Equilibrium in Auctions with Entry

By DAN LEVIN AND JAMES L. SMITH*

We model entry incentives in auctions with risk-neutral bidders and characterize a symmetric equilibrium in which the number of entrants is stochastic. The presence of too many potential bidders raises coordination costs that detract from welfare. We show that the seller and society can benefit from policies that reduce market thickness (i.e., the relative abundance of buyers). Our analysis extends well-known revenue-equivalence and ranking theorems but also demonstrates that variations in the auction environment affect optimal policies (e.g., reservation prices) in ways not anticipated by models that ignore entry. (*JEL* D44, L10)

Most of the auction literature assumes that the number of bidders, n, is determined exogenously. By ignoring the role of entry, the "fixed-n" paradigm simplifies comparisons of auction outcomes and performance across institutions.[1] The celebrated revenue-ranking theorems of William Vickrey (1961), Paul Milgrom and Robert Weber (1982), and related propositions regarding optimal design (see Charles Holt, 1980; Roger Myerson, 1981; John Riley and William Samuelson, 1981; Eric Maskin and Riley, 1984) illustrate the power of this approach. Its implications, however, are not necessarily consistent with more general models of equilibrium behavior. To fix n, for example, and then suggest that a seller would prefer one institution if it raises more revenue than another ignores the possibility that bidders might have less incentive to enter that auction in the first place.

An alternative approach is to assume that entry proceeds until the expected gains to all bidders are dissipated. If bidders incur real costs of entry (bid preparation, information processing, etc.), entry does not simply transfer all rent to the seller. The relative efficiency of institutions, and whether the seller should favor one mechanism over another, may depend on the extent of entry those mechanisms induce.

We model entry incentives in auctions with risk-neutral bidders, characterize equilibrium behavior, and examine the impact of induced entry on performance. There are N identical, potential bidders. Each must incur costs to enter a bid. If there is room for only $n < N$ bidders in the auction, the symmetric entry equilibrium involves mixed strategies: each potential bidder enters with probability q and stays out with probability $1-q$. Thus, in equilibrium n varies stochastically between 0 and N with probabilities that are determined endogenously by the seller's mechanism and other market factors.

This model departs markedly from the existing literature on entry and produces

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[1] If not fixed, n is assumed to vary exogenously, as in Steven Matthews (1987), R. Preston McAfee and John McMillan (1987a), and Ronald Harstad et al. (1990).

important new insights. Previous studies (Smith, 1982, 1984; Richard Engelbrecht-Wiggans, 1987, 1991; McAfee and McMillan, 1987b) assume that potential entrants use pure strategies, which produces a deterministic, asymmetric equilibrium in which exactly n bidders enter and $N-n$ stay out.[2] The process by which potential bidders divide into these two groups is not explained.

Our approach enriches the analysis of entry in several respects. First, by introducing mixed entry strategies we restore full symmetry to the equilibrium, which seems natural if potential bidders are assumed to be identical. Second, our model produces an equilibrium in which the number of actual bidders is stochastic and will on many occasions be "too large" or "too small"—even zero. Such outcomes are frequently observed in practice and create efficiency losses that impact the seller. We are able to investigate the ramifications of this phenomenon within an equilibrium framework. Third, and perhaps most significantly, we show that market thickness (i.e., the relative abundance of potential buyers) has an important and surprising impact on auction performance—one that sellers may dislike. Unlike previous models in which extra bidders have no real impact, we demonstrate that coordination costs associated with thick markets create incentives that may induce sellers to limit entry by restricting the pool of qualified bidders.

Our treatment of entry raises new issues in the analysis of auctions but also extends many well-known results from the fixed-n literature. We show, for example, that many of the celebrated revenue-equivalence and ranking theorems still hold after accounting for entry. We also show that variations in the auction environment affect mechanism design, but not necessarily as suggested by models that ignore entry. Because reservation prices tend to discourage entry, for example, they are never desired in independent private-value (IPV) auctions, where unrestricted entry is optimal; however, they can be useful in common-value (CV) auctions, where sellers generally want to deter entry.

I. The Entry Process

Consider a single item offered to group of N potential bidders. The offering proceeds in two stages. The second stage is where bidding transpires; an auction is conducted among n participants who have elected to incur the fixed cost (c) of entry.[3] The outcome of that auction is to allocate the item according to the rules of the seller's mechanism (m). In the first stage, and knowing what will ensue, each potential bidder decides whether or not to incur c and enter the second stage.

We maintain the following assumptions throughout:

ASSUMPTION 1: *The seller and all potential bidders are risk-neutral.*

ASSUMPTION 2: *The domain of possible values for the item (V) and the domain of estimates (x) are compact: $V \in [0, \bar{v}]$ and $x \in [0, \bar{x}]$.*

ASSUMPTION 3: *Information is symmetric; all bidders randomly draw values from the same distribution.*

ASSUMPTION 4: *The auction mechanism (m) and the number of potential bidders (N) are common knowledge, and the number of actual bidders is revealed prior to stage 2.*

[2]Engelbrecht-Wiggans (1987) presents an example in which entry probabilities are influenced by profitability, but they do not constitute a Nash equilibrium. Donald Hausch (1988) models entry stochastically, but his entry probabilities also violate equilibrium. Harstad's (1990) treatment of stochastic entry relies improperly on the assumption that the expected number of bidders always enter.

[3]Before entry, potential bidders may already have common information regarding the item. Expenditure c represents the cost of developing and evaluating private information, then preparing and delivering a formal bid.

ASSUMPTION 5: *The environment is such that a unique symmetric Nash equilibrium bidding function exists, which is increasing.*[4]

We denote by $E[\pi|n,m]$ each potential entrant's *ex ante* expected gain from entering, paying c, learning n, and bidding according to the symmetric Nash strategy implied by n and m.[5] If $E[\pi|n,m]$ is decreasing in n, there exists a unique integer, n^*, such that $E[\pi|n^*,m] \geq 0 > E[\pi|n^*+1,m]$. We are concerned with cases in which entry costs are low enough to admit some, but not all, potential bidders: $0 < n^* < N$, since only then is a model of the entry process important.

