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Multi-unit auctions with uniform prices★

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Summary. Auctions in which individuals can purchase more than one unit of the good being sold differ in striking ways from multi-unit auctions in which individuals may purchase only one unit. The uniform price auction in particular frequently yields Nash equilibria in which bidders underbid for their second unit and therefore pay very low prices for the good. This paper characterizes equilibria for the uniform price auction.

JEL Classification Numbers: D44, D82, C72.

Introduction

The recent upsurge of interest in the study of auctions is due both to the important economic role played by auctions and to their links to fundamental theoretical issues in information economics and institutional design. Research has been devoted to auctions for government procurement, for public resources – timber from national forests, offshore oil, and most recently electromagnetic frequency spectrum – and for Treasury bills. Although most of the theoretical work examines the sale of a single object, many of the most important auctions – including examples cited above – involve the simultaneous sale of multiple identical objects. One of the two common forms of simultaneous multi-unit auctions is the "uniform price auction," in which high bids win the units, but all units are sold for the same price. (In the other form, the "pay-your-bid" auction, each unit is sold at its corresponding bid.) The uniform price auction has been frequently recom-

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mended by economists because of its supposedly desirable incentive properties; the Treasury's experimental adoption of the uniform price auction was based on such recommendations.

In theoretical work on multi-unit auctions, it has generally been assumed that no bidder purchases more than one unit. Most of our intuition and almost all of the policy prescriptions have been based on this simplifying assumption. In this paper we show that dropping the assumption of single unit purchases dramatically changes the results of these models. Thus the intuition based on the bulk of the existing literature is misleading.

This paper characterizes Nash equilibria in undominated strategies for the uniform price auction in the independent private values case. We assume that bidders may purchase up to two units apiece. We provide general results concerning monotonicity of equilibrium strategies, and we show that purchasers' lower bids will be shaded strictly below their valuation of the good. We also provide some strong results with respect to symmetric auctions: As long as at least three units are being sold, the equilibrium strategy is strictly monotonic except possibly for pooling at the lowest permitted price (which we take to be zero throughout the paper).

By extending the analysis beyond the single-unit-purchase case, we discover that in many situations uniform price auctions exhibit a significant and potentially undesirable property: they tend to encourage bids of zero. We provide conditions in which zero bids arise. We also state conditions under which it is guaranteed that pooling at zero does not occur. We provide conditions under which each bidder bids zero on the second unit with probability one. We also investigate an example in which each bidder wishes to purchase three units, in order to show how the zero-bid results may generalize.

Zero-bids have several interesting consequences. In extreme circumstances they mean that the uniform price is zero in the auction. Of more practical interest is the fact that bidders bid zero (in practice, don't bid at all) on units for which they have positive value, even when bidding is costless. This suggests that we may observe fewer bids in such uniform price auctions than in other forms of multi-unit auctions, a potentially testable implication.

Our results on strict monotonicity do not apply when only two units are being auctioned. For the case of two units we provide necessary and sufficient conditions for a strategy to be a symmetric equilibrium, and we use these conditions to generate examples of pooling at zero or pooling in the interior of the space of bids. In some situations, the unique equilibrium is a discontinuous function; in other situations there are a continuum of equilibria.

Our work, along with other recent work, makes it apparent that auctions in which individuals can purchase more than one unit differ in striking ways from auctions studied before. In contrast to our paper, Back and Zender [3] focus on the common values case and constant returns. Our paper is closest to recent work by Noussair [16]. He provides conditions for an equilibrium within a more limited category of symmetric equilibria. Among our examples are equilibria for cases in which no equilibrium exists in the class he examines.[1]

The paper is organized as follows: Section 1 presents the model and an introductory example. Section 2 characterizes undominated strategies. Section 3 provides characterizations of symmetric equilibria (which are generalized to asymmetric equilibria in the Appendix).

As long as the number of bidders is at least as large as the number of objects, there can be pooling at zero. Section 4 provides results which show when pooling at zero will occur. In this section we provide simple sufficient conditions for the existence of an equilibrium in which all bidders bid truthfully on the first unit and zero on the second unit. Section 5 provides additional results for the case where two units are auctioned, including methods for generating equilibria in that case. Section 6 provides examples. The final section provides a brief discussion, including a review of the ways that allowing bidders to win more than one object makes such a fundamental change in auction behavior.

1 The model

An auctioneer is to auction M units of a good, where $M \geq 2$; each bidder wishes to purchase two units of the good. There are $N + 1$ bidders for the good, where $2(N + 1) > M$, so that not all bidders can be satisfied.

The rules of the auction are as follows: After observing his valuations, each bidder n submits two bids c_1^n and c_2^n. Without loss of generality we order the bids so that $c_1^n \geq c_2^n$. The goods are awarded to the M highest bids; and for each good received, the recipient pays an amount equal to the highest losing bid. The reserve price is zero. (In other words, if fewer than M non-negative bids are submitted, the non-negative bids win at a price of zero.[2])

All bidders maximize expected profits. We let v_1^n represent the value to player n of receiving one unit of the good and we let v_2^n represent the additional value from receiving a second unit of the good; with probability one, $v_1^n \geq v_2^n > 0$. Let $G(v_1, v_2)$ represent the distribution of an individual's pair of valuations; this distribution is identical over all bidders and each bidder's pair of bids is independent of all other bidders' bids.[3] We write the two marginal distributions as $G_1(v_1)$ and $G_2(v_2)$. We assume the marginal

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[1] Other recent examinations of multi-unit auctions with multiple purchase include Ausubel and Cramton [2], Daripa [4], Katzman [10], Pesendorfer and Swinkels [17], Rothkopf et al. [18], and Tenorio [19].

