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A city is not a semilattice either

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Abstract. In 1965 Christopher Alexander took the original step of analysing the city in graph theoretical terms and concluded that its historical or natural form is a semilattice and that urban planners of the future should adhere to this model. The idea was well received in architectural circles and has passed without serious challenge. In this paper, the value of such analysis is once again emphasized, although some of Alexander's arguments and his conclusions are refuted. Beginning with an exposition of the relationship between the graph theoretical concept of a tree, and the representation of a tree by a family of sets, we present a mathematical definition of a semilattice and discuss the 'points' and 'lines' of a graph in terms of a city, concluding that it is neither a tree nor a semilattice. This clears the ground for future graphical analysis. It seems that even general structural configurations, such as graphs or digraphs with certain specified properties, will fail to characterize a city, whose complexity, at this stage, may well continue to be understood more readily through negative rather than positive descriptions.

Purpose

In his article "A city is not a tree" Christopher Alexander (1965) made a significant contribution to urban studies by analyzing the City in a novel fashion. His concern was to find the 'ordering principle', or animating character, of historic cities. This involved trying to pinpoint the discrepancy between 'natural' cities, those which had arisen spontaneously, and 'artificial' cities, those which had been planned from scratch. In other words, he was seeking to circumscribe that 'inner nature' which contemporary planners have often failed to incorporate within their designs. For this purpose he applied graph theoretical analysis to the study of urban systems in the hope of enhancing the understanding of the design process. Our article, while in no way wishing to undermine the originality of his approach and the valuable insights he gave by raising the question in this manner, takes issue with his principal conclusion that a city is a semilattice, as well as with some of his other assertions.

Critique of 'A city is not a tree'

We may begin by agreeing with Alexander that the mind has difficulty in grasping complex realities *in toto* because it operates by a rational process which could be depicted as a tree. This 'tree' is an abstract mathematical concept taken from graph theory and will be discussed more precisely below. As regards a city two factors are involved: its concrete complexity and our abstract understanding of that complexity. At times Alexander appears to confuse the two for he says in one place that "The tree of my title... is the name for a pattern of thought" (1965, page 58), and in another "It is the name for an abstract structure" (1966, page 47). However, he goes on to draw a diagram of the city of Cambridge, which is clearly intended to represent an actual and natural entity. A blueprint certainly originates in the mind, but it does not represent the series of logical steps which may have given it birth; rather, it

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depicts some *thing* whether potential or actual. The dichotomous nature of discursive thought[1] is fundamental to rational intellect and cannot be changed. What planners can do is to improve their understanding of the intricacies of civic phenomena and also approach the design process in a more intuitive way.

Secondly, whereas Alexander is obviously sensitive to social needs, his approach to civic problems is, notwithstanding, rather materialistic. Obviously planners must study cities primarily from a *physical* standpoint and for this reason it is understandable that *Design* (1966, page 46) should be so interested in his work, for they say: "The principles he ... describes, and the analytic methods he adopts, are applicable at all levels of design". Nevertheless each designer should have an empathy for *society* and it is perhaps the lack of such feeling that distinguishes modern architects from those of the past.

Those who wish to produce buildings of beauty often look to the past for examples they may emulate. Whereas it is certainly justifiable to derive inspiration from historical forms, we cannot return to, or merely repeat, the monuments of another civilization. Alexander (1966, page 47) is aware of this and he states that our task is to search "for the abstract ordering principle which the towns of the past happened to have, and which our modern conceptions of the city have not yet found". He continues by saying that contemporary designers "fail to put new life into the city, because they merely imitate the appearance of the old, its concrete substance: they fail to unearth its inner nature".

This 'inner nature' cannot be found in the physical aspect of a city which, whether spontaneous or planned, is an artifact and as such is only natural in a qualified sense. It must therefore be found in the other element of cities, the citizens themselves. Many architects have emphasized that 'the city is people', but it is surely more than this: it is civic society animated by its culture, which finds outward expression in urban forms characteristic of a given civilization.