A symmetric entry equilibrium must yield the same probability of entry for all potential bidders. For $q^* \in (0,1)$ to constitute a mixed-strategy equilibrium, each potential entrant must be indifferent between entering or not (i.e., an entrant's *ex ante* expected gain must be zero):

$$(1)\quad \sum_{n=1}^{N}\left[\binom{N-1}{n-1}(q^*)^{n-1}(1-q^*)^{N-n}E[\pi|n,m]\right] = 0$$

where the first terms in the bracket give the binomial probability that exactly $n-1$ rivals also enter, giving n participants in total.[6]

The value q^* that satisfies (1) characterizes equilibrium in mixed strategies. The number of actual bidders follows a binomial distribution with mean $q^*N = \bar{n}$ and variance $(1-q^*)\bar{n}$ determined by m, N, and c.

We allow the seller's choice of mechanism to include any rule by which a bidder wins and pays for the item only if his bid is the highest. The mechanism might entail an entry fee e, which denotes an *ex ante* admission fee paid to the seller, or reservation prices $R = \{R_1, \dots, R_N\}$, where R_i represents the common-knowledge reserve price enforced by the seller if n bidders enter. For clarity, we will denote the seller's mechanism by $m(R,e)$.

Throughout this paper we assume that the seller's valuation is zero. From this and Assumption 5 it follows that if $R = 0$ the mechanism is *ex post* efficient since the item will then necessarily be allocated to the highest valued use. By setting R above zero, the seller might block a profitable trade. We let T represent the event that trade occurs and let $T_n(R_n)$ represent the probability of trade given n and the seller's mechanism.

Since the entry fee is paid before bidders obtain estimates, it does not screen bidders with low valuations.[7] The fee does enable the seller to influence entry as desired under any mechanism he might choose. This seems like a powerful tool, but we show later that he might not be inclined to use it. We say the seller permits "free entry" if $e = 0$.

Using symmetry, a bidder's *ex ante* expected profit, conditional on entering an auction where trade occurs as one of n bidders, can be written as $(V_n - W_n)/n - (c+e)$, where V_n is the expected value of the item to the highest bidder and W_n is the expected payment this bidder makes to the seller, both conditional on trade occurring under the given mechanism with n bidders. We use Ω to denote $\{m(R,e), c, N\}$ and $B_i(q,\Omega)$ to denote the ith bidder's expected profit from entering when all $N-1$

[4]If additional nonsymmetric equilibria exist, we will assume that individual behavior conforms to the symmetric equilibrium.

[5]In certain markets, entrants do not learn their number before bids are tendered. At least for the IPV case, however, this variation would not alter our results (see footnote 15).

[6]Requiring symmetry means that (1) is satisfied by a unique q^*. In principle, many asymmetric equilibria may exist (e.g., the one where n^* bidders enter with probability 1 and $N - n^*$ with probability zero). Similarly, if n^*-1 bidders enter with probability 1, there exists an entry probability $0 < q < 1$ for the remaining $N - n^* + 1$ bidders that would constitute equilibrium. However, we cannot have asymmetric equilibria in mixed strategies of the form $0 < q_i < q_j < 1$ since (1) is identical for each bidder and strictly decreasing in q.

[7]Samuelson's (1985) model of "interim" entry costs, which are not incurred until after each bidder learns his value, does screen low valuations. Within that framework, the total investment in information is independent of entry decisions.

rivals are using arbitrary entry probability q:

$$(2)\quad B_i(q,\Omega) = \sum_{n=1}^{N}\left[\binom{N-1}{n-1}q^{n-1}(1-q)^{N-n}T_n(R_n)(V_n - W_n)/n\right] - (c+e).$$

If the ith bidder also elects to enter with probability q, the expected profit of all N parties must be:

$$(3)\quad B(q,\Omega) = NqB_i(q,\Omega) = \left[\sum_{n=1}^{N} p_n T_n(R_n)V_n - \bar{n}c\right] - \left[\sum_{n=1}^{N} p_n T_n(R_n)W_n + \bar{n}e\right]$$

where p_n denotes the binomial probability that exactly n bidders enter in total. The seller's expected revenue is

$$(4)\quad \Pi(q,\Omega) = \sum_{n=1}^{N} p_n T_n(R_n)W_n + \bar{n}e.$$

Total social welfare is the sum of all expected gains:

$$(5)\quad S(q,\Omega) = B(q,\Omega) + \Pi(q,\Omega) = \left[\sum_{n=1}^{N} p_n T_n(R_n)V_n - \bar{n}c\right].$$

Equations (2)–(5) hold for any q, in or out of equilibrium. However, the best response for the ith bidder is to enter ($q_i = 1$) if $B_i(q,\Omega) > 0$ or to stay out ($q_i = 0$) if $B_i(q,\Omega) < 0$. The bidder is content to use q^* if and only if

$$(6)\quad B_i(q^*,\Omega) = 0$$

which defines the symmetric entry equilibrium, $q^* = q(\Omega)$.

The intuition that higher entry costs or fees reduce q^* and higher payoffs increase q^* is correct within limits.[8] To see why, let ρ denote the correlation (induced by the seller's mechanism) between the expected profit of an entrant and the number of rivals he faces. We can then show the following lemma.

LEMMA 1: *$\rho \lessgtr 0$ is necessary and sufficient for:*

$$\partial q^*/\partial e \lessgtr 0$$
$$\partial q^*/\partial c \lessgtr 0$$

and

$$\partial q^*/\partial(V_i - W_i) \gtrless 0 \qquad (i = 1,\ldots,N).$$

(See Appendix A for the proof.)

