

Nonmetric test of the minimax theory of two-person zerosum games

(conflict/group dynamics/Nash equilibria)

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ABSTRACT As an experimental test of the minimax theory for two-person zerosum games, subjects played a game that was especially easy for them to understand and whose minimax-prescribed solution did not depend on quantitative assumptions about their utilities for money. Players' average relative frequencies for the moves and their proportions of wins were almost exactly as predicted by minimax, but subject-to-subject variability was too high. These results suggest that people can deviate somewhat from minimax play since their opponents have limited information-processing ability and are imperfect record keepers, but they do not stray so far that the difference will be noticed and their own payoffs will be diminished.

1. Introduction

von Neumann and Morgenstern's minimax solution (1) is generally accepted as the correct way to play two-person zerosum games. It states that a player should evaluate each possible strategy by its lowest expected value to the player over all possible strategies by the opponent and should choose a strategy whose lowest expected value is maximum. von Neumann and Morgenstern (1) give a justification for this policy as that of rational players; but does it describe the behavior of real players? A number of experiments have been reported but the results have been rather negative (see, for example, ref. 2 and Section 6). The issue of the empirical validity of the minimax theory is important since many models in the social sciences, particularly economics, are based on the minimax theory or its generalization for nonzerosum games, the theory of Nash equilibria (3).

A problem in empirical research has been the design of an experiment that accurately tests the theory. Here I describe an experimental game chosen to avoid two previous difficulties. First, the game allows calculation of the solution without assumptions about the exact shape of the players' utility functions for money. Second, the game is easy for the subjects to comprehend; in fact, it is unique in being the simplest nontrivial game possible according to a definition of simplicity to be given in Section 3. My subjects' behavior was close to minimax, and I suggest that this confirmatory evidence should weigh strongly against past failings of the theory, on the grounds that the design used here is more appropriate.

2. The Problem of Utility Measurement

The strategies recommended by the minimax solution depend on the subjects' utilities for the money payoffs. The utilities may be different from the payoffs themselves, so any empirical test must either determine or make some assumption about the players' utility functions. To my knowledge, all past researchers have assumed explicitly or implicitly that

utility in zerosum games depends only on the player's own payoff and is a linear function of that payoff. This seems counterintuitive since it rules out such motives as a desire to equalize winnings or to beat the opponent. Also, I have seen no empirical evidence that utility is linear in money even if the range of payoffs is restricted to small amounts.

A different approach would be to assess each player's utility function individually and then design a game matrix with payoffs calculated to be zerosum in utilities. I know of no research that has done this, and very likely the reason is that utilities would be different in a game than in the single-person decision situation used for calibration. When people interact various competitive or altruistic motives arise so their utilities come to depend on their own and their opponent's payoffs as well.

The experiment described here uses a different approach. A matrix is constructed with the property that a player's minimax strategy is invariant over a set that includes all reasonable utility functions. The game is shown below. Each player has four moves, choosing one of the four rows or one of the four columns, and each cell indicates the payoff of player 1 in cents for that joint choice. (I am free to choose wins and losses of nickels since any game for larger or smaller amounts that can be transformed into the game below is regarded as identical.) The payoff for player 2 at an outcome is always the negative of the payoff for player 1.

		Move of player 2			
Move of player 1	1	+5	-5	-5	-5
	2	-5	-5	+5	+5
	3	-5	+5	-5	+5
	4	-5	+5	+5	-5
	5	-5	-5	-5	-5

Minimax theory prescribes that each player should use the mixed strategy vector (0.4, 0.2, 0.2, 0.2), that is, assign these four probabilities to each of the four rows or columns. In this case player 1 will win 40% of the time and have an expected payoff of -2 cents. This solution holds for any pair of utility functions $u_1(x_1, x_2)$ and $u_2(x_1, x_2)$, where x_1 and x_2 are the money payoffs in the matrix. The premises required for this result are only that each player would rather win an amount than lose that amount, that utility depends only on the payoffs and not on extra-game theoretic features such as the cell in which the payoff vector appears, and that the subjects regard their opponents to be gaining or losing utility increments that are proportional but opposite to their own.

