Asymmetric Equilibria in the War of Attrition

Barry Nalebuff and John Riley†

Department of Economics, U.C.L.A., Los Angeles, California 90024, U.S.A.

In Maynard Smith's seminal analysis of the war of attrition the gains to competition are assumed to be public knowledge. As a result, the evolutionary equilibrium is a mixed strategy. More recent work has emphasized the role of private information (degree of hunger etc.) in generating an evolutionary equilibrium in pure strategies, under the assumption that competitors are observationally identical. In this paper it is shown that, for the war of attrition with private information, there is, in general, a continuum of asymmetric equilibria. Thus, even with only a payoff-irrelevant observational difference between potential competitors, very asymmetric behavior is evolutionarily viable.

In the original formulation of the war of attrition by Maynard Smith (1974) two contestants each value a prize equally. The opportunity cost of competing is an increasing function of the length of the contest and each contestant must "decide" when to concede.

As Maynard Smith showed, the equilibrium strategy in this game is a mixed strategy in which the average time costs incurred are equal to the value of the prize. That is, on average, a contestant gains nothing from competing and is therefore equally well off always refusing to enter a contest.

This rather unsatisfactory feature of the equilibrium disappears as soon as the assumption of symmetry is replaced by the assumption that potential contestants in general differ in their valuations (or time costs), and that each contestant knows only his own valuation. As established by Bishop & Cannings (1978), if the distribution of valuations is the same for each contestant there is an equilibrium concession function $T(v)$, which maps each possible valuation onto the time of concession, generating positive expected gains to the contestants.

However Bishop & Cannings did not consider the possibility of asymmetric equilibria. In Riley (1980) an example is presented in which valuations are distributed exponentially. It is shown that there is a one parameter

† Harvard University and U.C.L.A.

family of asymmetric equilibrium bid functions and the conjecture is made that this is an illustration of a general proposition.

In this paper it is shown that there is indeed a continuum of equilibria and the nature of the asymmetries are characterized. This conclusion is in sharp contrast with the results of Maynard Smith & Parker (1976) and Hammerstein & Parker (1982). These authors focus on contests in which the identity of the contestant benefitting more from winning is public knowledge. Here we consider situations in which such identification is, at best, imperfect.

Of course, with a homogeneous population, asymmetric equilibria are impossible since there are no observable characteristics upon which to condition behavior. However, even within a species, complete homogeneity is a rather extreme assumption. For example, age is a commonly observable difference.

Among the family of asymmetric equilibria are some in which one sub-class of agents is very aggressive while the second-class is very passive; the aggressive sub-class almost always wins. Such equilibria can therefore explain the evaluation of "pecking orders" based on observable characteristics within a species. However, the theory also suggests that these pecking orders will not be absolute. Instead equilibrium involves occasional serious challenge from agents lower in the pecking order.

A similar argument holds for competition between species. Indeed it is tempting to suggest that the existence of highly asymmetric equilibria explains, in part, the remarkable degree of specialization in nature.

To illustrate the point, suppose mutation results in regional variations in color differences emerging within a bird population. At the boundary of two such regions, if, instead of competing for territory on all levels of trees, the two different color types begin to specialize, each aggressively defending a different part of trees, both are better off since the asymmetric equilibrium involves lower average costs of combat. But once asymmetry of behavior emerges, the pressure of selection can begin to operate to create two very different sub-species.

In the following section we begin by briefly reviewing the war of attrition and then present the main result on the existence of a continuum of equilibria. Certain characteristics of these equilibria are also identified.

Section 2 concludes with some remarks about extensions of the model and also discusses how economists have used the war of attrition to explain competitive behavior.

1. The War of Attrition

Consider two observably different populations which may or may not belong to the same species. We shall refer to these two groups of agents as

class 1 and class 2. Let the value of some prize (food or territory) for member of class i be v_i. We assume that valuations vary across members of each sub-class and define $F_i(v_i)$ to be the probability that the valuation of a member of class i is v_i or less.

Throughout, we assume that the function $F_i(v_i)$, is a member of the family of distribution functions, $\mathcal{F}$, defined as follows:

DEFINITION 1: FEASIBLE DISTRIBUTION FUNCTIONS

The class distribution function $F \in \mathcal{F}$ if F is a strictly increasing and continuously differentiable mapping from $[0, \alpha] \to [0, 1]$.