[2] In case the lowest winning bid ties with the highest losing bid, the remaining objects are allocated randomly to the individuals making the tied bid; higher bids receive the good with certainty; lower bids do not receive the good.

[3] Note that we do *not* assume that a bidder's two valuations are independent. In fact, the analysis is general enough to encompass situations where there is a positive probability that the two valuations are equal – for example, situations of perfect correlation.

distribution of v_1 has a density $g_1(v_1)$ with the interval $[0, \bar{v}]$ as its support. Thus $G_1(v_1) = G(v_1, v_1)$ and $G_2(v_2) = G(\bar{v}, v_2)$.

A natural example of these valuations arises when each individual has two possible uses to which the good may be put, where the value in each use is drawn independently from the distribution $H(\cdot)$. If the bidder receives one unit, he will put it in the higher valued use; if he receives two units, he will put the second in the lower valued use. Then the distribution of v_1 becomes the distribution of the first order statistic of two draws from the distribution H; and the distribution of v_2 is the distribution of the second order statistic. More generally we could imagine that v_1 and v_2 are the first and second order statistics from i draws from distribution $H(\cdot)$. In examples below we will frequently use this structure, which we will describe as preferences generated by "independent draws."

We focus on the strategy of player 1. (When no confusion will arise, we omit superscripts.) Let b_1 and b_2 respectively denote the $M-1$th and Mth highest of other players' bids; these are the ones which determine whether player 1 will win zero, one or two units.[4] Given the strategies of the other players, let the function $F(b_1, b_2)$ denote their joint distribution and let $F_1(b_1)$ and $F_2(b_2)$ denote their marginal distributions;[5] note that $\Pr\{b_1 < b_2\} = 0$.

1.1 An example

The following example illustrates the uniform price auctions can have undesirable properties even in the most simple and natural of cases. Suppose two units are being auctioned. For each bidder, the valuations are two independent draws from a uniform distribution on the interval $[0, \bar{v}]$. Then it can be verified that the following is a symmetric Nash equilibrium strategy:

$$(c_1, c_2) = (v_1, 0) \ .$$

In words, the bidder bids "truthfully" for his first unit and zero for his second unit. Thus, if there are two bidders, the price paid in equilibrium is zero with certainty. Moreover, as is shown in section 6, this is the unique equilibrium in continuous strategies.

2 General characteristics of equilibria

We let U denote the expected payoff for the player, given others' strategies, his own valuations and his own bids. As long as the distribution of oppo-

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[4] If there are not sufficient bids in excess of zero, we define b_1 and b_2 to be equal to zero. More generally, if there are not sufficient bids in excess to the reserve price (zero or otherwise) we would define b_1 and b_2 to be equal to the reserve price.

[5] In the appendix, when we address the question of ties in bidding the definition of the marginal distributions will be refined.

nents' bids is non-atomic, so that there is zero probability of ties,[6] a player's expected payoff is dependent on other bidders' strategies only through their effects on b_1 and b_2.

Theorem 2.1. *Unless there is a positive probability that* $c_i = b_j (i,j = 1,2; j \neq i)$, *expected payoff is defined by the continuous function*

$$\begin{aligned} U = & \iint_{\{(b_1,b_2)|b_2<c_1\}} [v_1 - \max\{c_2, b_2\}] \, dF(b_1,b_2) \\ & + \iint_{\{(b_1,b_2)|b_1<c_2\}} [v_2 + c_2 - 2b_1] \, dF(b_1,b_2) \ . \end{aligned} \tag{2.1}$$

Proof. Define the function Ω as follows:

$$\Omega = \begin{cases} 0 & \text{if } c_1 < b_2 \\ v_1 - b_2 & \text{if } c_2 < b_2 < c_1 \\ v_1 - c_2 & \text{if } b_2 \leq c_2 < b_1 \\ v_2 + v_1 - 2b_1 & \text{if } b_1 < c_2 \end{cases}$$

This function describes player 1's payoff as long as it is not the case that one of the player's bids is tied for both highest losing and lowest winning bid. As long as the probability of such ties is zero,

$$U = \int \Omega(c_1, c_2, b_1, b_2) \, dF(b_1, b_2)$$

which simplifies to (2.1). The integrals in this expression are continuous unless there is a discontinuity in the distribution at the boundary of integration – that is, unless there is a positive probability that $c_1 = b_2$ or $c_2 = b_1$. □

We confine our investigation to the following set of strategies:

Set S of Strategies: The higher of the two bids equals the higher of the two valuations. The lower of the two bids is non-negative and no greater than the lower of the two valuations.