It will be a vain quest to look for an unchanging principle in artifacts such as buildings and roads which are necessarily transient. The only constant is a certain structure in civic society persisting beneath the ever changing cultural patterns that give it life (Rockey, 1973). Alexander neglects this human aspect though he does criticize plans for not corresponding to social realities.

One's philosophy of civic design will be determined by the things that are taken to be fundamental or, as Alexander (1966, page 48) puts it: "whatever picture of the city someone has is defined precisely by the subsets he sees as units". For him these units are physical fixed entities such as street furniture, plus the human relationship to these, which together form a dynamic system. Such a conception may be challenged on two counts. First, the units he selects are not peculiar to the city and hence cannot lead to its specific principle or nature. And second, the fixity of one part of his system minimizes the essential dynamism of city life, which comes not from a man's relationship towards an inanimate object, but from his relationship with other citizens. It is a two-way process that cannot always be depicted by a semilattice. We agree that the fixed part of his system (1966, page 48) is of special interest to the designer, but only because it disposes towards human interaction by facilitating or hindering social exchange. It must be conceded that personal meetings do take place at such units as traffic lights and newsstands but, whereas these are an

(1) Its tree-like nature may be demonstrated by graphing a series of syllogisms. Even spurious arguments are formed in this way, as the following example shows:

Trees are patterns of thought,
But the graphs of artificial cities are trees,
Therefore the graphs of artificial cities are patterns of thought.

interesting aspect of civic life, they are primarily chance occurrences, of lesser importance, and therefore incidental to the argument.

Our major objection to Alexander's thesis is his contention (1966, page 54) that the "natural structure" of cities "is in every case a semi-lattice" and it is with this assertion that the rest of our article is chiefly concerned. To exemplify these criticisms it is necessary to understand precisely what the mathematical terms 'tree' and 'semilattice' mean.

Graphical explanation of trees

The title of Alexander's article places his topic squarely within graph theory, yet he introduces another branch of mathematics, namely set theory, to support his argument. Furthermore he tries to induce a mathematical theory of the city from a certain number of particular and limited examples.

We prefer to begin by understanding what 'tree' signifies and then see how this may depict civic and other urban phenomena. Alexander defines trees in terms of sets and hence bypasses graph theory and the usefulness it brings to bear on structural models. Unfortunately his use of the logically equivalent set theoretical formulation of trees, and later of semilattices, avoids the natural and intuitively simple and meaningful formulation of these intrinsically structural configurations in terms of graph theory. Further, and more important for empirical applications, the theorems of graph theory are thereby overlooked. Examples of such applications are given in the book by Harary et al (1965), which will also be useful in clarifying many of the concepts presented below.

We now develop a self-contained elementary exposition of the relevant concepts from graph theory, so that the reader will not have to refer to the mathematical literature.

By definition, a tree is a *connected graph* without *cycles*. What is the meaning of these three words? Space does not permit us to develop an axiomatic treatment of these concepts; the subject of trees is presented more deeply elsewhere (Harary, 1969, chapter 4). Hence we shall illustrate the answer intuitively, rather than rigorously, by drawing the eleven different graphs having four points. These will show clearly the distinction between connected and disconnected graphs.

For purposes of referring to these eleven graphs, we shall read figure 1 in columns from left to right. Thus the first five graphs are *disconnected*, as at least one point or line of each is isolated, whereas the remaining six are *connected*. Technically the first graph is called *totally disconnected* and the last one is called *complete*. Between these two extremes lie the various other possibilities for graphs with a given number of points. Formally we define a *graph* as consisting of a finite nonempty set V of 'vertices' or 'points' together with at most one 'line' joining each pair of distinct points.