This invariance over utility functions occurs because there are only two types of outcomes, a win and a loss, and the minimax solution is invariant for a positive linear transformation of either player's utilities. If their utility functions satisfy $u_i(5, -5) > u_i(-5, 5)$, then positive linear transformations can always be performed to bring the players' utilities into coincidence with the money values. An experimenter using the payoffs to solve the game will get the same result as the subjects using the utilities provided they follow minimax theory.

Another way of looking at how games of this type avoid utility function assumptions is to regard utility as a surface,

a function in the dimensions of the two money payoffs. Traditional methodology assumes that a player's utility $u_i(x, y)$ is a linear function of that player's payoff x , i.e., a plane whose height is constant for the other player's payoff. It is always possible to fit such a plane if there are only two points $(a, -a)$ and $(b, -b)$ through which it must pass, so it does not matter if the true utility function is quite irregular elsewhere. If there were three such points, that is, three possible outcomes of the game, it would not be possible in general to replace an arbitrary function with a plane of the given form and derive identical strategy predictions. There are other games than the one above that are invariant over such changes in utility functions—any game in which each player has exactly two payoff levels will do—but the above game is most appropriate for an experiment because it is the smallest nontrivial one, as will be shown in the next section.

3. The Problem of Simplicity

Past experiments have used games that tax the comprehension of subjects, in my view. In some ways a 2×2 game is simpler than the situations people confront in their daily lives but a laboratory experiment involving conflict over numerical payoffs is an unfamiliar setting. Many subjects have difficulty processing numbers, especially when they must look at them from their own and their opponent's viewpoint simultaneously. (A 2×2 zerosum game with a solution in mixed-strategy probabilities requires the subject to set up and solve a linear equation involving the four payoffs.)

One way to simplify their task is to restrict the payoffs to two levels as described above. The game becomes a purely structural entity involving the relationship of wins and losses, so the subjects are freed from having to consider relative magnitudes. In line with past experiments I will have each player move only once. I will look for games with the smallest number of strategies, eliminating any dominated or duplicated strategies.

Of course if the game is trivial to solve one will not be testing the full logic of the minimax solution, so I will also eliminate games that are completely symmetrical in strategies. An example would be the children's game "scissors, paper, and stone" (4), where a choice of scissors beats paper, a choice of paper beats stone, and a choice of stone beats scissors, leading to the obvious mixed strategy solution $(1/3, 1/3, 1/3)$.

These ideas are formalized in the following requirements:

Condition 1: The game is in normal (matrix) form.

Condition 2: There are exactly two levels of payoff for each player.

Condition 3: It is not true that a player has two identical strategies.

Condition 4: Neither player has a dominated strategy.

Condition 5: The game is not completely symmetrical in strategies.

Condition 6: Any other game satisfying *Conditions 1–5* has at least as many strategies for each player.

It is surprising that there is a unique game satisfying *Conditions 1–6*, assuming of course that games equivalent under positive utility transformations or permutations of the players or strategies are regarded as identical. It is the game depicted in *Section 2*. Uniqueness can be verified relatively easily by hand by constructing four undirected graphs for $n = 1, \dots, 4$, where the nodes represent the 2^n possible rows or the game matrix and an edge joins two nodes, if it is possible for the two rows to occur in a game matrix satisfying *Condition 4*. One then determines the maximal complete subgraphs and eliminates games with dominating strategies for the column player and also eliminates duplicate and completely symmetrical games. For the case $n = 4$, it is helpful to break the search into a series of smaller tasks by

looking for possible games according to the number of wins in each row for the row chooser, e.g., for 4×4 games those with 1, 1, 1, and 1 row-chooser win in the four rows, those with 1, 1, 1, and 2 wins, etc.

4. Procedure

The game was played by 50 students working in 25 pairs. They were recruited from the Northwestern University student body by posted advertisements and personal contacts. Each served in only one session, and students who knew each other were not allowed to participate in the same pair. The sessions lasted about half an hour, and subjects received their winnings as payment.

The players sat opposite each other at a table. Each held four cards: joker, ace, two, and three. A large board across the table prevented them from seeing the backs of their opponent's cards. The experimenter read the following instructions.

We are interested in how people play a simple game. I will give each of you the rules of the game, then have you play about 15 hands to make sure you are clear about the results. Then you will play a series of hands for money at 5¢ per hand.

