Hausdorff, and that M is the set of regular, countably additive probability measures on the Borel subsets of X .)

Suppose that for every $\mu \in M$, there exists a sequence $\{f_n\}$ of elements of $C_f = \{g \in C(X): g \geq f\}$ such that

$$(A.1) \quad \int_X f(x) d\mu = \lim_n \int_X f_n(x) d\mu.$$

Then,

$$\int_X f(x) d\mu = \inf_{g \in C_f} \int_X g(x) d\mu, \quad \text{for every } \mu \in M,$$

implying that $F(\mu)$ is u.s.c. in μ on M , being the infimum of a collection of continuous functions. Thus, it suffices to establish that for each $\mu \in M$, there exists a sequence of continuous functions $\{f_n\}$ majorizing f and satisfying (A.1). Since we have not found precisely this result in the literature, we provide a proof.

Fix $\mu \in M$. Since f is u.s.c., it is measurable. Hence, because μ is regular, Lusin's theorem (see, e.g., Cohn (1980, Theorem 7.4.3)) gives for every $n \geq 1$, a compact subset X_n of X such that $\mu(X_n) > 1 - (1/n)$ and f is continuous on X_n . Since f is u.s.c., we may assume without loss that $f(x) < 0$ for all $x \in X$. For each $n \geq 1$, let $g_n(x) = f(x) + (1/n)$ if $x \in \bigcup_{k=1}^n X_k$, and $g_n(x) = 1$ otherwise. Consequently, $g_n \geq g_{n+1} > f$ for all n , and $g_n(x) \rightarrow f(x)$ for all $x \in \bigcup_{k=1}^\infty X_k$ and so for μ a.e. $x \in X$. Hence,

$$(A.2) \quad \lim_n \int_X g_n(x) d\mu = \int_X f(x) d\mu,$$

by the monotone convergence theorem.

Since each g_n is lower semicontinuous, while f is upper semicontinuous and $g_n > f$, Dowker's theorem (Theorem VIII.4.3 of Dugundji (1989)) implies that for each $n \geq 1$, there exists $f_n \in C(X)$ such that $g_n > f_n > f$. This, together with (A.2) establishes (A.1) and the result. *Q.E.D.*

The Pay-Your-Bid Multi-Unit Auction

Below, we establish the claims made in Section 5 of the main text concerning this auction. All claims but the last employ the assumption that all bidders' marginal valuation functions are strongly increasing, i.e., they are nondecreasing, and they strictly increase whenever at least one signal strictly increases and no signal decreases. Also, all claims but the last concern the game in which pure strategies are constrained to be nondecreasing bid functions in which no bidder bids above his value on any unit.

³⁵Note that α is finite since for all n , $f_n(y_n) \geq g_1(y_n)$, and $g_1 \in C(Y)$ is bounded on Y .

CLAIM 1: *In a mixed strategy equilibrium, no bidder places ex ante positive probability on any single bid.*

PROOF OF CLAIM 1: First, note that whenever at least one of his values is positive, a bidder must earn a positive payoff in equilibrium, since he can always do so by bidding half his value on each unit. This wins with positive probability since all others might have values near zero and would then bid below him on these units. Note that this also implies that a zero bid is submitted on a unit if and only if the unit is valued at zero, and that units with positive value are won with positive probability.

Suppose, by way of contradiction, that there is a largest $k \geq 1$ such that bidder one places positive probability (ex ante) on some k th-unit bid, \bar{b} . Since values are zero with probability zero, this implies that $\bar{b} > 0$. Now, fix $\epsilon \in (0, \bar{b})$ and suppose that some other bidder places competing bids between $\bar{b} - \epsilon$ and \bar{b} with positive probability.³⁶ If ϵ were small enough, this could not be a best reply. The reason is that bids near \bar{b} can only come from units valued above some $\bar{v} > \bar{b}$ since every positive valuation must earn a positive surplus bounded away from zero. Consequently, the other bidder can strictly increase the probability of winning an additional unit with value at least \bar{v} by raising his bid just above \bar{b} . When ϵ is small enough, the discrete jump in the probability of winning more than compensates for the (arbitrarily) small increase in the bid. We conclude that there exists $\epsilon > 0$ such that other bidders place competing bids outside the interval $(\bar{b} - \epsilon, \bar{b})$ with probability one. But this means that on a set of signals having positive probability, bidder one can reduce his k th-unit bid slightly below \bar{b} without reducing his probability of winning. (This is feasible since, by definition of k , bidder one's bid on the $k + 1$ st unit is without loss strictly below \bar{b} on this positive probability set.) Since his probability of winning is positive (recall that positive bids win with positive probability) this strictly improves his payoff, contradicting the equilibrium hypothesis and completing the proof.