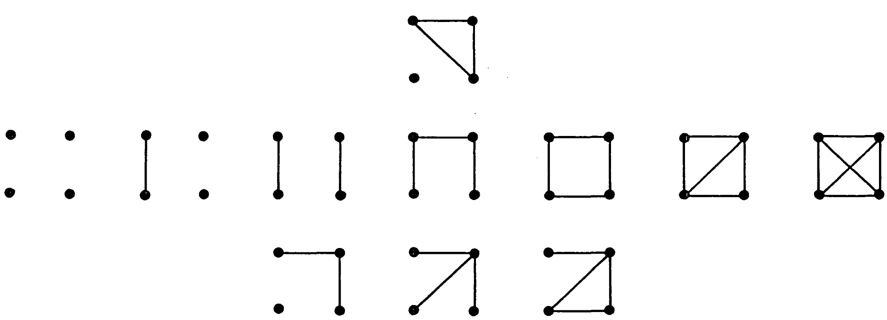

Figure 1. The graphs with four points.

Cycles are drawn like polygons (figure 2) and indeed some graph theorists call them by that name.

With these concepts clearly understood, the formal definition of a *tree* as a connected graph with no cycles makes sense. We can now see that in figure 1 there are just two trees among the 4-point graphs, namely the sixth and seventh. These are simultaneously the last two graphs with three lines and the first two of the connected graphs. If, however, we consider 5-point graphs we find that there are now three trees, as is shown in figure 3.

Alexander presents us with a different kind of tree: he refers to a more complicated form 'oriented from a point', as depicted in figure 4. The difference between figures 4a and 4b is formal: in the former, arrows indicate the direction of every line from the top point, 1, whereas in the latter this direction is always understood to be oriented downwards. Strictly speaking, figure 4 shows a *rooted tree*, in which one of the points (here point 1) is distinguished from the others. In figure 4a this distinction is seen by the fact that all lines are oriented away from the root (and hence downward) whereas in figure 4b the root point is encircled.

Since Alexander introduces the question of sets, we will show how trees and set patterns interrelate. The unique pattern of sets equivalent to the rooted tree of figure 4 is shown in figure 5. The set pattern of figure 5 is arrived at in the following way:

The root point 1 at the top of figure 4 is represented by drawing a large ellipse and labelling it 1, as in figure 5. The root of the tree then becomes the enclosing ellipse of the family of sets.

Points 2 and 3 of figure 4, which are adjacent to point 1, are represented by sets 2 and 3 within the confines of ellipse 1.

The remaining sets 4, 5, 6, and 7 are located in a similar fashion.

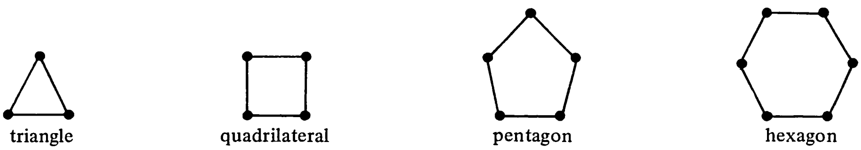

Figure 2. The four smallest cycles.

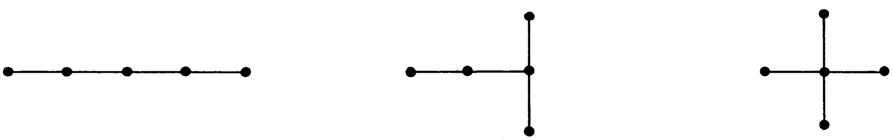

Figure 3. The trees with five points.

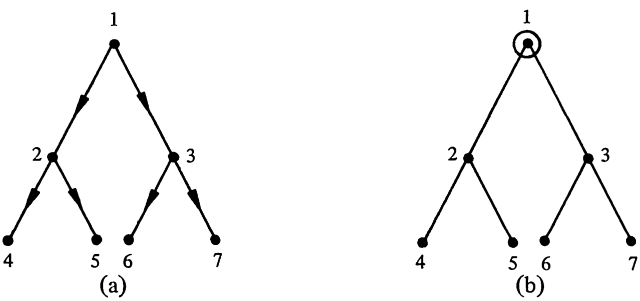

Figure 4. Two depictions of the same rooted tree.