The rules are as follows:

1. Each player has four cards—ace, two, three, and a joker.

2. Each player will start with \$2.50 in nickels for the series of hands.

3. When I say "ready" each of you will select a card from your hand and place it face down on the table. When I say "turn," turn your card face up and determine the winner. (I will be recording the cards as played.)

4. The winner should announce, "I win" and collect 5¢ from the other player.

5. Then return the card to your hand.

Are there any questions?

Now to determine the winner . . . [Subjects were shown a placard giving these rules, which were read aloud to them.]

[One subject's name] wins if there is a match of jokers (two jokers played) or a mismatch of number cards (two, three, for example).

[Other subject's name] wins if there is a match of number cards (three, three, for example) or a mismatch of a joker (one joker, one number card).

Thus the game was presented in English without a matrix, and the subjects learned the rules by practice. The subjects played 15 times for practice and then 105 times for real money, proceeding at their own speed. They were not told the number of hands.

Based on some pretrials, the device of having the players themselves figure out who won seemed to be useful in that it increased their involvement in the game and caused them to focus their attention on each other rather than on the experimenter. If they happened to make an error in determining the winner, the experimenter corrected them.

In a post-session questionnaire, all subjects answered that they had understood the rules of the game well.

5. Results

The number cards (ace, two, and three) are strategically equivalent to each other so each should be used with equal probability. Therefore, I will usually group these three moves into one in the analysis and look only at the relative counts of jokers versus number cards.

There were 5250 total moves made in the experiment (50 subjects $\times$ 105 plays each). Numbers of jokers for each role are listed in Table 1. (The complete data may be obtained from B.O.N.) The proportion of jokers was 0.396 compared to minimax theory prediction of 0.400. Looking at the two types of players separately, those in the role of player 1 chose proportion 0.362 jokers, and those in the role of player 2 chose 0.430, compared to minimax theory predictions of 0.400 for both.

These values are close to the predictions of the theory, and the discrepancies are not statistically significant. Using a *t*-test for comparing a sample of unknown variance to a mean of 0.400, the two-tailed *t*-values are 0.77 ($n = 50$), 0.051 ($n = 25$), and 0.15 ($n = 25$), respectively.

The standard deviation of the number of jokers used from subject to subject is greater than that predicted by the minimax theory. For players 1, 2, and both combined, the standard deviations are 9.8, 10.6, and 10.8 jokers, respectively, compared with the value 5.02 predicted by the assumption of independent sampling from a binomial distribution with probability of success, 0.4. These three values are significantly high ($P < 0.01$) using χ^2 tests. To model this behavior, I could regard each subject as choosing beforehand a value of P , the probability of a joker, from a β distribution and using it through the games. (The β was chosen because it is on the interval $[0, 1]$, easy to manipulate, and assumes a variety of shapes.) If the mean P is 0.400, the parameters of the β can be written r and $3r/2$. Applying maximum likelihood methods using the 50 observed frequencies of jokers gives $r = 12$. The upper and lower quartiles of such a β distribution are $P = 0.34$ and $P = 0.46$, so we estimate that 50% of the subjects have values of P in this interval in contrast to minimax theory that says that all have P equal to exactly 0.4.

Another difference from the predictions of minimax is found in the numbers of runs, i.e., unbroken strings of jokers or of number cards. Many subjects produced unusually high numbers of runs, meaning that they had too many and, therefore, too short runs, i.e., a tendency to switch back and forth between the types of cards more quickly than if their choices were independent. The number of runs arising from a long random sequence of independent choices is approximately normal; so for each player, the expectation and variance of that player's number of runs, given the observed proportion of jokers used, was calculated; and the observed proportion of runs was converted to a *z* score. The mean was $z = 0.843$ compared to a null hypothesis expectation of $z = 0$, indicating significantly more runs than expected given independence ($P < 0.001$).

Looking at the frequencies of the three types of number cards, we can judge the prediction that they are produced with equal likelihood. Subjects in the role of player 1 produced 578 aces, 565 twos, and 532 threes, and those in the

role of player 2 had 593 aces, 470 twos, and 446 threes; the latter triple being significantly different from the prediction of equiprobability by a χ^2 test ($P < 0.001$). The best explanation we can offer is that the players were attracted by the powerful connotations of an ace, and in hindsight it may have been a mistake to use this card in the experimental design.