Without manipulating reservation prices, there is almost no scope for $\rho \geq 0$. In the class of IPV auctions described by Riley and Samuelson (1981), in which the winner pays on average the second-highest value, ρ must be negative. The same is true of any mechanism in CV auctions so long as competition does not rapidly drive *down* the price paid by the winning bidder (typically it would be driven up). Therefore, for the remainder of this paper we shall assume:

ASSUMPTION 6: *For any mechanism the seller would choose, $\rho < 0$ whenever $R = 0$.*

By definition, induced entry drives out all expected profit to the bidders. Thus, $B(q^*,\Omega) = 0$, and the seller's expected revenue constitutes total social welfare:

$$(7)\quad \Pi(q^*,\Omega) = S(q^*,\Omega) = \sum_{n=1}^{N} p_n T_n(R_n)V_n - \bar{n}c.$$

This shows that all entry costs are deducted from the seller's return, as originally noted by Kenneth French and Robert McCormick (1984). But (7) also shows that the seller pays for the possibility that his mechanism

[8] We are grateful to an anonymous referee who helped clarify our statement of these results.

leaves the item unsold, either due to zero entry or the inefficiency of reservation prices.[9]

Since $B(q^*, \Omega) = 0$ in equilibrium, only the seller's payoff is affected by mechanism design. That does not mean that the seller's optimal mechanism necessarily maximizes social welfare. To clarify, consider a social planner who, after selecting a mechanism, could impose any q, even one the bidders dislike. The planner need not settle for voluntary entry if he wields independent control. Despite this fundamental difference, we show that the seller and planner favor mechanisms that induce identical entry behavior.

PROPOSITION 1: *Any mechanism that maximizes the seller's expected revenue also induces socially optimal entry. Such a mechanism might involve entry fees, but not reservation prices.*

PROOF:

The social optimum is defined, for given N, by the set $\{q^s, R^s, e^s\}$ that maximizes (5).[10] We claim that e^s can be chosen arbitrarily since, for given q, it represents a transfer that does not affect social welfare.[11] That $R^s = 0$ follows from the *ex post* inefficiency of any positive reservation price, as shown by $\partial S(q, \Omega)/\partial R_n \leq 0$ for all $R_n > 0$ (see Appendix B). Thus, q^s is determined by maximizing (5) subject to $T_n(0) = 1$, for all n. By definition,

$$S(q^s, R^s, e^s, N, c) \geq S(q^*, R, e, N, c)$$
$$= \Pi(q^*, R, e, N, c)$$

for all $\{R, e\}$, which bounds the seller's profit. By Assumption 6 and Lemma 1, q^* varies monotonically with e between 0 and 1. By continuity we can, given $R = 0$, then select e^* to induce $q^* = q^s$. It follows that

$$\Pi(q^*, R = 0, e^*, N, c)$$
$$= \Pi(q^s, R^s, e^s, N, c)$$
$$= S(q^s, R^s, e^s, N, c).$$

The seller maximizes expected revenue by designing $m(R = 0, e^*)$ to reproduce the social optimum.

This brings us to the celebrated revenue-equivalence theorem, which identifies a class of efficient mechanisms (first-price, second-price, English, Dutch; all with zero reservation prices) that produce identical bidder payoffs with fixed n. We know by (6) that any two mechanisms which, for given n, offer bidders equal payoffs must also induce equal entry. Furthermore, (7) implies that any two mechanisms that share identical entry probabilities and reservation prices must generate equal expected revenues for the seller. This establishes the following proposition, which extends revenue equivalence to models with entry.

PROPOSITION 2: *Any two mechanisms that are revenue-equivalent with fixed n and R remain revenue-equivalent with induced entry.*

We have not yet characterized the seller's optimal entry fee, except to note that it maximizes social welfare. But whether the seller would encourage (subsidize) or dis-

[9]The separate effects can be isolated by rewriting (7) as

$$\Pi(q^*, \Omega) = (1 - p_0) \sum_{n=1}^{N} V_n p_n / (1 - p_0) - \bar{n}c$$
$$- \sum_{n=1}^{N} V_n p_n [1 - T_n(R_n)].$$

The first term is the probability that at least one bidder enters multiplied by the expected value of the item to the highest bidder given entry and trade. Subtracted from that are expected information costs and the expected cost of reservation prices.

[10]Parameters q, R, and e are the only aspects of the mechanism that affect welfare; the form of payment is irrelevant.

[11]The distribution of profits between bidders and seller would not be independent of the chosen entry fee, nor do we claim (yet) that the parties would voluntarily participate under the terms required to produce the social optimum.

courage (tax) entry remains unclear. The answer to this question requires more specific information regarding the auction environment, as we demonstrate next.

A. *Common Values*

In the CV model, first introduced by Robert Wilson (1977), the value of the item to the winner, V, is independent of the number of bidders. We have already established that the seller's optimal reservation price is zero, so $T_n(R^s)=1$ for all n. Consequently, social welfare in CV auctions can be written as follows:

$$(8)\quad S(q,R^s,e)=(1-p_0)V-qNc = \left[1-(1-q)^N\right]V-qNc$$

with $\partial S/\partial q = N[(1-q)^{N-1}V-c]$, and $\partial S^2/\partial q^2<0$. Thus, e^* must induce entry such that at q^*, $\partial S/\partial q$ vanishes:

$$(9)\qquad (1-q^*)^{N-1}V=c.$$

PROPOSITION 3: *In CV auctions the seller should discourage entry by charging a positive entry fee but no reservation price. Without the entry fee, entry would be excessive from social and private points of view.*

PROOF:

We use the fact that q^* depends on e through (6) to substitute for c in (9):[12]

$$(10)\quad (1-q^*)^{N-1}V = \sum_{n=1}^{N}\left[\binom{N-1}{n-1}\frac{(q^*)^{n-1}(1-q^*)^{N-n}(V-W_n)}{n}\right]-e^*.$$

We then solve (10) for e^*, using the fact that $W_1=0$ to eliminate terms of order $n=1$:

$$(11)\qquad e^*=\sum_{n=2}^{N}p_n(V-W_n)/\bar{n}>0$$

with the inequality coming from individual rationality, which requires $V-W_n\geq 0$ for all n.[13] Now, since $e^*>0$, $R^s=0$ (which implies $\partial q/\partial e<0$), and since $S(q,\Omega)$ is concave in q and does not depend on e directly, we know that $\partial S/\partial q<0$ evaluated at $R=e=0$. This establishes the proposition.[14]

Excessive entry has been examined in other market settings. Gregory Mankiw and Michael Whinston (1986), for example, find excessive entry in markets where individual incentives include a "business-stealing" effect. That intuition applies here since the CV auction allocates an indivisible item, which makes the business-stealing effect paramount. The remedy of taxing entry, however, must balance benefits against potential costs. Any tax that reduces q also raises the probability of no entrants. This suggests another interpretation of Proposition 3: to mitigate the business-stealing effect in CV auctions, the seller would willingly institute policies that increase the risk of no trade.