We choose units so that the valuation v_i is measured in units of time cost. Then if "combat" ends at time t the agent conceding has a payoff of $-t$ while the agent remaining has a payoff of $v_i - t$. Since we shall consider equilibria in which the time of concession, $T_i(v_i)$, is a strictly increasing function of v_i we can ignore the possibility that contestants concede simultaneously.[†]

Rather than work directly with the concession functions $\langle T_1(v_1), T_2(v_2) \rangle$ it proves more convenient to define the inverse functions

$$y_i(t) = T_i^{-1}(t), \qquad i = 1, 2$$

where $y_i(t)$ is the valuation of an agent in class i who concedes at time t. We shall refer to $\langle y_1(t), y_2(t) \rangle$ as the concession value functions of the two classes.

We now show that any $\langle y_1(b), y_2(b) \rangle$ satisfying a system of ordinary differential equations and associated boundary conditions constitutes a pair of concession value functions.

PROPOSITION 1: SUFFICIENT CONDITIONS FOR AN EQUILIBRIUM

If $\langle y_1(t), y_2(t) \rangle$ is a solution to

$$\begin{cases} \text{(a)} & y_1(t) F_2'(y_2(t)) y_2'(t) = 1 - F_2(y_2(t)) \\ \text{(b)} & y_2(t) F_1'(y_1(t)) y_1'(t) = 1 - F_1(y_1(t)) \end{cases} \tag{1}$$

such that

$$\min \{ y_1(0), y_2(0) \} = 0 \tag{2}$$

then $\langle y_1(t), y_2(t) \rangle$ is an equilibrium pair of concession value functions.

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† Fudenberg & Tirole (1983) show that a necessary condition for equilibrium is that $T_i(v_i)$ should be a strictly increasing continuously differentiable function. Similar arguments for a model of sealed bidding can be found in Maskin & Riley (1984).

Proof: Suppose $\langle y_1(t), y_2(t) \rangle$ satisfies all the hypotheses of the Proposition. From equation (1), it follows that for all $t > 0$ $y_i(t)$ is strictly increasing and differentiable. Thus if agent 2 concedes according to $T_2(v)$, the distribution of his concession times can be written as $F(y_2(t_2))$. Then if agent 1 concedes at time s his expected gain is

$$\Pi_1(s; v_1) = \int_0^s (v_1 - t_2) \, dF_2(y_2(t_2)) - s(1 - F_2(y_2(s))).$$

Differentiating by s, agent 1's expected gain to increasing s is

$$\frac{\partial \Pi_1}{\partial s}(s; v_1) = v_1 F_2'(y_2(s)) y_2'(s) - (1 - F_2(y_2(s))). \tag{3}$$

Substituting for $y_2'(s)$ from equation (1), we obtain

$$\frac{\partial \Pi}{\partial s}(s; v_1) = \left(\frac{v_1 - y_1(s)}{y_1(s)} \right)(1 - F_2(y_2(s))). \tag{4}$$

By hypothesis $y_2(s)$ is a strictly increasing function. Hence for any $s > 0$, $1 - F_2(y_2(s)) > 0$. Then

$$[v_1 - y_1(s)] \frac{\partial \Pi_1}{\partial s}(s; v_1) \geq 0.$$

Moreover the inequality is strict for all s such that $y_1(s) \neq v_1$. Thus agent 1's optimal response is indeed to choose s_1 so that $y_1(s_1) = v_1$. A symmetric argument establishes that $v_2 = y_2(s_2)$ also defines an optimal response for the second agent.

To complete the proof we must consider the endpoints of the equilibrium concession value functions. By requiring condition (2), we rule out solutions to equation (1) with $y_1(0)$ and $y_2(0)$ both positive. For if this were the case, both agents would concede at time zero with finite probability. But then an agent with a positive valuation would gain by waiting infinitesimally to see if his adversary concedes immediately.

By an almost identical argument, it is sufficient at the upper endpoint, for the two contestants to have the same maximum concession time $\bar{t}$,† that is,

$$y_1(\bar{t}) = y_2(\bar{t}) = \alpha. \tag{5}$$

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† If agents of class 1 never concede after time $\bar{t}$ then it cannot be optimal for agents of class 2 to wait until time $\bar{t} + \Delta$ and then concede; they would save costs by conceding at $\bar{t} + \Delta/2$. Thus class 2 agents concede at time $\bar{t}$ with finite probability. But then a class 1 agent is strictly better off waiting until after $\bar{t}$, when his valuation is sufficiently close to α, rather than conceding just prior to $\bar{t}$ as hypothesized.

Since $y_1(t)$ and $y_2(t)$ satisfying equation (1) are both increasing, either

(i) $y_1(t^*) = \alpha$ for some i and $t^* \in (0, \infty)$

or

(ii) $y_1(t)$ and $y_2(t)$ are bounded away from α for all t.