Note that in this strategy, by definition, there is no shading of the higher of the player's two bids. We will call an equilibrium in which all players choose strategies in set S an *S-equilibrium*. The following lemma demonstrates that set S is in fact precisely the set of undominated strategies:[7]

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[6] The issue of ties raises important technical questions and greatly complicates many of the formulas. In order to ease the burden on the reader we will focus in this section on the cases where the probability of a tie is zero. Nonetheless, in the mult-unit context ties cannot be ruled out a priori. The calculations in the appendix 8.1.1 demonstrate that our key results continue to hold even when ties are not ruled out in advance.

[7] We thank Jeroen Swinkels for pointing out this result. The proof provided here is valid given zero probability of ties. A few extra steps are necessary to cover the case of ties, as discussed in the appendix.

Lemma 2.2. *Set S is the set of undominated strategies.*

Proof. Clearly bids below zero are weakly dominated. Expression (2.1) is weakly decreasing in c_2 for $c_2 \geq v_2$, as can be seen most easily by considering the formula for Ω in the proof above. Thus bidding $c_2 > v_2$ is a weakly dominated strategy. Finally, given c_2 expression (2.1) attains a maximum over all $c_1 \geq c_2$ when $c_1 = v_1$. Thus any strategy not in S is weakly dominated by a strategy in S. Now suppose a strategy s_1 in S is dominated by some other strategy. Then it must be dominated by another strategy s_2 in S. To show a contradiction, simply take a valuation where the two bids are not identical, and a distribution for other players' bids such that s_1 is the price setting bid and the next lower bid below s_1 is just barely below it. ☐

By showing that this dominance is strict in the presence of "trembles," Appendix 8.1.2 demonstrates[8]

Theorem 2.3. *Every perfect equilibrium is an S-equilibrium.*

In an S-strategy, the function U takes a simple form:

Theorem 2.4. *As long as there is zero probability that $c_2 = b_1$, expected payoff is defined by the function*

$$U(c_2, v_1, v_2) = \int_{c_2}^{v_1} F_2(b_2)\,db_2 + 2\int_0^{c_2} F_1(b_1)\,db_1 + (v_2 - c_2)F_1(c_2)\ . \quad (2.2)$$

In the text we will suppress dependence on v_1 and simply write $U(c_2, v_2)$.

Proof. If $c_1 = v_1$, the bidder is indifferent to winning or losing the first unit when $c_1 = b_2$ so that formula (2.1) remains valid even if the tie $c_1 = b_2$ occurs with positive probability. Substitution of v_1 reduces the formula to

$$U = v_1F_2(v_1) - c_2F_2(c_2) - \int_{c_2}^{v_1} b_2\,dF_2(b_2) + (v_2 + c_2)F_1(c_2) - 2\int_0^{c_2} b_1\,dF_1(b_1)$$

Then integration by parts yields formula (2.2). ☐

In calculations a key role will be played by the derivative of U with respect to c_2. As long as $\Pr\{c_2 = b_1\} = 0$ and F_1 is differentiable at c_2, U is differentiable at c_2 and from (2.2) we have

$$\frac{\partial U}{\partial c_2}(c_2, v_2) = (v_2 - c_2)\frac{dF_1}{dc_2}(c_2) + F_1(c_2) - F_2(c_2) \quad (2.3)$$

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[8] In [7] the result is extended to an auction with an arbitrary number of purchases by each bidder, as are the first-order conditions noted below. Restricting attention to perfect equilibria is not innocuous; [3], [6], and [14] show, in different contexts, that there are equilibria in which by making bids above their valuations, bidders in multi-object uniform price auctions increase their profits.

(Note the derivative does not depend on v_1). When U is differentiable, we have the necessary first order condition for c_2 to be an interior local maximum:

$$(v_2 - c_2)\frac{dF_1}{dc_2} + F_1(c_2) - F_2(c_2) = 0 \ . \tag{2.4}$$

The intuitive interpretation of the derivative is as follows: The benefit of a slight increase in the lower bid is an increase in the likelihood of receiving the second unit of the good just below b_1 to just above it. This benefit equals $v_2 - c_2$, the value of the good less the price paid (in this case c_2 is approximately b_1); the probability of this occurring for c_2 is proportional to dF_1/dc_2. Thus the first term is the first order approximation of the benefit per unit increase in the bid in the vicinity of c_2. The cost of the change is the increase in the bid times the likelihood that the bid will be the price setting bid, (so that the bidder will have to pay the increase for the first unit purchased). $F_2(c_2) - F_1(c_2)$ is the probability that a bid of c_2 is the price-setting bid; this quantity therefore represents the approximate cost per unit increase in the bid.

3 Characteristics of symmetric equilibria

The following fundamental result characterizing symmetric S-equilibria is a corollary of general results for all S-equilibria (symmetric and asymmetric). These more general results are stated and proved in Appendix 8.2.

Theorem 3.1. 1) *Suppose that at least three objects are being auctioned and the number of bidders is less than the number of objects. Then in any symmetric S-equilibrium, bids on the second object are strictly increasing in the valuation of the second object.*

2) *Suppose that at least three objects are being auctioned and the number of bidders is* at least as large as *the number of objects. Then in any symmetric S-equilibrium, bids on the second object are strictly increasing in the valuation of the second object, except that a range of low values may be pooled at zero.*

The intuition behind this result is clearest in the case where the lower of a player's bids is shaded below the corresponding valuation. The following theorem indicates that under very mild conditions on the distribution of valuations such shading occurs. Again, the proof is in the appendix.