[12]The seller's optimal policy is $R=0$, thus $T_n(R_n)=1$ in (6).

[13]Since $V\geq W_n$ for all n, as long as W_n varies with n, e^* is strictly positive in (11). Although (11) bounds e^* away from zero, it does not provide a closed-form solution since the p_n and $\bar{n}$ which appear on the right-hand side vary with e^* via q.

[14]The optimal entry fee is the weighted average of what a seller could charge in a corresponding set of fixed-n auctions. With n fixed, a seller could charge at most $e_n=[(V-W_n)/n]-c$ without imposing expected losses on the bidders. Suppose the seller announces that e_n is the fee he will charge in the auction with entry if n bidders enter. From the perspective of a bidder who would enter, the probability of being charged e_n is

$$\Pr[n-1\text{ rivals}]=\binom{N-1}{n-1}q^{n-1}(1-q)^{N-n}$$

and the expected fee would be:

$$E[e_n]=\sum_{n=1}^{N}\binom{N-1}{n-1}q^{n-1}(1-q)^{N-n}\left[\frac{V-W_n}{n}-c\right]$$

$$=\sum_{n=1}^{N}p_n\frac{V-W_n}{qN}-c$$

$$=e^*+(1-q^*)^{N-1}V-c=e^*,\text{ using (9).}$$

What if the seller cannot charge entry fees? Would he then want to use distortive reservation prices to discourage entry instead? To answer, we must weigh all consequences of increasing a typical component, R_j, above zero:

$$(12) \quad d\Pi/dR_j = (\partial\Pi/\partial q)(\partial q/\partial R_j) + \partial\Pi/\partial R_j.$$

Through entry, the reservation price has a feedback effect on revenue, $(\partial\Pi/\partial q)\times(\partial q/\partial R_j)$, which is typically positive in the CV case: the reservation price discourages entry, which in turn reduces information costs and raises expected revenue. If the *direct* effect of the reservation price is also positive holding n constant ($\partial\Pi/\partial R_j > 0$), then entry reinforces the seller's incentive to use it. Thus, we are able to prove the following:

PROPOSITION 4: *If entry fees are not allowed, the seller gains by introducing at least $R_1 > 0$ in CV auctions.*

PROOF:

In proving Proposition 3 we showed that $\partial S/\partial q < 0$ at $R = 0$, which implies $\partial\Pi/\partial q < 0$. We show in Appendix A that $\partial q/\partial R_1$ evaluated at $R = 0$ is

$$(13) \quad \partial q/\partial R_1 = \frac{p_1(1-q)\partial[T_1(R_1)W_1]\partial R_1}{N\,\mathrm{Cov}_m}$$

where Cov_m is the covariance (under mechanism m) between an entrant's expected profit and the number of his rivals, which we have assumed to be negative given $R = 0$. Since $\Pi = \Sigma p_n T_n(R_n)W_n$, we also know that

$$(14) \quad \partial\Pi/\partial R_1 = p_1\partial[T_1(R_1)W_1]/\partial R_1.$$

Finally, we show in Appendix B that $\partial[T_1(R_1)W_1]/\partial R_1 > 0$ at $R = 0$, which by (13) and (14) implies $\partial\Pi/\partial R_1 > 0$ at $R = 0$.

While the use of reservation prices against single bidders may appear obvious, it is not universal. Later we show that sellers in IPV auctions would never impose reservation prices—not even against single bidders. Proposition 4 does *not* say that reservation prices are generally desirable for $n > 1$. As we demonstrate in Levin and Smith (1993), for large enough n the direct effect of $R_n > 0$ is negative in CV auctions; thus use of higher-order reservation prices to discourage entry might backfire.

Next we reexamine the revenue ranking of CV auction mechanisms. Milgrom and Weber (1982) have shown that, with n fixed, a risk-neutral seller in CV auctions prefers simple second-price mechanisms to first-price mechanisms , and so forth. This ranking is not affected by entry, as we demonstrate next.

PROPOSITION 5: *The revenue ranking of any two CV auction mechanisms that do not entail reservation prices or entry fees is preserved with equilibrium entry.*

PROOF:

Suppose that, for fixed n, the seller prefers mechanism a to b, neither of which entails entry fees or reservation prices. The seller's expected revenue is concave in q, and by Proposition 3, both mechanisms induce excessive entry. The seller favors the mechanism that produces smaller q. Since the seller's expected revenue is higher (for given n) under mechanism a, the expected profit of each bidder must be lower. By Lemma 1, mechanism a would then induce smaller q than mechanism b.

This proof hinges on the tendency of both mechanisms to induce excessive entry. If they do not, the fixed-n ranking can easily be reversed. For example, consider again mechanisms a and b now accompanied by a fixed reservation price large enough to reduce entry in both auctions below the socially optimal level. If mechanism a would be favored by the seller with n fixed, equilibrium entry must satisfy: $q_a < q_b < q^s$. Mechanism b, which induces greater entry, comes closer to the optimum and must now be preferred. Entry has reversed the fixed-n ranking.

Proposition 5 extends many results from the previous literature. Its implications include seller preference for second-price mechanisms over first-price mechanisms in affiliated CV auctions with entry (cf. Milgrom and Weber, 1982 theorem 15) and seller preference for truthful disclosure of his private information (cf. Milgrom and Weber, 1982 theorems 8, 9, 13, 16, and 17).