The proportions of wins by each player were strikingly close to the predicted values. Proportions were (0.401, 0.599) compared to the minimax prediction (0.400, 0.600) for players 1 and 2, respectively.

A finding of interest is that there was no statistically significant evidence of differential skill in playing the game. Skill would mean that some of the subjects could exploit the other's deviation from minimax by estimating the other's individual moves or noticing statistical tendencies, and so would do significantly better than 40% wins, while in consequence others would do worse. This would show up as a higher than expected variance in the number of wins from subject to subject. Of course variance in the number of wins for the two types of players will be identical so I calculated it for those in the role of player 1. The sample standard deviation of the number of wins was 6.7, which is not significantly different from 5.02 according to a χ^2 test ($P > 0.1$). The value 5.02 is the standard deviation of a binomial distribution with parameters $P = 0.4$ and $n = 50$. This means that the typical player 1 differed from the expectation of 40% wins by about 6.4% wins rather than 4.8% as would be expected by chance.

If a player uses a certain move and wins, is that move more likely than average to be repeated in the next game? For my subjects it was somewhat less likely. If a and b are the probabilities that players 1 and 2, respectively, use a joker, then the probabilities that a move that has just won will be repeated immediately are $a + b - 2ab$ and $1 - a - b + 2ab$ for the two players, given that successive moves are independent. The parameters a and b were estimated by the observed relative frequencies of jokers used, and for each player these numbers were compared with the observed number of repetitions of winning moves. Of the 50 players, 18 repeated more often than expected, and 32 repeated less often—a difference that is statistically significant ($P < 0.05$). A large majority of subjects, 19 out of 25, in the role of player 2 tended to avoid repeating a move after a win.

My players seemed to feel that a move that had just succeeded should be avoided. This behavior is related to the single-person decision phenomenon of the gambler's fallacy in which a decision maker expects a coin to land "heads" after a run of "tails," and to the negative recency phenomenon in probability learning research (5). In a probability-learning experiment, subjects must guess which of two outcomes of a random process will occur. It is observed that they tend to switch their guesses following a success.

6. Discussion

Since game theory deals with social interaction rather than individuals in isolation, I feel the most accurate test of the theory involves situations where subjects face other subjects, rather than preprogrammed strategies. Many researchers

Table 1. Numbers of jokers and wins for each subject

(19, 37, 41)	(46, 58, 31)	(57, 58, 41)	(35, 76, 44)	(49, 47, 36)
(41, 47, 44)	(32, 37, 50)	(34, 31, 49)	(31, 36, 38)	(44, 43, 40)
(32, 39, 39)	(51, 45, 57)	(28, 56, 43)	(32, 34, 49)	(48, 39, 34)
(46, 40, 35)	(38, 43, 35)	(41, 43, 51)	(34, 53, 41)	(45, 52, 46)
(39, 34, 50)	(48, 36, 42)	(17, 44, 47)	(27, 39, 52)	(35, 62, 35)

Numbers in parentheses are the number of jokers used by player 1, the number of jokers used by player 2, and the number of wins by player 1. The range is (0–105, 0–105, 0–105), and the prediction of minimax is (42, 42, 42).

have had subjects play against computers or stooges, and while this is useful in increasing experimental control, it may not elicit natural game-playing behavior. An experimenter cannot program a computer to behave like a natural opponent without knowing how an opponent behaves; and this knowledge is, after all, the goal of the experiment.

Another disadvantage of having subjects play preprogrammed strategies is that the subjects are usually isolated in cubicles. They do not see an opponent (of course because there is none) and so feel less involved in a competition. I believe subjects should face their opponent, as they did in the present experiment. Some past experiments that were intended to test the minimax theory misinformed subjects in just the opposite way, telling them they were in a random environment when in fact there was a real opponent. In other cases they were not told the payoffs beforehand. Although the results may be interesting in other ways I do not believe they are valid tests of the theory of games.