It will be noted that by such means one can pass *from* a tree *to* a set pattern. It is possible, however, to reverse the process and construct a tree from a set pattern. Alexander adopts the latter course, which we criticize as reversing the proper mathematical process since it introduces graphs artificially as patterns of sets, thereby losing the intuitive advantages and structural concepts provided by graph theory. This further complicates the issue by introducing an unnecessary medium between a graph and the phenomenon it represents.

It is obvious that some road systems may be depicted by trees, as can organizational charts showing lines of authority, and so forth. But this is not the case for a phenomenon so complicated as a "living city", which Alexander (1966, page 51) contends "*is and needs to be a semi-lattice*" (our italics). Before this proposition can be understood, much less argued, one must know just what a semilattice is.

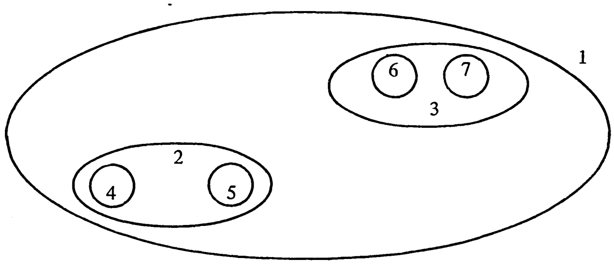

Figure 5. A representation of the tree of figure 4 by a family of sets.

Graphical explanation of semilattices

In order to define a semilattice it is best to explain the concept of a mathematical 'lattice' first, and then reduce the two essential axioms by one, thus getting 'half a lattice' or a 'semilattice'.

Mathematically a *lattice* may be defined in a formal axiomatic manner as *a partially ordered set of points in which every two points have a least upper bound* (LUB) *and a greatest lower bound* (GLB). Using the standard terminology of Birkhoff (1967), we may say that a *semilattice* is *a partially ordered set of points in which every two points have a* LUB. We should point out that the presence of a GLB is deliberately excluded from this definition. It therefore follows that every lattice is a semilattice but not vice versa. In this sense, a semilattice is a more general mathematical structure than a lattice. Every rooted tree (figure 4) is likewise a semilattice, but the converse is not true.

We must now explain each element in the definition of a lattice. A *partially ordered set* consists of a set V of vertices or *points* and a binary relation R (which is a set of ordered pairs u, v, where both u and v are points in V) which satisfy the following three properties, in which $u\mathrm{R}v$ means that u is in the relation R to v:

1. The relation is *irreflexive*: no point is in relation R to itself.
2. The relation is *asymmetric*: if $u\mathrm{R}v$, then v is *not* in the relation R to u.
3. The relation is *transitive*: for every three distinct points u, v, w, if $u\mathrm{R}v$ and $v\mathrm{R}w$, then $u\mathrm{R}w$.

For a detailed explanation of these three properties of relations with illustrations, see Harary et al (1965, chapter 1).

In these terms a *directed graph*, or more briefly a *digraph*, D, is simply defined as an irreflexive relation. Thus D may or may not be symmetric and also is not necessarily transitive. An *oriented graph* is an asymmetric digraph; it has no symmetric pairs of arcs joining the same two points.

The discussion of the other two elements, namely the LUB and GLB, will be facilitated by introducing a structural concept. A *directed path* from one point i to another point j of a digraph is illustrated in figure 6. In this case we say that there is a sequence of directed lines, also called *arcs*, and that point j is *reachable from* point i. Note that all points of a path are taken as distinct.

It is instructive to compare the three types of structure: lattice, semilattice, and oriented graph, all of which are special classes of digraphs, and this is done in figure 7.

It can be seen that the LUB of points u and v in figure 7 is point w because a directed path exists from w to u and from w to v, making it an *upper* bound. Further, every other point that can reach both u and v, can also reach w, making it a *least* upper bound. Similarly, in the tree of figure 4 the LUB of points 4 and 5 is point 2 because this point can reach both 4 and 5, and every other point (such as point 1) that can reach 4 and 5 can also reach 2. It is therefore evident that the LUB of points 3 and 4 is 1.