B. *Private Values*

1. *Independent Private Values.*—Now consider IPV auctions, for which we will show that the seller does not want to discourage entry.[15] His optimal policy is given by e^*, which we have argued must satisfy the necessary condition for a social optimum: $\partial S(q,\Omega)/\partial q = 0$. $R^s = 0$ implies $T_n(R_n) = 1$ for all n, so using (5), $S(q,\Omega)$ can be written as

$$(15)\quad S(q,\Omega) = \sum_{n=1}^{N} \binom{N}{n} q^n (1-q)^{N-n} V_n - qNc.$$

On taking the partial derivative we obtain:

$$(16)\quad \partial S(q,\Omega)/\partial q = \left[\sum_{n=1}^{N} p_n V_n (n - qN) - qNc(1-q)\right] \Big/ q(1-q).$$

PROPOSITION 6: *Optimal entry, for society and the seller, occurs in* IPV *auctions when the seller charges no entry fee or reservation price.*

PROOF:

Since $R^s = 0$ by Proposition 1, we need only show that (16) vanishes at $e = 0$. Using (3) and equilibrium condition (6) under the presumption that $e = 0$ implies

$$(17)\quad qNc = \sum_{n=1}^{N} p_n (V_n - W_n).$$

However, in IPV auctions the expected payment made by the highest bidder is on average equal to the second-highest valuation, which permits us to write:[16]

$$(18)\quad V_n - W_n = n(V_n - V_{n-1}).$$

Substituting this into (17) gives: $qNc = \sum_{n=1}^{N} p_n n(V_n - V_{n-1})$, which, when substituted back into (16) gives

$$(19)\quad \partial S(q,\Omega)/\partial q = \frac{\sum_{n=1}^{N} p_n V_n (n - qN) - (1-q)\sum_{n=1}^{N} p_n n (V_n - V_{n-1})}{q(1-q)}$$

$$= \left[\sum_{n=1}^{N} p_n n V_{n-1} - \sum_{n=1}^{N} p_n V_n q(N-n)/(1-q)\right] \Big/ q$$

$$= \left[\sum_{n=1}^{N} p_n n V_{n-1} - \sum_{n=1}^{N-1} p_{n+1}(n+1) V_n\right] \Big/ q$$

[15] In the risk-neutral IPV case, our equilibrium analysis and results hold even if bidders do not learn the number of their rivals after entering, since entrants are indifferent about whether the actual number is revealed or not (see Matthews, 1987; McAfee and McMillan, 1987a; Harstad et al., 1990). We thank a referee for calling our attention to this generalization. However, if bidders are risk-averse, their *ex ante* expected utility (and the entry equilibrium) is affected by the assumption that n is revealed.

[16] This property holds for all IPV mechanisms in which the bidder who values the object most highly is certain to receive it and any bidder who values the object at its lowest possible level has an expected payment of zero (Milgrom and Weber, 1982 theorem 0). The expected value of the item to the bidder holding the highest of n values is, by definition, V_n:

$$V_n = \int_{\Sigma v} v n f(v) F(v)^{n-1} dv.$$

Since expected payment equals the expected value of the second-order statistic, we have

$$W_n = \int_{\Sigma v} v n(n-1) f(v)[1-F(v)] F(v)^{n-2} dv$$

$$= n \int_{\Sigma v} v(n-1) f(v) F(v)^{n-2} dv - (n-1)\int_{\Sigma v} v n f(v) F(v)^{n-1} dv$$

$$= nV_{n-1} - (n-1)V_n$$

which implies

$$(V_n - W_n) = n(V_n - V_{n-1}).$$

where the last equality is due to

$$q(N-n)p_n/(1-q) = (n+1)p_{n+1}.$$

We claim that (19) vanishes since $V_0 = 0$.[17]

COROLLARY: *In IPV auctions with entry, the seller should not resort to reservation prices, even if he cannot charge entry fees.*

This finding concurs with McAfee and McMillan (1987b) and Engelbrecht-Wiggans (1991) but contradicts a major theorem of the fixed-n IPV literature: that a distortive reservation price always helps the seller (cf. Jean-Jacques Laffont and Maskin, 1980; Milton Harris and Artur Raviv, 1981; Myerson, 1981; Riley and Samuelson, 1981). The intuition behind our result, first suggested by Engelbrecht-Wiggans (1991), lies in (18). When one bidder joins an IPV auction when $n-1$ are present, the social gain is simply $(V_n - V_{n-1} - c)$, whereas the individual bidder's gain is $(V_n - W_n)/n - c$. Due to (18), the two always coincide. No matter how many rivals are expected, in IPV auctions the private gain from further entry corresponds exactly to that of society (and the seller). Since the seller has no reason in IPV auctions to discourage entry (unlike the CV case), the reservation price loses its only appeal.[18]

How can this be reconciled with our understanding of reservation prices in fixed-n models? Our model reduces to the fixed-n case when $n^* > N$. Therein, each bidder has positive expected profits (assuming $e = 0$) even in equilibrium. Social welfare would surely fall if any bidder elected to exit, but since each bidder has a positive margin of profit, the seller may safely try to appropriate some part of it without triggering exit. Thus, the optimality of reservation prices when n is fixed.

2. *Affiliated Private Values.*—How robust is the finding of optimal entry within the private-values paradigm? Since we found incentives for excessive entry in the CV case, where strong affiliation exists, there may be a presumption that extensions of Proposition 6 (optimal entry) could be defeated by strong affiliation among private values.[19] The analogy is misleading. Excessive entry may arise in the APV case, but it is not triggered by the degree of affiliation.

We approach the problem using a particular model of affiliated values drawn from previous literature (see Milgrom, 1981; Engelbrecht-Wiggans, 1991; Levin and Smith, 1993). Let z represent an unknown parameter of the distribution that generates private values, and assume that all bidders hold identical prior beliefs regarding this parameter, represented by the density $g(z)$. To create affiliation, we imagine that the private values held by the bidders $\{x_1, \dots, x_n\}$ are conditionally independent and identically distributed, given z.