Theorem 3.2. *Suppose that for each bidder, the distribution of v_2 conditional on v_1 has support $[0, v_1]$. Then in any symmetric S-equilibrium, the second bid is strictly below the second value for all valuations in the open interval $(0, \bar{v})$.*

The key intuition underlying Theorem 3.1 is that it is a suboptimal strategy for a bidder to shade his lower bid to c_2 if there is a positive probability of tying with b_1, the $M - 1$th highest of opponents' bids. If he were to do so, an infinitesimal increase in the bid would break the tie, thereby giving a first order improvement in expected payoff. Suppose that for all players in a symmetric equilibrium the bid c_2 is identical for some interval of valuations. The conditions listed in part 1 of Theorem 3.1 are sufficient for there to be a positive probability that this bid is the $M-1$th of the opponents' bids. On the other hand, if there are at least as many bidders as units then there are enough first bids to win all the units. With positive probability no second bid wins. But if the number of units exceeds the number of bidders then at least one second bid must win.

The only tie that affects the payoff is a tie between b_1 and c_2. A tie between c_1 and b_1 has no effect on the outcome; nor does any tie with b_2. In short, the only way that there can be a symmetric equilibrium in which a range of different valuations is associated with the same second bid is for there to be zero probability that those bids tie for receipt of the good. [Thus the theorem does not preclude the possibility of more general forms of pooling when only two units are auctioned (see section 5).]

4 Zero bids

One of the most striking possibilities that arises in the case of uniform price auctions is the tendency for extreme pooling of second bids at zero. In this section we will provide conditions under which it is an equilibrium for all bidders to bid $(v_1, 0)$ – that is, to bid their valuation on the first unit and to bid zero on the second unit. In effect, the bidders bid only on a single unit; we call this a "single unit bid" equilibrium. In some cases "single unit bid" equilibria have a sales price of zero with probability one. Since we have already shown that pooling does not occur when there are more units than bidders, we know in advance that "single unit bid" equilibria will only arise when the number of bidders is at least as great as the number of units on auction.

Lemma 4.1. *It is an optimal response to bid zero for the second unit for all valuations $v_2 < v$ if and only if*

$$(v - c_2)F_1(c_2) + \int_0^{c_2} (2F_1(b) - F_2(b))\, db \leq 0 \text{ for all } c_2 \text{ in } (0, v] \tag{4.1}$$

Proof. The inequality is a simple transformation of the statement that bidding zero on the second unit is more profitable than bidding c_2. □

Note that if the inequality is satisfied for some v, then zero is also an optimal response for all lower valuations. In the case where all individuals bid truthfully on the first unit and zero on the second unit, we have that F_1 is the $M-1$th order statistic from N draws from G_1 and F_2 is the Mth order statistic:

$$F_1(x) = \sum_{n=N-M+2}^{N} \binom{N}{n} (G_1(x))^n (1 - G_1(x))^{N-n} \tag{4.2}$$

$$F_2(x) = \sum_{n=N-M+1}^{N} \binom{N}{n} (G_1(x))^n (1 - G_1(x))^{N-n} \tag{4.3}$$

Thus necessary and sufficient conditions for existence of a "single unit bid" equilibrium are that condition (4.1) holds for $v = \bar{v}$, when F_1 and F_2 are defined by (4.2) and (4.3). This formula (4.1) can easily be verified for specific distributions, but it is also helpful to have sufficient conditions which allow us readily to predict distributions which yield "single unit bid" equilibria. The following result is based on the first-order conditions for the auction.

Theorem 4.2. *Bidding zero on the second unit for all v_2 in $[0, v^*]$ is an optimal response to the same behavior by other bidders, provided the number of bidders is at least as large as the number of goods and*

$$(M-1)(v^* - c)\frac{g_1(c)}{1 - G_1(c)} \leq 1 \tag{4.4}$$

for all $c \in [0, v^*)$.

Proof. Let f_1 represent the derivative of F_1. From formula (2.3) it follows that a sufficient condition for it to be optimal to make the second bid equal to zero is that

$$0 \geq [F_1(c) - F_2(c)] + f_1(c)(v - c) \tag{4.5}$$

for all v and $c \in [0, v^*)$. By manipulating equations (4.2) and (4.3), we find that as long as $N + 1 \geq M$,

$$F_2(x) - F_1(x) = \binom{N}{N-M+1}(G_1(x))^{N-M+1}(1 - G_1(x))^{M-1}$$

$$f_1(x) = N\binom{N-1}{M-2}(G_1(x))^{N-M+1}(1 - G_1(x))^{M-2}g_1(x) \ .$$

Thus inequality (4.5) can be rewritten as

$$\binom{N}{N-M+1}(G_1(c))^{N-M+1}(1 - G_1(c))^{M-1}$$
$$\geq N\binom{N-1}{M-2}(G_1(c))^{N-M+1}(1 - G_1(c))^{M-2}(v - c)g_1(c)$$

which simplifies to the formula of the theorem. □

Corollary 4.3. *If there are at least as many bidders as units and condition* (4.4) *holds for $v^* = \bar{v}$, then there is a "single unit bid" equilibrium.*

In other words, if the hazard function for the high valuation for any individual is sufficiently low then the optimal bid is zero. Note that when the number of bidders equals the number of objects, $F_2(x) = 1$, and all the formulas of the proof continue to hold. In this case, all the objects sell for zero with certainty.