On the other hand, the GLB of the two points u and v is defined analogously by directional duality, that is, the considerations resulting from reversing all the orientations on the arcs. For example, point x of figure 7a is the GLB of u and v, but in figure 7b the points u and v do not have a GLB. The lack of a GLB in this latter instance makes it an 'upper semilattice' since every pair of points does have a LUB. If this graph were inverted, it would then become a 'lower semilattice'.

In the oriented graph of figure 7c, the points u and v have neither a LUB nor a GLB. Hence it is neither a lattice nor a semilattice. One example of this is to be found in the gridiron street plan, often called the 'Manhattan Plan', common to many cities. Alexander's semilattice model is inadequate to realize even this simple physical aspect of a city. By and large, a network of intersecting roads forms a graphical structure more general than a semilattice.

The lack of a GLB is the distinguishing factor between a lattice and a semilattice. It will be noted that Alexander (1966, page 49) says: "A collection of sets forms a semi-lattice if and only if, when two overlapping sets belong to the collection, then the set of elements common to both also belongs to the collection". This statement shows clearly his conception of trees and semilattices in terms of *sets*, whereas we think of these and other structures in terms of *graphs*. The advantage of our approach is that the graph theoretical definitions of a semilattice, a tree, or any directed graph, readily lend themselves to potentially more searching analyses. This is due to the geometric representation of a graph, which results in an enhanced understanding of the abstract mathematical concepts and thereby enables the nonmathematician to make more effective use of the model.

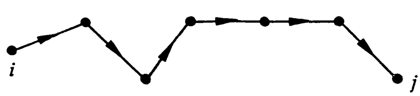

Figure 6. A directed path.

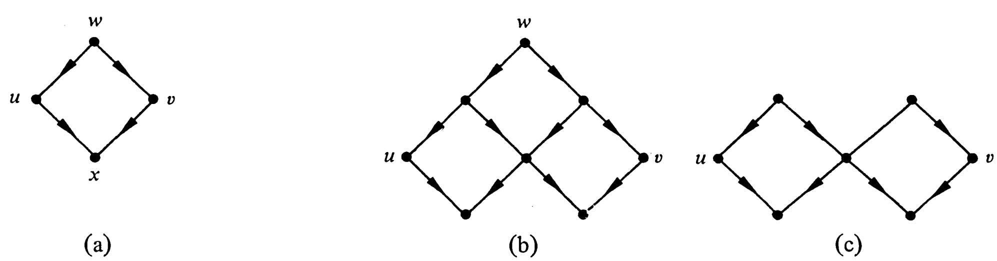

Figure 7. (a) Lattice, (b) semilattice, and (c) oriented graph.

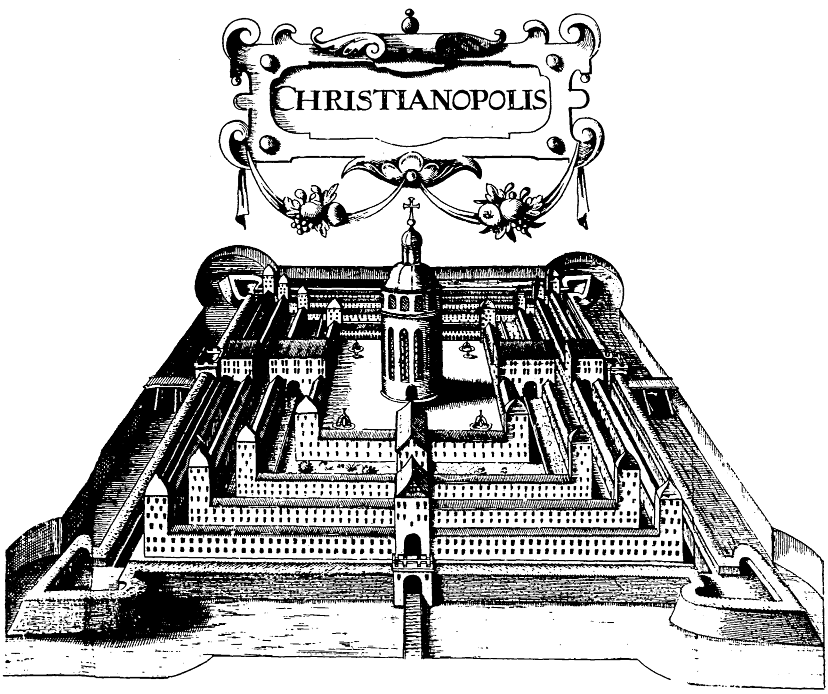

Figure 8. Christianopolis. (Reproduced by kind permission of the Curators of the Bodleian Library.)