With this formulation, (18) holds also for the affiliated model, which means that private and social gains from entry still coincide, but only for second-price mechanisms.

PROPOSITION 7: *If private values are affiliated (conditionally independently and identically distributed), then free entry with no reservation price is optimal for society and the seller under a second-price mechanism, but excessive under a first-price mechanism.*

PROOF:

We first confirm the validity of (18) under a second-price mechanism. The expected value of the item to the bidder holding the

[17] The second derivative of $S(q, \Omega)$ is negative at the point $e = 0$, which ensures at least a local maximum. It would be sufficient to demonstrate that $S(q, \Omega)$ is quasi-concave to ensure that a global optimum has been attained.

[18] The intuition in McAfee and McMillan (1987b) is different. Sellers use reservation prices in the fixed-n literature, they argue, to extract profits from bidders at equilibrium. Since entry drives bidders' profits to zero anyway, nothing is left to be extracted by the reservation price. We show that sellers in CV auctions may want to use reservation prices anyway.

[19] We thank an anonymous referee for suggesting this line of inquiry.

highest signal is, by definition,

$$V_n = \int_{\Sigma z} \left[\int_{\Sigma v} vnf(v|z)F(v|z)^{n-1}\,dv \right] g(z)\,dz.$$

But the winning bidder in a second-price auction pays the amount of the second-highest signal, which allows us to write

$$\begin{aligned} W_n &= \int_{\Sigma z} \left[\int_{\Sigma v} vn(n-1)f(v|z) \right. \\ &\qquad \left. \times [1-F(v|z)]F(v|z)^{n-2}\,dv \right] g(z)\,dz \\ &= n\int_{\Sigma z} \left[\int_{\Sigma v} v(n-1)f(v|z)F(v|z)^{n-2}\,dv \right] g(z)\,dz \\ &\quad -(n-1)\int_{\Sigma z} \left[\int_{\Sigma v} vnf(v|z)F(v|z)^{n-1}\,dv \right] g(z)\,dz \\ &= nV_{n-1} - (n-1)V_n \end{aligned}$$

which extends (18) under second-price mechanisms to the APV case. This part of the proof can then be completed by following the same steps as in IPV. Regarding the second part, entry under first-price mechanisms is greater since for fixed n the seller prefers second-price to first-price mechanisms (see Milgrom and Weber, 1982 theorem 15), which means that the expected profit of each bidder must be higher under first-price mechanisms. By Lemma 1, first-price mechanisms must then induce larger q.

To summarize, whether values are common or private, affiliation ensures that second-price mechanisms induce less entry than first-price mechanisms. In the CV case, free entry is excessive under both mechanisms, but less so under second-price mechanisms. In the private-values case, free entry is optimal under second-price mechanisms but excessive under first-price mechanisms.

II. Market Thickness and Coordination Costs

What is the optimal number of potential bidders? Is more necessarily better? And does the seller's interest match that of society? The presumption of previous studies, that superfluous bidders simply stay out and become irrelevant to the outcome, overlooks coordination problems that arise when "too many" participants are involved. Our approach permits this effect to be investigated, with new and surprising results.

Recall that n^* is the point of transition between pure and mixed entry strategies. When $N > n^*$, symmetric equilibrium requires each potential bidder to enter with probability $q < 1$. Since each entry decision is taken independently, the number of entrants can range from 0 to N. Many of those realizations are unfavorable for the seller and society. The question is whether the weight of unfavorable outcomes varies systematically with N. We will show that it does and will demonstrate that limiting the number of potential entrants reduces coordination costs and benefits the seller and society.

To begin, we note that the expected number of bidders induced by a given mechanism varies ambiguously with N and is by itself an incomplete indicator of seller or social welfare. Since any increase in N is typically offset by a reduction in q^*,[20] the net change in the expected number ($\bar{n} = qN$) may be slight and ambiguous in sign. Even when the effects are exactly offsetting, social welfare would be affected since the variance of n, given by $(1-q)\bar{n}$, still rises (with $\bar{n}$ fixed) as q falls. The presence of more potential bidders increases the likelihood of extreme outcomes. The potential cost of this is easy to demonstrate in the case of CV auctions.

PROPOSITION 8: *As the number of potential bidders increases beyond n^* in CV auctions, the probability of no entry also increases if the seller is using an optimal mechanism.*

[20] It is easy to show that a sufficient condition for $\partial q/\partial N < 0$ is that $V_n - W_n$ be decreasing in n. Details are omitted for brevity.

PROOF:

Let q_N^s be the optimal probability of entry when the number of potential bidders is N. Thus, $(1-q_N^s)^N$ is the probability of no entry. From (9), $(1-q_N^s)^{N-1}=c/V$; thus q_N^s declines with N and $(1-q_N^s)^N = (1-q_N^s)(c/V)$ must increase with N.

When there are many potential entrants, not only is the number of bidders stochastic and therefore too small or too large on occasion; for a given mechanism the number can be too small or too large on average. One relevant benchmark is provided by n^*, which corresponds to the socially optimal number of bidders for a broad class of private-value auctions.[21] In many examples we find that $\bar{n}$ can be made either smaller or larger than n^* simply by varying the magnitude of entry costs.[22]

The coordination problems we have discussed impose specific costs on society that stem from mixed entry strategies. Thus, the transition to stochastic entry which occurs at $N=n^*$ has broad welfare implications.