CLAIM 2: *If in some mixed strategy equilibrium, the pure strategy $b^*(x) = (b_1^*(x), \dots, b_k^*(x))$ is a best reply for bidder 1 for every signal $x \in [0, 1]^m$, then each $b_k^*(\cdot)$ must be nondecreasing.*

PROOF OF CLAIM 2: Let x^0 and $x^1 \in [0, 1]^m$ be two distinct signals of bidder 1 with $x^1 \geq x^0$. Let $b^0 = b^*(x^0)$ and $b^1 = b^*(x^1)$. We wish to show that $b^1 \geq b^0$.

Let us proceed by induction on the components of the vectors of bids. Suppose we have already shown that $b_1^1 \geq b_1^0, \dots, b_k^1 \geq b_k^0$. (When $k = 0$ this statement is empty.) To complete the induction we need only show that $b_{k+1}^1 \geq b_{k+1}^0$. So, assume, by way of contradiction, that $b_{k+1}^1 < b_{k+1}^0$.

Let $j' \geq 1$ be the largest j such that $b_{k+1}^1 < b_{k+1}^0, \dots, b_{k+j}^1 < b_{k+j}^0$. Let

$$\tilde{b}^0 = (b_1^0, \dots, b_k^0, b_{k+1}^1, \dots, b_{k+j'}^1, b_{k+j'+1}^0, \dots, b_k^0),$$

and let

$$\tilde{b}^1 = (b_1^1, \dots, b_k^1, b_{k+1}^0, \dots, b_{k+j'}^0, b_{k+j'+1}^1, \dots, b_k^1).$$

We claim that both \tilde{b}^0 and \tilde{b}^1 are feasible bid vectors. That is, their components are nonincreasing. Since b^0 and b^1 are feasible, the feasibility of \tilde{b}^0 follows from the string of inequalities $b_k^0 \geq b_{k+1}^0 > b_{k+1}^1$ and $b_{k+j'}^1 \geq b_{k+j'+1}^0 \geq b_{k+j'+1}^1$, where the first string follows by assumption, and the second by the definition of j' . Similarly, the feasibility of \tilde{b}^1 follows from the strings of inequalities $b_k^1 \geq b_k^0 \geq b_{k+1}^0$ and $b_{k+j'}^0 > b_{k+j'+1}^1 \geq b_{k+j'+1}^0$, where the first string follows from the induction hypothesis, and the second from the definition of j' .

³⁶Two bidders' bids (on perhaps different units) are *competing*, if, with positive probability, the higher of the two bids determines who wins one of the objects. Thus, with two units and two bidders only one bidder's first unit bid and the other's second-unit bid are potentially competing, while with two units and three bidders all pairs of bids, except two second-unit bids, are potentially competing.

So, b^i must yield an expected payoff at least as large as \tilde{b}^i when bidder 1's signal is x^i , $i = 0, 1$. These two best reply conditions combine to yield the following inequality

$$\sum_{j=k+1}^{k+j'} [P_j(b_j^0) - P_j(b_j^1)][v_j(x^0) - v_j(x^1)] \geq 0$$

where $P_j(b)$ denotes the probability that bidder 1 wins a j th unit when his bid on that unit is b .³⁷