To rule out the latter, consider any subinterval $[a, b]$ of $(0, \infty)$. Since $y_2(t)$ is increasing it follows from equation (1b) that

$$\frac{y_2(b)F_1'(y_1)y_1'(t)}{1 - F_1(y_1)} > 1 > \frac{y_2(a)F_1'(y_1)y_1'(t)}{1 - F_1(y_1)}.$$

Integrating we obtain

$$y_2(b) \ln \left[\frac{1 - F_1(y_1(a))}{1 - F_1(y_1(b))} \right] > b - a > y_2(a) \ln \left[\frac{1 - F_1(y_1(a))}{1 - F_1(y_1(b))} \right].$$

If (ii) holds the left hand expression is bounded from above. But this is impossible since b can be made arbitrarily large, contradicting the first inequality. Then suppose $y_1(t^*) = \alpha$. It follows that, as $b \to t^*$, the right hand expression increases without bound. Therefore, from the second inequality, $t^* = \infty$.

Finally, a symmetric argument using equation (1a) shows that

$$y_1(b) \ln \left[\frac{1 - F_2(y_2(a))}{1 - F_2(y_2(b))} \right] > b - a.$$

Then, as $b \to \infty$, the left hand expression must also increase without bound.
To summarize, we have proved that

$$\lim_{t \to \infty} y_1(t) = \alpha = \lim_{t \to \infty} y_2(t).$$

Hence condition (5) is satisfied,   Q.E.D.

Having provided sufficient conditions for equilibrium, it remains to show that there is a continuum of pairs of concession value functions $\langle y_1(t),$ $y_2(t) \rangle$ which satisfy these conditions. Before providing a general demonstration, we consider the special case in which valuations are distributed uniformly on $[0, 1]$, that is

$$F_i(v) = v, \qquad i = 1, 2.$$

Then the system of equations (1), can be rewritten as

$$y_1(t)y_2'(t) = 1 - y_2(t)$$
$$y_2(t)y_1'(t) = 1 - y_1(t).$$

$$(6)$$

Dividing the second by the first and rearranging we obtain

$$\frac{1}{y_2(1-y_2)}\frac{dy_2}{dy_1}=\frac{1}{y_1(1-y_1)}. \tag{7}$$

Thus equation (6) implicitly defines a mapping from the valuations of class 1 into the valuations of class 2. Integration of equation (7) yields

$$\ln\left(\frac{1-y_2}{y_2}\right)+k=\ln\left(\frac{1-y_1}{y_1}\right).$$

The set of solutions is indexed by the constant of integration, k. When $k=0$ then by symmetry, $y_1=y_2$ for all t; substitution back in equation (6) leads to $-y_i(t)-\ln[1-y_i(t)]=t$. Inverting, it follows that the symmetric equilibrium concession function is

$$T_i(v_i)=-v_i-\ln(1-v_i), \qquad i=1, 2.$$

By contrast, when $k<0$ then $y_1(t)>y_2(t)$ for all t. Since for any t the concession value of agents in class 1 is larger than the concession value of agents in class 2, the probability that a member of class 1 will concede by time t is also larger. Class 1 are therefore very passive in comparison with class 2. Of course with $k>0$ the opposite is true. The range of equilibria is illustrated in Fig. 1.

FIG. 1. Alternative equilibria for identical uniform distributions of value.

One interesting feature of this example is that for any k, the mapping $y_1 \to y_2$ passes through $(0,0)$. Therefore the probability of immediate concession by either class is zero. This turns out to be a property of equilibrium for some, but not all distributions. In fact we shall show that the probability

of immediate concession is zero if and only if the two distribution functions F_1, F_2 are in the set $\mathcal{F}_0$ defined as follows.†

<div align="center">DEFINITION 2: PARTITION OF $\mathcal{F}$</div>

If

$$H_i(y) \equiv \int_y^\beta \frac{F_i'(x)\,dx}{x(1 - F_i(x))}$$

increases without bound as $y\downarrow 0$ then $F_i \in \mathcal{F}_0$. Otherwise the integral has some finite limit as $y\downarrow 0$.

With both F_1 and F_2 in $\mathcal{F}_0$ the family of equilibria have the qualitative properties of the mappings $y_1 \rightarrow y_2$ as depicted in fig. 1.‡ With only F_1 in $\mathcal{F}_0$ the family of equilibria have the qualitative properties of the mappings depicted in Fig. 2(a). Note that $y_2(0) = 0$ and $y_1(0) > 0$ so that all those in class 1 with valuations less than $y_1(0)$ concede immediately. The third possibility, with neither F_1 or F_2 in $\mathcal{F}_0$, is depicted in fig. 2(b). We now summarize this formally.