PROPOSITION 9: *The level of social welfare generated by optimal auctions decreases monotonically as N increases beyond n^*.*

PROOF:

Assume $N>n^*$, which implies that $q^s = q^s(N)<1$. Suppose provisionally that the number of potential entrants drops by 1 to $N-1$, but that each remaining member continues to use $q^s(N)$. From (5), the impact on social welfare is:

$$(20)\quad \Delta S = \sum_{n=1}^{N} p_n V_n - \sum_{n=1}^{N-1} \phi_n V_n - q^s c$$

where

$$\phi_n = [(N-1)!/n!(N-1-n)!](q^s)^n(1-q^s)^{N-1-n} = p_n(N-n)/N(1-q^s).$$

After substituting for ϕ_n in (20) and simplifying, we obtain

$$(21)\quad \Delta S = \sum_{n=1}^{N} p_n V_n \frac{(n-q^sN)}{(1-q^s)N} - q^s c = (q^s/N)\,\partial S/\partial q = 0$$

where we have used (16) and the fact that $\partial S/\partial q = 0$ at q^s. Thus, dropping one potential entrant while holding entry probabilities constant leaves social welfare unchanged.[23] If we now relax the constraint on q^s, it will adjust to the new social optimum based on $N-1$, and by construction S must increase.

The coordination problem grows with the number of potential bidders. Proposition 9 demonstrates that each step taken to eliminate the source of the coordination problem (successively reducing N) enhances social welfare.

COROLLARY: *The expected revenue of any seller who uses his optimal mechanism increases monotonically as the number of potential bidders decreases toward n^*.*

Since a seller's optimal mechanism reproduces the social optimum, his preference for N must exactly match that of society. If the number of potential bidders is too high, the seller can mitigate the coordination problem *ex ante* by restricting the number

[21] See footnote 16 and our account of second-price APV auctions.

[22] Specific examples are omitted to conserve space, but details are available from the authors upon request.

[23] Let q^s represent the socially optimal entry probability for N potential entrants and consider the decision of the Nth, when $N-1$ are already using q^s. The gains from the Nth party entering must exactly offset the costs of his entering. If this were not true, society would gain by his being either definitely in or definitely out, and it could not be optimal for him to use q^s also. Since the expected gains from his entering equal the expected costs, nothing is lost if he is eliminated from the market, so long as the remaining $N-1$ members continue to use q^s.

of potential entrants qualified to bid.[24] Whether the seller (or society) would ever gain by suppressing the number of potential entrants below n^* depends on the environment. By definition the Nth entrant receives nonnegative gains when $N \leq n^*$. Since that gain corresponds exactly to the social gain from entry in IPV auctions (see above), by pushing N below n^* the seller would reduce social welfare and his expected revenue. In CV auctions, social gains are zero (and therefore smaller than social costs) for all $n \geq 2$; thus reductions beyond n^* are beneficial.

Proposition 9 and its corollary cast new light on the influence of market thickness, which in our model can be defined as $\theta = N/n^*$. The thicker the market, the larger is the number of redundant bidders, and the greater are the penalties that society and the seller pay for permitting randomized entry. Demand and supply factors influence market thickness separately and therefore exert distinct effects on the magnitude of coordination costs. For example, holding the number of potential bidders (demand side) constant, any change in the nature of the item (supply side) that increases the relative magnitude of c will reduce n^* and thereby raise θ. Thus, sellers (or procurers) who deal in unique or technical items that are by nature relatively hard to evaluate are more likely to gain by restricting the number of qualified bidders than sellers (or procurers) who deal in items whose value is straightforward.

III. Concluding Remarks

Most of the previous auction literature assumes that the number of bidders is given. Endogenizing entry is a natural step, one that is necessary to complete our understanding of how auction design affects performance. We have introduced a model of induced entry that differs from previous work in several important ways. First, we maintain symmetry throughout, which limits potential entrants to mixed entry strategies. One consequence is that the number of bidders is stochastic—even in equilibrium—with distribution determined endogenously by characteristics of the seller's mechanism and other market factors. This can account for the variability in number of bidders frequently observed within repeated auctions of similar items, even the failure of any bidders to participate. Despite this new element, we are able to show that many revenue-equivalence and ranking results from the fixed-n literature generalize to auctions with entry. When values are affiliated, for example, second-price mechanisms induce less entry than first-price mechanisms, which always works to the seller's advantage.

Second, our treatment of induced entry generates new and unexpected insights. Reservation prices are seen strictly as instruments that discourage entry, which may or may not be beneficial depending on the environment. Third, and perhaps most interesting, is the realization that since stochastic entry creates coordination problems whose cost mounts as the thickness of the market increases, the mere existence of more potential bidders can impose costs on society and the seller, whether they are actually bidding or not.

These insights should direct empirical study of auction markets along entirely new lines of research. For example, holding the number of potential bidders and the value of the item constant, we have shown that coordination problems impose greater costs on the sellers of items that are complex, unique, or otherwise costly to evaluate. Those sellers have the greatest incentive to adopt institutions designed to prequalify bidders and limit participation. Conversely, holding complexity (or entry costs) constant, any sudden decrease in the value of an offering (like the plunge in value of offshore

[24] In CV auctions, this result relies heavily on the seller's ability to charge entry fees, which are part of his optimal mechanism. In IPV auctions, the ability to charge entry fees is almost inconsequential, since they are not part of the optimal mechanism when $N > n^*$. However, the seller would need them to capture the small surplus that may still accrue to bidders when $N = n^*$. Thus, sellers who cannot charge entry fees in IPV auctions might prefer $n^* + 1$ potential bidders to n^*, but never more.

oil leases that occurred after 1981) reduces the number of bidders that fit profitably in the auction, which simultaneously increases market thickness, the size of welfare losses that stem from stochastic entry, and the seller's incentive to limit participation.

