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Optimal bidding in multi-unit auctions with many bidders[☆]

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Abstract

We derive the optimal bid functions for discriminatory and competitive multi-unit private value auctions, assuming bidders act as price-takers. The results suggest that the competitive auction is superior to the discriminatory auction, both in terms of efficiency and simplicity.

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1. Introduction

Unlike single-unit auctions, where the seller offers only one unit of an indivisible good, multi-unit auctions are not very well researched.[1] Harris and Raviv (1981) showed that many results on single-unit auctions generalize quite easily provided bidders demand just one unit.[2] However, as they already emphasized, in many applications bidders may buy many units. In that case, the computation of equilibrium bidding strategies becomes extremely complicated, and only a few results are available.[3]

In this paper, we assume price-elastic demand functions, and derive optimal bids for the two most prevalent multi-unit private value auctions, assuming that bidders act as 'price-takers'.[4]

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[1] For a survey see, for example, Milgrom (1989).

[2] Compare, also, Spindt and Stolz (1989).

[3] Maskin and Riley (1989) generalized the theory of optimal auctions to include the multi-unit case for private value auctions with symmetric, independent and risk-neutral bidders. Bolle (1992) developed the generalization of the Vickrey auction, which assures that *truth-revealing* is a dominant strategy.

[4] Scott and Wolf (1979) study optimal bidding strategy in the discriminatory Treasury bill auction. As regards the similarity of the assumptions, their model is the common value counterpart to the private value model presented here. However, Scott and Wolf failed to derive general properties of the optimal bidding behavior.

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This assumption is appropriate if there are a large number of bidders and a large number of units for sale. An important application of multi-unit private value auctions with many bidders is securities repurchase agreements that provide commercial banks with short-term liquidity, see Nautz (1994).

The most frequently used multi-unit auctions are the discriminatory auction, where every successful bidder pays exactly his bid, and the so-called competitive auction where all successful bidders pay the market clearing stop-out price. For these two auctions we derive the optimal bids, and show that Pareto-efficiency can only be assured by competitive auctions.

2. Assumptions

Suppose the seller sets a grid of prices $p_0 < p_1 < \cdots < p_{k+1}$.[5] He invites bidders to submit bid functions in the form of demand schedules

$$B(p_0) \geq B(p_1) \geq \cdots \geq B(p_{k+1})$$

that state how many units $B_i := B(p_i)$ the bidder is willing to buy at price p_i. The inverse function $Z_B := B^{-1}$ is the bidder's *revealed* marginal willingness to pay, which may, of course, deviate from his true marginal willingness to pay. Let $D(p)$ be the true demand function and $Z := D^{-1}$ the corresponding true marginal willingness to pay. $D(p)$ is taken to be strictly decreasing.

Aggregating the bid functions of all bidders allows the seller to compute the *stop-out price*, at which demand and supply are matched. Let F be the distribution function of the stop-out price, reflecting a bidder's subjective expectations about the probability $F_i := F(p_i)$ that the stop-out price is less than or equal to p_i. Since there are a large number of bidders it is plausible to assume that a single bidder behaves as if his actions do not affect these probabilities.[6] Without loss of generality we set $D(p_{k+1}) = B_{k+1} = F(p_0) = 0$.

Finally, bidders are assumed to be risk-neutral.

3. The optimal bid in a discriminatory auction

In a discriminatory auction, bidders are caught in a dilemma: in order to gain they have to understate their willingness to pay, but they thus risk going empty-handed. Fig. 1 illustrates a typical bid function, and the associated payoffs, depending upon the stop-out price.

If the stop-out price is p_1, the payoff is $A_1 + A_2 + A_3$; if it is p_2, the payoff reduces to $A_2 + A_3$, etc. Since we set $F(p_0) = 0$ the cumulated bid at p_0, B_0, has no impact for the expected payoff.

The bidder's expected payoff is

[5] For example, in financial markets bids often must be multiples of certain percentage points.

[6] Similar assumptions appear in Scott and Wolf (1979).