Corollary 4.4. *In an auction of two units, if the density $g_1(x)$ exists and is nondecreasing then there is a "single unit bid" equilibrium.*

Proof. If $g_1(x)$ is non-decreasing, then $1 - G_1(c) \geq (\bar{v} - c)g_1(c)$ for any v, c such that $0 \leq c \leq v$. Thus, $(M-1)(\bar{v} - c)g_1(c)/(1 - G_1(c)) \leq M - 1 = 1$, and the condition of the previous corollary is satisfied. □

In the case of independent draws from a distribution $H(x)$, $g(x)$ can be non-decreasing even if $h(x)$ is decreasing, for example $H(x) = \sqrt{x}$. If v is the highest of n independent draws from a uniform distribution, then $g_1(x)$ will be non-decreasing and there will be a "single unit bid" equilibrium in the two-object auction. Thus this corollary generalizes the initial example of the paper.

For $M > 2$, $g_1(\bar{v})$ must be unbounded for the condition (4.4) to be satisfied. (There may of course be other densities with 'single unit bid" equilibria which do not satisfy the sufficient conditions.) Although "single unit bid" equilibria may require stringent restrictions on the distribution functions for $M > 2$, pooling at zero over some part of the support is likely to be typical of a much wider range of distributions. In particular:

Corollary 4.5. *Suppose there are at least as many bidders as units and $g_l(x)$ is bounded by B for x in some interval $[0, x^*]$. Then for any v^* satisfying*

$$0 \leq v^* \leq \min\left\{x^*, \frac{1}{(M-1)B}\right\}$$

bidding zero on the second unit for all v_2 in $[0, v^*]$ is an optimal response to the same behavior by other bidders.

Proof. Suppose $c \leq v_2 \leq v^* \leq \min\left\{x^*, \frac{1}{(M-1)B}\right\}$. Then $1 \geq v^* B(M-1)$; therefore,

$$1 - Bc \geq (M-1)Bv^* - (M-1)Bc \geq 0$$

implying

$$1 \geq \frac{(M-1)B(v^* - c)}{1 - Bc} \geq \frac{(M-1)(v^* - c)g(c)}{1 - G(c)}$$

which suffices by Theorem 4.2. □

5 Generation of equilibria in the two-unit case

When only two units of the good are sold, as we have seen, special considerations arise. We cannot guarantee that strategies in S-equilibria are strictly increasing. In this section we provide necessary and sufficient conditions for equilibrium. In section 6 the conditions are used to generate of equilibria in which the following phenomena arise: pooling at intermediate levels in a continuum of equilibria and discontinuities in the equilibrium bid function. As will be apparent, we will be able to generate examples of each of these phenomena for open sets of parameter values.

We adopt the following notational conventions: For any function $\gamma(x)$ of a real number x, we let $\gamma(x-)$ denote $\lim_{\hat{x} \uparrow x} \gamma(\hat{x})$ and we let $\gamma(x+)$ denote

$\lim_{\hat{x}\downarrow x}\gamma(\hat{x})$. Given our results so far, we will henceforward commit the slight solipsism of identifying an S-strategy or symmetric S-equilibrium with a weakly-increasing function $\gamma(v)$ representing the lower bid as a function of the lower valuation, or equivalently, with the inverse function

$$\gamma^{-1}(c_2) \equiv \sup\{x|\gamma(x) < c_2\}$$

which indicates the highest valuation resulting in a bid below c_2.

In the case of two units for sale, if all opponents follow strategy $\gamma(v)$, the bid distributions take the following special forms:

$$F_1(c) = G_1^N(c) \tag{5.1}$$

$$F_2(c) = NG_1^{N-1}(c)(G_2(\gamma^{-1}(c)) - G_1(c)) + G_1^N(c) \tag{5.2}$$

Therefore the first order condition $\partial U/\partial c_2 = 0$ (equation (2.4)) takes the following simple form:

$$0 = NG_1^{N-1}(c_2)[G_1(c_2) - G_2(\gamma^{-1}(c_2)) + g_1(c_2)(v_2 - c_2)]\ . \tag{5.3}$$

Define

$$\Gamma(z,y) \equiv \int_0^z NG_1^{N-1}(x)[G_1(x) - G_2(y) + g_1(x)(y - x)]\ dx$$

While this expression is closely related to the equilibrium first order condition, note that the function is continuous and only depends on the exogenous information in G, the distribution of an individual's valuations. Γ is key to finding best responses in the two-good equilibrium.