Past experiments were surveyed according to these guidelines with the results shown in Fig. 1. I tried to include all game experiments that used two real people informed of their situation. This excluded, for one, a design using laboratory rats playing 2×2 games for food pellets (9). Further criteria were that subjects chose their moves directly, e.g., they did not choose mixed strategies that were later implemented by the researcher, and also that the game had two moves per player. This last requirement aimed at admitting studies that could be compared with the present experiment, and it

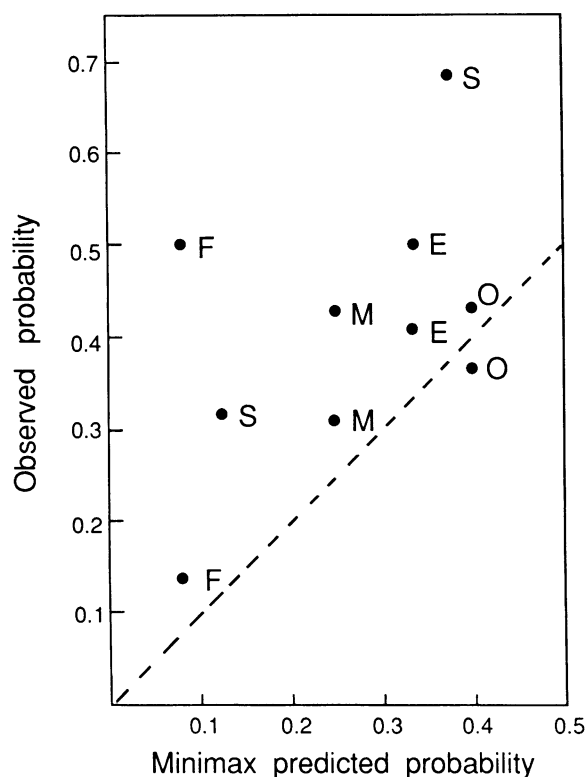


FIG. 1. From 2×2 zerosum game experiments, the predicted probability of the move with lower predicted probability is plotted versus observed probability. The values of Frenkel were reversed in the original publication due to a typographical error. E, Estes from figure 7 in ref. 6; S, Suppes and Atkinson from group E in ref. 7; M, Malcolm and Lieberman from ref. 8; F, Frenkel from game 75 in ref. 2; O, B.O.N. from the present study.

eliminated one 3×3 game design and several games with large strategy sets involving duels in continuous time and Colonel Blotto games of allocation.

Of each player's two moves, the move with the lower minimax-predicted probability had its value plotted on the horizontal axis. Each experiment then yielded two data points on the figure, one for each player. Clearly past results have gone against minimax theory, but I regard the present experiment as evidence that it may have some validity after all. The difference is due, I suggest, to the simplicity of the present game, the subjects' feelings of involvement due to the experimental design, and the test's freedom from metric assumptions about utilities.

Positive evidence for the theory was the correct proportions of strategies used and the correct proportions of wins. Negative evidence was the dependence among successive moves and the high variance in the proportion of jokers from subject to subject.

Thus the theory seemed validated by the large-scale statistics but not the finer ones. This is puzzling. How could the overall proportions have followed the theory when the individual moves that generated them did not? One theoretical explanation, which seems unlikely given people's psychological limitations, is that the subjects regarded the 105 plays as one large supergame and randomized over all possible strategies for this game, a set that includes some with nonindependent moves and greater than predicted variance in number of jokers. A more plausible explanation is that players were constrained to follow the minimax in its gross statistics because these were relatively observable by the opponent. However, at each move players felt free to invent patterns, follow hunches, or do other things that introduced dependencies and variance into the sequence of plays. They could do this without significant danger because the opponent had a limited ability to calculate all the relevant probabilities especially when only a small sample of moves was available. But a large deviation from the overall minimax proportion was easier to notice so players avoided the risk of loss by sticking close to the minimax proportions.

This is more plausible when one considers the feature of the minimax and Nash equilibrium that, if one player follows it, the opponent is often free to deviate without loss. The closer one player gets to the minimax, the less incentive there is for the other to follow it too. One can then expect a certain amount of variation, but it would be centered on the minimax probability. If this explanation were true, one would expect each player's proportion of wins to be as predicted. Neither would decide to go so far from the minimax that their proportion of wins decreased. This is precisely what was found: player 1 won almost exactly 40% of the time.

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