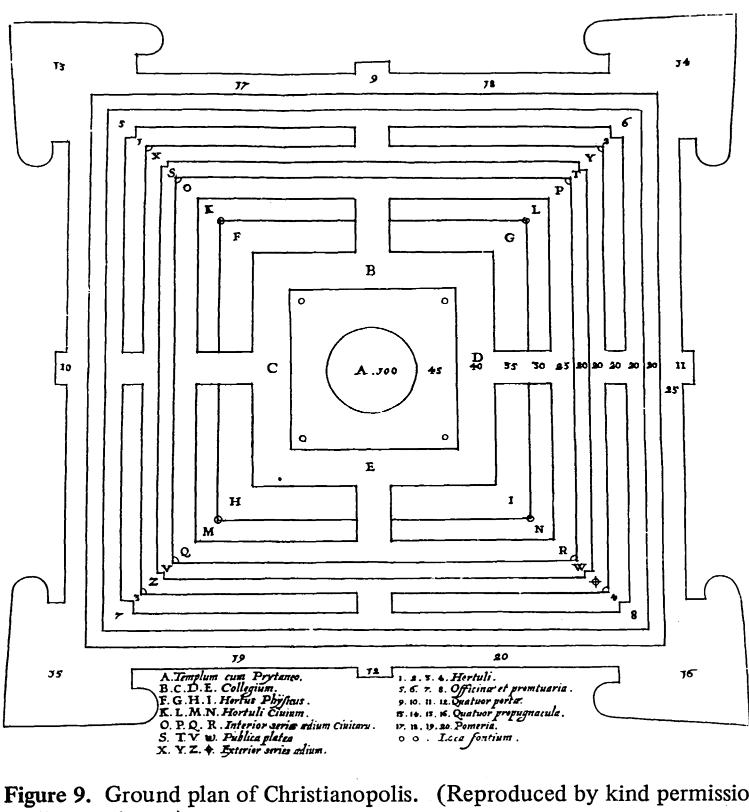

Figure 9. Ground plan of Christianopolis. (Reproduced by kind permission of the Curators of the Bodleian Library.)

Application of graph theory to a Utopian model

In order to demonstrate that it is possible to pass directly from plans to graphs, without the medium of sets, we take as an example Andreae's *Christianopolis* (1619). This is one of the many Utopian plans that have been forwarded over the centuries and it is interesting in that it represents a midpoint between Alexander's 'patterns of thought' and an actual city. It also shows how an apparently rigid plan need not be as constraining as might at first appear.

The sketches themselves (figures 8 and 9) were not intended to be plans for an actual city, but were ideas expressed symbolically and pictorially. By this means Andreae indicated that his ideal city should have a central and dominant religious and political area, should be protected by a wall, should have facilities in a certain relationship to one another, should have enclosures opening off main axes, and so forth.

In our structural model of this ideal city (figure 10), the *points* are the corridor routes or spaces of figure 9 and the *lines* are direct connections between these. The resulting *graph* is a semilattice because any two of the points have a LUB. It is this semilattice structure that Alexander advocates as an alternative to trees when he asserts that tree-like structural simplicity "is crippling our conceptions of the city" (1966, page 49). We agree, but go further in holding that a semilattice also is an inadequate representation.