Lemma 5.1. *If*

$$c_2 \in \arg\max_{c\in[a,b]} \Gamma(c,v)$$

and all opponents are bidding according to $\gamma(\cdot)$ *and*

$$\gamma(v-) \le c_2 \le \gamma(v+)$$

then

$$c_2 \in \arg\max_{c\in[a,b]} U(c,v_2).$$

Proof. Note that with only two units in the auction and given the assumptions imposed on the distribution G, $\Pr\{c_2 = b_1\} = 0$ and U is absolutely continuous in c. We have, a.e.,

$$\frac{\partial\Gamma}{\partial c}(c,v) = NG_1^{N-1}(c)[G_1(c) - G_2(v) + g_1(c)(v - c)] \tag{5.4}$$

$$\frac{\partial U}{\partial c}(c,v) = NG_1^{N-1}(c)[G_1(c) - G_2(\gamma^{-1}(c)) + g_1(c)(v - c)] \tag{5.5}$$

so

$$\frac{\partial\Gamma}{\partial c}(c,v) - \frac{\partial U}{\partial c}(c,v) = NG_1^{N-1}(c)[G_2(\gamma^{-1}(c)) - G_2(v)]\ . \tag{5.6}$$

Therefore,

$$U(c_2,v) - U(\tilde{c},v) = \int_{\tilde{c}}^{c_2} \frac{\partial U}{\partial c}\, dc \tag{5.7}$$

$$= \int_{\tilde{c}}^{c_2} \frac{\partial \Gamma}{\partial c}\, dc - \int_{\tilde{c}}^{c_2} NG_1^{N-1}(c)[G_2(\gamma^{-1}(c)) - G_2(v)]\, dc.$$

If $\tilde{c} < c_2$, then for any $c \in (\tilde{c}, c_2)$, the bracketed term in the integrand is non-positive. If $\tilde{c} > c_2$, it is non-negative. Either way

$$U(c_2,v) - U(\tilde{c},v) \geq \int_{\tilde{c}}^{c_2} \frac{\partial \Gamma}{\partial c}\, dc = \Gamma(c_2,v) - \Gamma(\tilde{c},v)\ .$$

Since $\Gamma(c_2,v) > \Gamma(\tilde{c},v)$, we know that $U(c_2,v) > U(\tilde{c},v)$. □

The result has the following intuitive interpretation: Imagine that opponents adopted a strategy from the following family of inverse bid functions, indexed by v:

$$\hat{b}_v^{-1}(c) = \max\{v, c\}$$

– in other words, the strategy under which bidders whose second valuation is greater than or equal to v bid their valuations; bidders whose second valuation is less than v bid zero. Then if the bid c is an optimal response to this strategy, it is an optimal response to any equilibrium strategy in which valuations above v yield bids above c and valuations below v yield bids below c.

The necessary and sufficient conditions for an S-strategy $\gamma(v)$ to yield a symmetric S-equilibrium are closely linked to the requirement that $\gamma(v)$ be a local maximizer to $\Gamma(\cdot, v)$, over intervals of the form $[\gamma(v-\epsilon), \gamma(v+\epsilon)]$ (for completeness, we take $\gamma(-\epsilon) = 0$ and $\gamma(\bar{v}+\epsilon) = \bar{v}$). First we state and prove the necessary condition:

Theorem 5.2. *In the two-unit case, if $\gamma(\cdot)$ is a symmetric S-equilibrium then for each v in the open interval $(0, \bar{v})$ there exists $\epsilon > 0$ such that*

$$\{\gamma(v-), \gamma(v), \gamma(v+)\} \subseteq \underset{c \in [\gamma(v-\epsilon), \gamma(v+\epsilon)]}{\arg\max}\ \Gamma(c,v).$$

Proof. Suppose $\gamma(\cdot)$ is a symmetric S-equilibrium. Pick any v and let $c = \gamma(v)$ and $\bar{c} = \gamma(v+)$. We will demonstrate that there exists ϵ such that c (and $\bar{c}$, when γ is right discontinuous) maximizes $\Gamma(., v)$ on an interval $[c, \gamma(v+\epsilon)]$; the demonstration for the interval $[\gamma(v-\epsilon), c]$ is analogous. We will state the proof assuming the density g is continuous; the extension is tedious but straightforward.

Since the directional derivatives for U with respect to c exist for all values of c, a necessary condition for $\gamma(v)$ to be an S-equilibrium is that for all v

$$\frac{\partial U}{\partial c}(\gamma(v)+, v) \leq 0$$

or, from (5.4),

$$NG_1^{N-1}(c)[G_1(c) - G_2(\gamma^{-1}(c+)) + g_1(c)(v-c)] \leq 0$$

From Theorem 3.1, γ must be weakly increasing. If the function $\gamma^{-1}(\cdot)$ is discontinuous to the right at c, then for some $\epsilon > 0, \gamma(v) = \gamma(v+\epsilon)$ and the result holds trivially. Therefore assume $\gamma^{-1}(c+) = \gamma^{-1}(c)$. Then (5.5) implies

$$\frac{\partial \Gamma}{\partial c}(\gamma(v)+, v) \leq 0$$

so that there is an interval $[c, c+\delta]$ on which c is maximal for Γ. If γ is continuous from the right, then there is an $\epsilon > 0$ such that $c + \delta \geq \gamma(v+\epsilon)$ and we are done. On the other hand if γ is discontinuous, then there is an interval on which $\gamma^{-1}(c) = v$; $\bar{c}$ is the supremum of this interval. On this interval $\partial\Gamma/\partial c = \partial U/\partial c$ (as can be verified from (5.6)) so c is maximal for Γ throughout this interval. Moreover, since U is continuous, $\bar{c}$ must also be maximal for $U(., v)$. Repeat the argument of the proof replacing c with $\bar{c}$ and conclude that there is an interval $[\bar{c}, \bar{c}+\delta]$ on which $\bar{c}$ is maximal for Γ, and therefore both c and $\bar{c}$ are maximal for Γ on $[c, \bar{c}+\delta]$. Since $\bar{c}$ is the left limit of $\gamma(v+)$ there is an $\epsilon > 0$ such that $\bar{c} + \delta > \gamma(v+\epsilon)$, and we are done. □