Now, by definition of j' , $b_j^0 > b_j^1$ for every j over which the above sum is taken. Consequently, because the probability of winning a unit is nondecreasing in one's bid for that unit, we have $P_j(b_j^0) - P_j(b_j^1) \geq 0$ for every such term appearing in the sum. Also, because v is strongly increasing, $v_j(x^0) - v_j(x^1) < 0$ for every such term appearing in the sum. Consequently, each term in the above sum is nonpositive and so each term must be zero. We conclude that $P_j(b_j^0) = P_j(b_j^1)$ for every $j = k + 1, \dots, k + j'$. Now, because $x^1 \neq 0$, bidder 1 has a strictly positive value for every unit when his signal is x^1 . Consequently, each probability $P_j(b_j^0) = P_j(b_j^1) > 0$ since, as we argued in the proof of Claim 1 above, bidders with a positive value on a unit must, in equilibrium, win that unit with positive probability. But this implies that bidder 1, when his signal is x^0 , can reduce his bids on units $j = k + 1, \dots, k + j'$ from b_j^0 to b_j^1 without decreasing the (positive) probability of winning any object, thereby strictly increasing his payoff. Thus, \tilde{b}^0 is a profitable deviation from b^0 , contradicting the best-reply definition of $b^0 = b^*(x^0)$ and completing the induction.

CLAIM 3: *An equilibrium when strategies are constrained to be nondecreasing is an equilibrium without this restriction.*

PROOF OF CLAIM 3: Given his signal and the equilibrium mixed strategies of the others, Claim 1 implies that each player's payoff is a continuous function of his vector of bids. Consequently, there is an optimal vector of bids, $(b_{i1}^*(x_i), \dots, b_{iK}^*(x_i))$ for every signal x_i . By Claim 2, each $b_{ik}^*(\cdot)$ is nondecreasing.

CLAIM 4: *Every mixed strategy equilibrium assigns probability one to a single pure strategy.*

PROOF OF CLAIM 4: To keep the notation from obscuring the point, suppose there are two units for sale, so that $K = 2$, and that the signal space is two-dimensional so that a bidder's signal takes the form $(x, y) \in [0, 1]^2$.³⁸ Consider any selection, $(b_1(\cdot), b_2(\cdot))$, from bidder 1's best reply correspondence. For any fixed $\alpha \in (0, 1]$, $b_1(x, \alpha x)$ is nondecreasing in x , and must jump whenever bidder 1 possesses multiple best first-unit bids at $(x, \alpha x)$. (Otherwise we could construct a selection from the best-reply correspondence that fails to be nondecreasing.) Since there can be but countably many such jumps, we conclude that for every $\alpha \in (0, 1]$ and all signals of the form $(x, \alpha x)$, there is a unique first-unit best reply for bidder 1 for all but countably many $x \in [0, 1]$. A similar argument applies to bidder 1's second-unit bid. Consequently, by Fubini's theorem, the mixed strategy equilibrium is essentially pure in that for every bidder and almost every signal, in equilibrium the bidder places probability one on a single pair of bids. Moreover, by selecting an arbitrary best reply at (the measure zero set of) signals where multiple best replies exist, we obtain a pure strategy equilibrium in which the bidding functions are nondecreasing.

CLAIM 5: *The assumption that the marginal valuation functions are strongly increasing can be reduced to nondecreasing.*

³⁷Because a bidder's j th unit bid must be his j th highest bid, this probability does not depend on the values of his other bids.

³⁸The argument extends naturally to the K -unit, m -dimensional signal setting.

PROOF OF CLAIM 5: We have established that when all bidders' marginal valuation functions are strongly increasing, the auction possesses a nondecreasing pure strategy equilibrium. Suppose now that each $v_k^i(\cdot)$ is merely strictly increasing. For $\delta > 0$, let $w_k^i(x) = v_k^i(x) + \delta \mathbf{1} \cdot x$ for every signal $x \in [0, 1]^m$, where $\mathbf{1}$ denotes the m -vector of 1's. Consequently, each $w_k^i(\cdot)$ is strongly increasing, and the auction with these functions as marginal valuations then possesses a nondecreasing pure strategy equilibrium. But for any fixed $\varepsilon > 0$, this nondecreasing equilibrium will be an ε -equilibrium of the original auction (with marginal valuations $v_k^i(\cdot)$) so long as $\delta > 0$ is small enough since payoffs under the w_k^i 's are uniformly close to those under the v_k^i 's. Hence, by Remark 3.1 following the proof of Theorem 3.1, the limit (which exists without loss) of these nondecreasing ε -equilibria of the original game, as ε tends to zero, is a nondecreasing equilibrium of the original better-reply secure game.

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