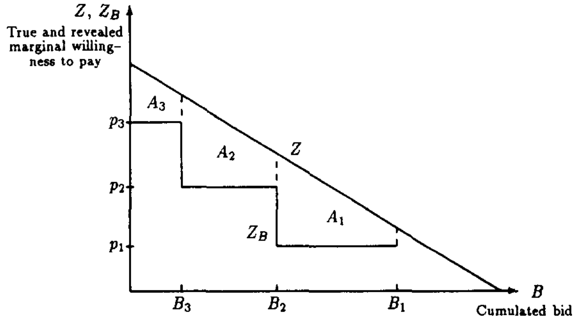

Fig. 1. The payoff in a discriminatory auction.

$$\prod(B_1,\ldots,B_k)=\sum_{i=1}^{k}[F_i-F_{i-1}]\sum_{j=i}^{k}A_j=\sum_{i=1}^{k}F_iA_i\,,$$

where

$$A_i=\int_{B_{i+1}}^{B_i}Z(s)\,\mathrm{d}s-p_i(B_i-B_{i+1})\,.$$

Note that this formula would also apply if a bidder exaggerated his demand.

The optimal bid maximizes $\Pi(B_1,\ldots,B_k)$ subject to the condition that the B_i's are a non-negative and decreasing sequence. This leads to the following Lagrange function:

$$L(B_1,\ldots,B_k,\lambda_1,\ldots,\lambda_k)=\sum_{i=1}^{k}F_iA_i+\sum_{i=1}^{k}\lambda_i(B_i-B_{i+1})\,. \tag{1}$$

The monotonicity of the demand and the convexity of the constraints assure that the following first-order conditions are sufficient for a maximum:

$$F(p_i)[Z(B_i)-p_i]+F(p_{i-1})[p_{i-1}-Z(B_i)]+\lambda_i-\lambda_{i-1}=0\,, \tag{2}$$

$$B_i\geq B_{i+1}\quad\text{and}\quad\lambda_i[B_i-B_{i+1}]=0\,, \tag{3}$$

$$\lambda_i\geq 0\,. \tag{4}$$

In order to prepare the characterization of the optimal bid function, we now introduce the following notation. Define p_{i*} as the highest price below p_i at which the monotonicity requirement $B_j\geq B_{j+1}$ is not binding:

$$i*:=\max\{j\mid j<i\text{ and }\lambda_j=0\}\,,\quad i=1,\ldots,k\,.$$

And, similarly, define p_{i^*} as the lowest price not below p_i at which the monotonicity requirement is not binding:

$$i^* := \min\{j \mid j \geq i \text{ and } \lambda_j = 0\} .$$

Of course, $\lambda_{i_*} = \lambda_{i^*} = 0$, and $B_i = B_{i_*+1} = \cdots = B_{i^*}$, due to (3).

Notice, p_{i_*} is well-defined if we set without loss of generality $\lambda_0 = 0$. And p_{i^*} is well-defined because the monotonicity requirement cannot bind for sufficiently high prices, because demand vanishes as we move towards p_{k+1}. Also, notice that $p_{i_*} = p_{i-1}$ and $p_{i^*} = p_i$ in the case where the monotonicity constraint is not binding at p_i.

Proposition 1. Consider a discriminatory auction. The optimal bid function is

$$B(p_i) = D\left(p_{i^*} + F_{i_*} \frac{p_{i^*} - p_{i_*}}{F_{i^*} - F_{i_*}}\right), \quad i = 1, \ldots, k . \tag{5}$$

Proof. First note that a bidder does not participate in the auction ($B_1 = 0$) if and only if he is sure that his reservation price lies above the stop-out price of the auction ($Z(0) \leq p_1$). In the following we assume therefore that $B_1 > 0$.

Next, assume that there is an index $i \in (1, \ldots, k)$, where $F(p_i) - F(p_{i-1}) = 0$. Then (2) leads to

$$0 > p_{i-1} - p_i = \frac{\lambda_{i-1} - \lambda_i}{F(p_i)} \geq \frac{-\lambda_i}{F(p_i)} .$$

Therefore $\lambda_i > 0$, and we conclude by (3)

$$F(p_i) = F(p_{i-1}) \Rightarrow \lambda_i > 0 \Rightarrow B_i = B_{i+1} . \tag{6}$$

In words, the bid placed at price p_i is zero. This is plausible, because it is pointless to bid at the higher price p_i instead of at p_{i-1} if one cannot increase the chance of 'winning'.