Appendix A: Comparative Statics of Entry

Using (2), the differential of (6) with respect to e and q can be written as

$$\text{(A1)}\quad \frac{1}{q(1-q)} \times\left\{\sum_{k=0}^{N-1}\theta_k[k-q(N-1)]T_{k+1}(R_{k+1}) \times\left[\frac{V_{k+1}-W_{k+1}}{k+1}\right]\right\}dq-de=0$$

where θ_k represents the probability that a bidder who does elect to enter will meet exactly k rivals:

$$\theta_k=\binom{N-1}{k}q^k(1-q)^{N-1-k}.$$

The summation over k in (A1) is simply the covariance, denoted Cov_m, between a bidder's expected profit, given that he enters, and the number of rivals he faces. Solving for $\partial q/\partial e$ yields:

$$\text{(A2)}\quad \partial q/\partial e=\frac{q(1-q)}{\mathrm{Cov}_m}.$$

Taking differentials with respect to the other variables yields similar results:

$$\text{(A3)}\quad \partial q/\partial c=\frac{q(1-q)}{\mathrm{Cov}_m}$$

$$\text{(A4)}\quad \partial q/\partial(V_j-W_j)=-\frac{p_jT_j(R_j)(1-q)}{N\,\mathrm{Cov}_m}.$$

This establishes Lemma 1. Using similar steps we also obtain

$$\text{(A5)}\quad \partial q/\partial R_j =-\frac{p_j(1-q)\partial[T_j(R_j)(V_j-W_j)]/\partial R_j}{N\,\mathrm{Cov}_m}.$$

As shown in Appendix B, $\partial[T_j(R_j)V_j]/\partial R_j$ vanishes at $R=0$, and so (A5) evaluated at $R=0$ and $e=0$ yields (13).

Appendix B: Evaluation of $\partial T_n(R_n)V_n/\partial R_n$

To establish $\partial S(q,\Omega)/\partial R_n\le 0$, it is sufficient by reference to (5) to show: $\partial T_n(R_n)V_n/\partial R_n\le 0$ for all $R_n>0$. This will be true for any auction in which the n actual bidders receive private information (values, estimates, etc.) distributed over a finite interval $[0,\bar{x}]$. We shall denote the ith bidder's information by x_i with density $f(x_i)$, and define $x_1^n=\max\{x_i\}$ with density $h(x_1^n)$. Also, for all $R_n\in[0,E(V|x_1^n=\bar{x})]$, define $g_n(R_n)=\inf z\ge 0$ such that $E[V_n|x_1^n=z]\ge R_n$. It is then clear that if and only if $x_1^n\ge g_n(R_n)$ will a trade occur since the highest signal (which maps to the highest bid due to the monotonicity of the bidding function) must be at least $g_n(R_n)$ to generate a bid equal to or greater than the reservation price. Consequently, we can write

$$\text{(B1)}\quad T_n(R_n)V_n =\int_{g_n(R_n)}^{\bar{x}}\left[\int_{\Sigma V_n|x_1^n}V_nf(V_n|x_1^n)\,dV_n\right]h(x_1^n)\,dx_1^n$$

which, after differentiating, gives

$$\text{(B2)}\quad \partial T_n(R_n)V_n/\partial R_n =-g_n'(R_n)h(g_n(R_n)) \times\int_{\Sigma V_n|g_n(R_n)}V_nf(V_n|g_n(R_n))\,dV_n \le 0$$

where the last (weak) inequality is by the fact that every term in (B2) is nonnegative except the leading sign. Evaluated at $R_n>0$, very often (B2) will be strictly negative, although we require only a weak inequality to establish Proposition 1.

We now show that $\partial T_n(R_n)V_n/\partial R_n$ is identically zero when evaluated at $R_n=0$, as claimed in Appendix A. Consider first the IPV case, where $V_n = x_1^n$ when $R_n = 0$, which implies that $g_n(R_n) \equiv R_n$ and $g_n'(\cdot) = 1$. Moreover, $F(V_n | x_1^n) = 0$ if $V_n < x_1^n$ and $F(V_n | x_1^n) = 1$ elsewhere. Consequently,

$$\int_{\Sigma V_n | g_n(R_n)} V_n f(V_n | g_n(R_n))\, dV_n$$

$$= \int_{\Sigma V_n | R_n} V_n f(V_n | R_n)\, dV_n = R_n$$

where the last equality comes from integration by parts. Thus, (B2) gives:

$$\frac{\partial T_n(R_n) V_n}{\partial R_n} = -h(g_n(R_n)) g_n(R_n)$$

$$= -h(g_n(R_n)) R_n = 0$$

at $R_n = 0$.

Next consider the CV case. If

$$\int_{\Sigma V_n | g_n(R_n)} V_n\, dF(V_n | g_n(R_n = 0)) = 0$$

then (B2) is zero, and we are done. Assume the contrary:

$$\text{(B3)} \quad \int_{\Sigma V_n | g_n(R_n)} V_n\, dF(V_n | g_n(R_n = 0)) = \varepsilon > 0.$$

Our definition of $g_n(R_n)$ and (B3) together imply that $g_n(t) \equiv 0$ for all $t \in [0, \varepsilon]$ and $g_n'(0) = 0$ (i.e., the level of filtering does not change as R_j rises above zero since it is not binding there on any bidder). Finally, $g_n'(0) = 0$ causes (B2) to vanish.

Next, we demonstrate that $\partial T_1(R_1) W_1 / \partial R_1 > 0$ when evaluated at $R = 0$, $e = 0$. Start by writing $\partial T_j(R_j) W_j / \partial R_j$ as $W_j \partial T_j(R_j)/\partial R_j + T_j(R_j) \partial W_j / \partial R_j$. For $j = 1$, we have $W_1 = 0$ and $T_1(0) = 1$. Thus, we need only to sign $\partial W_1 / \partial R_1$. W_1 has a particularly simple form:

$$\text{(B4)} \qquad W_1 = \Pr[x \geq g_1(R_1)] R_1$$

where x is the signal held by the lone bidder and $g_1(R_1)$ is the screening level associated with the seller's reservation price. We differentiate (B4) to obtain

$$\text{(B5)} \quad \partial W_1 / \partial R_1 = \Pr[x \geq g_1(R_1)] \partial R_1 / \partial R_1$$

$$+ R_1 \partial \Pr[x \geq g_1(R_1)] / \partial R_1$$

$$= \Pr[x \geq g_1(0)] = 1$$

when evaluated at $R = 0$. Thus,

$$\partial T_1(R_1) W_1 / \partial R_1 > 0.$$

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