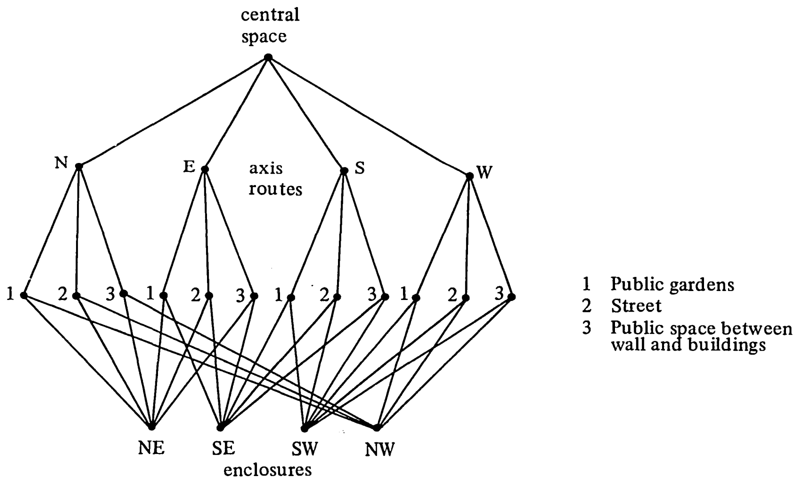

Figure 10. Graph of pedestrian routes from plan of Christianopolis.

Graphical representation of a city according to Alexander

We have already referred to the inconsistencies of Alexander's arguments in our critique; it now remains for the mathematical aspects of his graphs to be examined. Nowhere in his article does he give a consistent statement as to what his units represent: he is trying to define graphically the essence of a city without saying what its essential entities are. We have, however, been able to isolate three possible meanings for *graphs*, *lines*, and *points*, the latter corresponding to his sets and units.

In his first example (1966, page 48) Alexander chooses as his *points* such individual items as traffic lights, newsstands, drug stores, people, and even money. These are joined by *lines* in the form of sidewalks and roads which go to make up, by their interrelationships, a semilattice digraph. Since many such points are absent in historical cities, we hold that they are inessential to civic life as such. Furthermore they are not peculiar to urban areas and therefore can never be a specifying characteristic.

In another example (1966, page 49, figure 1) Alexander uses settlement clusters as his *points* and the connecting hierarchical road system as his *lines*. Both these elements he took from the artificial cities of recent creation, and he fashioned them into *graphs* which are trees. The road system is, of course, but one limited aspect of the entire city and it may well form a tree, but it does not follow that the more important social activities taking place within the city will do so also. A further objection to this particular example is that the community he depicts does not appear to be a city.

A third example of his inconsistency is to be found (1966, page 52, figure 12) where his *points* or 'units' correspond "to different kinds of centres for a single neighbourhood". In other words these are hubs of social, educational, and service activities, which may be represented in graphical form as a semilattice. He then goes on to draw the same area (1966, page 52, figure 13) from the redevelopment plan and in this instance uses whole neighborhoods as his *points*. It is not clear what the *lines* represent but the resulting *graph* has the form of a tree.

The fact that Alexander is willing to take traffic lights, people, settlement clusters, neighborhoods, and so forth as his *points* is indicative of an inconsistent treatment throughout the article to the mathematical model at hand. We have demonstrated that he uses graphs to portray three different types of structure with vastly different empirical significance.

In order to prove that certain artificial cities are trees, Alexander draws graphs of their physical layout. However, when seeking to prove that Cambridge, a natural city, is a semilattice, he does not adhere to the physical criterion he used in the first example but introduces another factor—points of activity, such as coffee bars. This is hardly a fair comparison since what appears tree-like on plan may be quite different when considered according to its social activities. Indeed Alexander himself shows this when drawing two different graphs for the Waterloo Area (1966, page 52, figures 12 and 13).

From the foregoing it is clear that Alexander adduces no positive proof to support his thesis that a city is a semilattice and he even admits (1966, page 55) that he cannot yet show plans or sketches depicting it as such. It is surprising that he did not make a comparative analysis of two plans, one of an artificial city and the other an historic one. A greater fault, however, is his limited conception of a city as such: he implies that even from a physical standpoint there are only two possible mathematical explanations. One of these is a tree, the other a semilattice.