FIG. 2. Alternative families of equilibria with V bounded from above.

<div align="center">PROPOSITION 2: CONTINUUM OF ASYMMETRIC EQUILIBRIA</div>

For all $F \in \mathcal{F}$ there is a one parameter family of equilibrium concession value functions $\langle y_1(t, k), y_2(t, k)\rangle$, such that:

If F_1, $F_2 \in \mathcal{F}_0$ the probability of immediate concession is zero ($y_i(0, k) = 0, \forall k$);

† An example of a family distributions $F \in \mathcal{F}_0$ is

$$F(v) = 1 - (1 - v)^a, \qquad a > 0.$$

An example of a family distributions $F \notin \mathcal{F}_0$ is $F(v) = v^c$, $c > 1$.

‡ However, there will not be a symmetric equilibrium, with $y_1 = y_2$, unless $F_1 \equiv F_2$.

If $F_1 \in \mathscr{F}_0$ and $F_2 \notin \mathscr{F}_0$ the probability of immediate concession is positive for members of class 1 and zero for members of class 2;

If F_1, $F_2 \notin \mathscr{F}_0$ the probability of immediate concession is always zero for one class and strictly positive for the other class in all but one equilibrium.

Proof: From equation (1)

$$\frac{y_2'(t)}{y_1'(t)} = \frac{y_2(1 - F_2(y_2))}{F_2'(y_2)} \frac{F_1'(y_1)}{y_1(1 - F_1(y_1))}. \tag{9}$$

Since $y_1(t)$ and $y_2(t)$ are both increasing functions, equation (9) implicitly defines a first order ordinary differential equation for y_2 as a function of y_1. To prove the Proposition we must show that there is a one parameter family of solutions to this differential equation.

Rearranging equation (9) we obtain

$$\frac{F_2'(y_2)}{y_2(1 - F_2(y_2))} \frac{dy_2}{dy_1} = \frac{F_1'(y_1)}{y_1(1 - F_1(y_1))}. \tag{10}$$

From Definition 2 we have

$$H_i'(y) = -\frac{F_i'(y)}{y(1 - F_i(y))}.$$

We can therefore rewrite equation (10) as

$$\frac{d}{dy_2} H_2(y_2) \frac{dy_2}{dy_1} = \frac{d}{dy_1} H_1(y_1).$$

Integrating we obtain

$$H_2(y_2) = H_1(y_1) + k.$$

Since both H_1 and H_2 are strictly decreasing functions we can define the increasing function

$$y_2 = H_2^{-1}(H_1(y_1) + k). \tag{11}$$

For $y > \beta$

$$H_i(y) < -\int_\beta^y \frac{F_i'(x)\, dx}{y(1 - F_i(x))}$$

$$= \frac{1}{y} \ln\left(\frac{1 - F_i(y)}{1 - F_i(\beta)}\right).$$

Thus, as $y \to \alpha$ and $F_i(y) \to 1$, $H_i(y) \to -\infty$. It follows that, for all k, the mapping $y_1 \to y_2$ must pass through the point (α, α).

If F_1 and $F_2 \in \mathscr{F}_0$, then $H_i(y)$ increases without bound as y declines to zero. Hence, for all k, equation (11) passes through $(0, 0)$. When $F_1(v) \equiv F_2(v)$ and $k = 0$, then $y_1(t) = y_2(t)$. This is the unique symmetric equilibrium examined by Bishop & Cannings (1978). However, even with $F_1(v) = F_2(v)$ (which implies that $H_2(y) = H_1(y)$) there are a continuum of equilibria; when $k > 0$, then $y_1(t) > y_2(t)$ and the second class of agents are the "aggressors"—the opposite is true if $k < 0$.

Next suppose $F_1 \in \mathscr{F}_0$ and $F_2 \notin \mathscr{F}_0$. From Definition 2, $\lim_{y \downarrow 0} H_1(y) = \infty$ while $H_2(0)$ is finite. Then there can be no point $(0, y_2)$ satisfying equation (10). It follows that equation (11) must, for all k, intersect the y_1 axis as depicted in fig. 2(a).

Finally, with F_1 and $F_2 \notin \mathscr{F}_0$, both $H_1(0)$ and $H_2(0)$ are well defined. Each member of the family of functions given by equation (11) then intersects one of the axes as depicted in Fig. 2(b),   Q.E.D.

Note that in each case, the equilibrium involves "aggressive" behavior by one class of agents and "passive" behavior by the other class for small or large values of k. In the first case both classes compete but one class almost always concedes very quickly. In the other two cases one of the two classes concedes immediately with high probability.