Additional necessary conditions are associated with the endpoints of the support: An analogous proof demonstrates that $\gamma(\bar{v}-)$ is a local maximizer of $\Gamma(., \bar{v})$; $\gamma(0+) = 0$ by the definition of an S-strategy.

Our sufficient condition requires that $\gamma(v)$ be a local maximizer *uniformly* on $[0, \bar{v}]$:

Theorem 5.3. *Suppose* $\gamma(\cdot)$ *is a weakly increasing function with* $0 \leq \gamma(v) \leq v$. *In the two-unit case, if there exists* $\epsilon > 0$ *such that*

$$\gamma(v) \in \underset{c \in [\gamma(v-\epsilon), \gamma(v+\epsilon)]}{\arg\max} \Gamma(c, v).$$

is satisfied for every v *in the open interval* $(0, \bar{v})$ *then* $\gamma(v)$ *is a symmetric S-equilibrium.*

Proof. For every c' in $[0, \gamma(v))$, there exists a $v' \leq v$ such that

$$c' \in [\gamma(v'-\epsilon), \gamma(v')].$$

By lemma 5.1,

$$U(\gamma(v'), v') \geq U(c', v').$$

Therefore

$$U(\gamma(v'), v) \geq U(c', v).$$

Then we show that for v'' closer to v,

$$U(\gamma(v''), v) \geq U(\gamma(v'), v).$$

and given the uniform value of ϵ a series of such steps will demonstrate

$$U(\gamma(v), v) \geq U(c', v).$$

Repeat the process for c' in $(\gamma(v), \bar{v}]$. □

In calculating equilibria, the following set is particularly useful:

$$C^*(v) = \arg\max_{c\in[0,\bar{v}]} \Gamma(c, v)$$

(We know that such a set exists for each v.) We call $C^*(v)$ an *increasing correspondence* if

$$v_2 > v_1$$

$$c_1 \in C^*(v_1)$$

and

$$c_2 \in C^*(v_2)$$

together imply

$$c_2 \geq c_1 \ .$$

The following are direct consequences of the previous theorems:

Corollary 5.4. *Suppose $C^*(v)$ is an increasing correspondence. Then $c^*(v)$ is a symmetric S-equilibrium if it is a selection from $C^*(v)$. Suppose in addition that $\Gamma(., v)$ is quasi-concave for each v. Then $c^*(v)$ is a symmetric S-equilibrium only if it is a selection from $C^*(v)$.*

Corollary 5.5. *Let $\{v_1, v_2, \ldots, v_k\}$ be a finite set of points, such that $C^*(v)$ is an increasing correspondence in some neighborhood of each v_i. It is a symmetric S-equilibrium for each player to adopt strategy $c^*(v)$ if c^* is a weakly increasing function with the following characteristic: for any v, either*

1. *$c^*(v) \in C^*(v)$, or*
2. *for some i, j, $v \in (v_i, v_j)$ and $c^*(v_i) = c^*(v_j) = c^*(v)$.*

The final corollary of this section demonstrates that gaps in the bidding are likely to be common in many examples.

Corollary 5.6 *Suppose that $c^*(\cdot)$ is an equilibrium strategy and $c^*(v_2) > c^*(v_1)$. If $g_1(\cdot)$ is continuous and increasing throughout the interval $[c^*(v_1), c^*(v_2)]$, then on the interval $[v_1, v_2]$, there is a discontinuity in c^*.*

Proof. By Theorem 5.2, any bid $c > 0$ which occurs in equilibrium at a point of increase of $c^*(\cdot)$ is an interior local maximum of Γ. Thus

$$NG_1^{N-1}(c)[G_1(c) - G_2(v) + g_1(c)(v - c)] = 0$$

Since c is interior, $G_1(c)$ is positive, so the bracketed term must equal zero. Moreover, the second order necessary conditions require that the expression be non-increasing in some neighborhood of c. Since $c \leq v$ this in turn requires that g_1 be non-increasing. □

As a consequence, if g_1 is increasing on the interval $[0, v)$ then in any equilibrium strategy, either there are an infinite number of discontinuities arbitrarily close to zero, or there is an interval $[0, \epsilon)$ in which the bid is 0. In the case of independent draws, $g_1 = 2hH$ so that g_1 is increasing at zero as long as h' is bounded below at zero. Thus in the case of independent draws with two units for sale there will typically be some pooling of bids at zero.