Basis for graphical representation of the city as a whole

There can be no argument about the extraordinary complexity of civic life; what is in question is the means by which this may be portrayed graphically. More precisely, it is the essential points or features that are so elusive. The major distinction within the city is between the buildings and streets on the one hand, and the people organized as a civic society on the other. From these two divisions other important features emerge which must all be considered if a true mathematical model of the city is to be had.

It has been shown elsewhere (Rockey, 1973) that the city is a familiarly based heterogeneous society directed to the good life, though not necessarily achieving that end. Such a society calls for a proportionately complex settlement containing facilities appropriate to satisfy the higher cultural, educational, and religious activities that it houses. These requirements are as necessary today as they were for the first cities, although the manner of their expression will obviously vary with the age, its technology, and its culture.

On the social side the economic, recreative, and educational activities of the family, village, and town must be differentiated from those that are uniquely civic.

One must delineate, first of all, a social structure in which the individual can hope to find at least some scope for the realization of his talents and desires. Secondly one must attend to the activities of the arts, the judiciary, higher learning, commerce, ceremony, and public debate which, in their quantity and quality, are special features of the good life offered by cities. Physical manifestation of these activities is to be found in monuments, law courts, theaters, public forums, communication networks, and so on. In addition there needs to be other material resources capable of supporting economically nonproductive people, whose numbers will vary with the technology of the age.

It is beyond the scope of this paper to detail these features more precisely but we must note in passing that each corresponds to a certain structure, which can be depicted graphically. We have spoken in the abstract but nevertheless we would expect to find these characteristics present to greater or lesser degree in any particular city we may choose to examine. Mathematical models abstract from the material base, and therefore the structural model obtained should be relevant to all cities in an analogous fashion to the way in which Utopian plans are universally relevant.

The proper graphical representation of a city is a problem for which no easy answer can be found. Certain aspects may be represented by a tree or semilattice. A more complicated type of structure known as a 'social network' provides more promise as a realistic mathematical model when several different types of relations are involved. Such graphs, based on the concept of multiplexity, have been most successfully used by Mitchell (1969) in a study of social anthropology to describe multistranded social networks.

It is clear that graphs, digraphs, and social networks can be useful as carefully limited mathematical models for certain structural aspects of the city. Nevertheless, one must be cautious about letting enthusiasm replace scientific demonstration.

Disclaimer

It has been pointed out by Mitchell (1969) and others that 'social networks', regarded as the superposition of various interpersonal relations (such as *influences*, *supports*, *respects*, *likes*, *dislikes*, *communicates with*, and so forth) on the same group of people, can serve as a meaningful mathematical model for certain aspects of urban life. However, it would be presumptious to claim that 'the city is a directed graph with the following properties: ...'. A real city is such a complex human institution, consisting of an enormous array of interacting social, economic, and physical phenomena, that it will probably continue to defy mathematical categorization.

We agree that Alexander has shown convincingly that a city is not a tree. It is our conviction that "A city is not a semilattice either".

References

Alexander C, 1965 "A city is not a tree" *Architectural Forum* **122** April and May
Alexander C, 1966 "A city is not a tree" *Design* **206** 47-55
Andreae J V, 1619 *Reipublicae Christianopolitanae Descriptio* (Argentorati)
Birkhoff G, 1967 *Lattice Theory* (3rd edition) *American Mathematical Society Colloquium Publications* **25** (American Mathematical Society, Providence, RI)
Design 1966 "Editorial comment on 'A city is not a tree'" *Design* **206** 46
Harary F, 1969 *Graph Theory* (Addison-Wesley, Reading, Mass)
Harary F, Norman R Z, Cartwright D, 1965 *Structural Models: An Introduction to the Theory of Directed Graphs* (John Wiley, New York)
Mitchell J C, 1969 "The concept and use of social networks" in *Social Networks in Urban Situations* Ed. J C Mitchell (Manchester University Press, Manchester) pages 1-50
Rockey J R, 1973 *The City in Definition* doctoral thesis, Angelicum University, Rome