Moreover, each of the equilibria is a strong Nash equilibrium, that is, any strategy other than the equilibrium strategy of agent i strictly lowers this agent's expected return. Thus Maynard Smith's requirement for evolutionary equilibrium is satisfied.

This is not true of the limiting Nash equilibrium in which one class of agents threatens never to concede while the other class always concedes. For in this case a mutant conceding at any time $t > 0$ does equally well against a completely passive opponent. The equilibrium is therefore not evolutionarily stable.†

2. Extension and Other Applications

While we have modelled informational differences as arising from differences in the benefits to competition, it should be intuitively clear that systematic differences in the costs of competition will generate qualitatively similar conclusions. The crucial simplification is that the gains to waiting are, *ceteris paribus*, always higher for one member of a class than for another member, regardless of the time elapsed since the start of a contest.

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† In one shot economic models the limiting equilibrium is also less satisfactory in that if a member of the passive population does bid, it is no longer in the interest of the aggressive contestant to carry out his threat. In the terminology of game theory the equilibrium is not sub-game perfect. See Wilson (1983) for a more complete discussion of this point.

In this paper we have focused on animal conflict. However certain aspects of economic competition have essentially the same structure. Consider, for example, two firms working to create a patentable invention. The market value of this invention is V, the cost per period of firm i's research team, c_i, is a random draw from some distribution $F_i(c_i)$. Finally, the probability firm i will make the breakthrough at time t, given that there has been no prior breakthrough is π. This last element of the problem complicates the model somewhat but the essential ingredients are the same and again there is a continuum of equilibria.

A second model discussed by Nalebuff (1982) and Osborne (1985), examines competition between two agents when the rewards are delayed until agreement is reached. Initial demands are incompatible and agreement requires one side or the other to make a concession. Informational asymmetry is introduced by making the cost of conceding a random variable. As Nalebuff shows, this model can be formulated so that it is mathematically identical to the war of attrition.†

Finally, Fudenberg & Tirole (1983) have used a similar approach to model the possible exit from an industry by one of the two currently competing firms. One interesting conclusion of their paper is that as long as there is some probability of a positive payoff to *both* contestants, the equilibrium is unique. Their work suggests a variation of the war of attrition that leads to a unique equilibrium.

Imagine that there is some probability, p, that an animal is "irrational". By irrational, we mean that once engaged in a conflict the animal will *never* give in; the animal becomes enraged and is then willing to fight to its death. The fact that there is a positive probability that an animal will never concede leads to a unique outcome.

The probability that an animal will concede by time t is the chance that it is both rational and has a valuation less than $y_i(t)$. Thus, the distribution of concession times t is now

$$\hat{F}_i(y_i(t)) \equiv (1-p)F_i(y_i(t)). \tag{12}$$

Replacing F_1 and F_2 in equation (1) by $\hat{F}_1$ and $\hat{F}_2$ we can define, as before, a pair of equilibrium concession value functions $\hat{y}_1(t)$, $\hat{y}_2(t)$. However, now the concession time for an agent with a valuation of α is finite. To see this, note from equation (1) that

$$t(y_2 = \alpha) = \frac{\int_0^\alpha y_1 \hat{F}_1(y_2)\,dy_2}{1 - \hat{F}_2(y_2)} < -\alpha \ln[1 - \hat{F}_2(y_2)]\big|_0^\alpha < -\alpha \ln p.$$

† The model analyzed also includes the possibility of escalation of the conflict rather than concession. However, this too is incorporated without altering the underlying mathematical structure.

A similar argument shows that $\hat{H}_i(y_i)$ also converges in the limit as $y_i \uparrow \alpha$.

Since $\hat{H}_i(y_i)$ converges and

$$\hat{H}_1[y_1(t)] = \hat{H}_2[y_2(t)] + k, \tag{13}$$

There is no longer a one parameter family of concession bid functions passing through (α, α), and hence satisfying condition (5). Instead there exists a unique equilibrium which is determined by the k that solves

$$\hat{H}_1(\alpha) = \hat{H}_2(\alpha) + k. \tag{14}$$

Because the functions $\hat{H}_i(y)$ are strictly monotonic, the choice of k is unique.

If the distribution functions are the same then $H_1(y) = H_2(y)$, $k = 0$, and the symmetric equilibrium is the unique solution. The advantage of this reformulation is that it suggests a method for choosing one of the continuum of asymmetric equilibria when the density functions differ.†

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† For economic applications it should also be noted that the result does not actually depend on the existence of irrational behavior; it is sufficient for each class of agents simply to believe that there is some positive probability that its opponent is irrational.