Adding conditions (2) from $i_* + 1$ to i^* leads to

$$F(p_{i_*})[Z(B_i) - p_{i_*}] = F(p_{i^*})[Z(B_i) - p_{i^*}] , \tag{7}$$

which requires that one cannot gain from reshuffling demand from p_{i_*} to p_{i^*}. By (6) we know that $\lambda_{i^*} = 0$ implies $F(p_{i^*}) > F(p_{i_*})$. Therefore, we can solve (7) for $Z(B_i)$, and due to the monotonicity of demand for B_i:

$$Z(B_i) = \frac{F(p_{i^*})p_{i^*} - F(p_{i_*})p_{i_*}}{F(p_{i^*}) - F(p_{i_*})} = p_{i^*} + F_{i_*} \frac{p_{i^*} - p_{i_*}}{F_{i^*} - F_{i_*}} , \tag{8}$$

$$\Leftrightarrow$$

$$B_i = D\left(p_{i^*} + F_{i_*} \frac{p_{i^*} - p_{i_*}}{F_{i^*} - F_{i_*}}\right) . \quad \square \tag{9}$$

We mention that $B(p_1) = D(p_1)$, and $B(p_i) < D(p_i)$ for $i \geq 2$. That is, the marginal willingness to pay is understated at *all* prices, except at p_1, which is the lowest stop-out price to which the bidder attaches positive probability. This is easily verified for an interior solution,[7] where the optimal bid function is

$$B(p_i) = D\left(p_i + F_{i-1}\frac{p_i - p_{i-1}}{F_i - F_{i-1}}\right), \quad i = 1, \ldots, k . \tag{10}$$

Also note that an interior solution implies that there are no zero-bids between two non-zero bids.

If a bidder bids at only one price p, he must assume that a lower stop-out price has probability zero, i.e. $p = p_1$, and $p_2 + F_1[(p_2 - p_1)/[(F_2 - F_1)]$ must exceed his reservation price.

4. The optimal bid in a competitive auction

In a competitive auction, a bidder pays the stop-out price for every unit he receives. Therefore, the expected payoff is now given by

$$\prod(B_1, \ldots, B_k) = \sum_{i=1}^{k} (F_i - F_{i-1})\left(\int_0^{B_i} Z(s)\,\mathrm{d}s - p_i B_i\right) . \tag{11}$$

The monotonicity of demand guarantees that the maximization problem based on (11) is well-behaved, with the first-order conditions:

$$(F_i - F_{i-1})Z(B_i) - p_i = 0 , \quad i = 1, \ldots, k . \tag{12}$$

It follows immediately:

Proposition 2. Consider a competitive auction. The optimal bid function coincides with the true demand function:

$$B(p_i) = D(p_i) , \quad i = 1, \ldots, k .$$

Hence, in a competitive auction all bidders reveal their true demand. This result is independent of bidders' expectations about the stop-out price.

5. Conclusions

In this paper we derived the optimal bid functions for discriminatory and competitive multi-unit private value auctions, assuming that bidders are price-takers. This assumption seems particularly appropriate if the number of bidders is large as in certain financial auctions. Note that we assumed neither identical bidders nor independent demand functions. In

[7] A sufficient condition for an interior solution is that the distribution F is concave for prices higher than p_2.

particular, affiliated demand functions in the sense of Milgrom and Weber (1982) are not ruled out. We mention that the ranking of the two auction rules in terms of the seller's payoff depends upon the price-elasticity of demand as well as the expectations of bidders. This is also known from common value auctions (see Scott and Wolf, 1979).

Proposition 1 showed that successful bidding in a discriminatory auction requires experienced and well-informed bidders, whereas in a competitive auction bidders need not be particularly sophisticated, because it is optimal to simply bid one's true valuation. Furthermore, Pareto-efficiency is always assured in a competitive auction, whereas in a discriminatory auction Pareto-efficiency requires that all bidders have exactly the same expectations concerning the stop-out price. This suggests that the competitive auction is superior to the discriminatory auction, both in terms of efficiency and strategic simplicity.

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