6 Calculations

The results of section 4 argue that pooling at zero is common when bidders can make no more than two bids. The results of section 5 suggest the possibility of discontinuities and interior pooling in equilibria, and provide a method for generating such equilibria. This section illustrates four different possibilities with examples. We then conclude with an example in which each bidder may make more than two bids; this again gives rise to pooling, suggesting that the result is robust.

The first four examples each have $N+1$ bidders bidding for the two objects. In each of these examples, the bidders' values are independent draws from a cumulative distribution $H(x)$. Note therefore that $G_1(x) = H^2(x)$ and $G_2(x) = 2H(x) - H^2(x)$. The examples differ only in the choice of the distribution; changes in the distribution give rise to qualitatively different equilibria.

Example 1. Given an a in $(0, 1]$, define $H(x)$ to be any cumulative distribution such that

$$H(x) = \begin{cases} 0 & \text{for } x < 0 \\ x & \text{for } x \in [0, a] \end{cases}$$

In other words, H is uniform on the lower part of its support and arbitrary beyond. Theorem 4.2 leads us to suspect that there will be an equilibrium with some pooling at zero, and Corollary 4.4 assures that there will be a single unit bid equilibrium when $a = 1$. We now show that there cannot be any equilibrium bidding strategy b for the second unit such that b is continuous, strictly increasing and positive at any $x \in (0, a)$; for $a = 1$, this demonstrates the claim made in the example of the initial section: the only continuous strategy is the single-unit-bid strategy.

We show this by contradiction. Imagine that there were a strategy $b(x)$ and a $v_2 \in (0, a)$ such that $b(v_2) > 0$ and b is strictly increasing and continuous at v_2. Then b must satisfy the first order condition 5.3, which simplifies to

$$0 = b^2 - 2v + v^2 + 2b(v - b) = -b^2 + 2bv + v^2 - 2v \ .$$

But this is a quadratic function in b. For it to have a real valued solution, we need that $0 \leq (2v)^2 - 4(-1)(v^2 - 2v) = 8v(v - 1)$. This is clearly impossible, since $v < a \leq 1$. Therefore there cannot be such a strategy.

Example 2. Let

$$H = \begin{cases} 0 & \text{for } x \leq 0 \\ x^{1/3} & \text{for } 0 \leq x \leq 1 \\ 1 & \text{for } 1 \leq x \end{cases}$$

In this example, $g_1(x) = \frac{2}{3}x^{-1/3}$, which is unbounded at $x = 0$. Thus Corollary 4.5 does not apply. In particular, it leaves open the possibility that $b(v_2) > 0$ for all $v_2 > 0$. And indeed, that is what happens. For this example the first order condition is

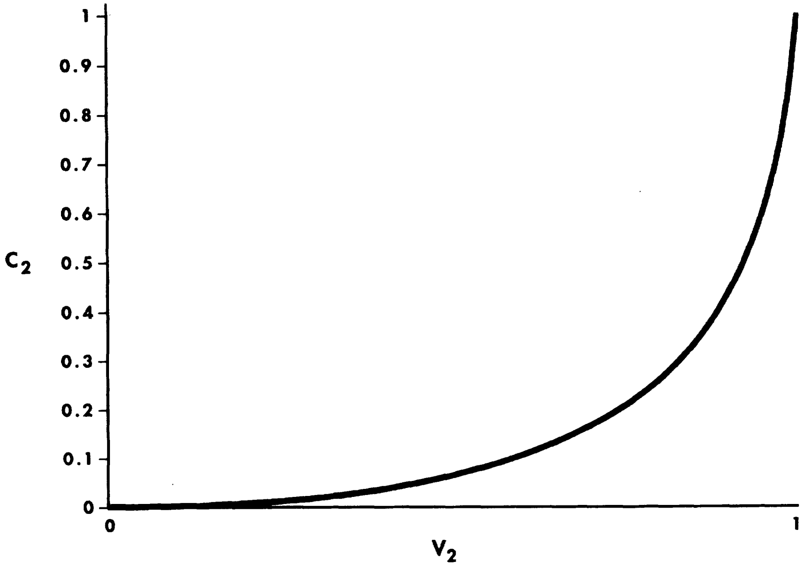

Figure 1

$$0 = b^{2/3} - 2v^{1/3} + v^{2/3} + \frac{2(v-b)}{3b^{1/3}} \ .$$

The best response set $C^*(v)$ has a single value for each v in $[0,1]$, and the function $b(v)$ so defined is strictly monotonic with $b(0) = 0$ and $b(1) = 1$. Thus it is an equilibrium, by Corollary 5.4. Its graph is depicted in Figure 1.

Example 3. Let $h(x) = 2(1-x)$ for $x \in [0,1]$ and zero elsewhere. Theorem 4.2 suggests that there might be an equilibrium with at least some pooling at zero. (For this example $g_1(x)$ attains its maximum value of about 1.54 at $x \cong 0.42$; thus the corollary suggests that there may be an equilibrium with $b(v) = 0$ for $v \in [0, 1/1.54] \cong [0, 0.65]$.) In fact there is an equilibrium with some pooling at zero, and a discontinuous jump to positive bids for sufficiently large values of v_2.

For this example, the best response set $C^*(v)$ has a single value in [0, 1] for almost all values of v. The function so defined is strictly monotonic, with $b(v) = 0$ for $v \in [0, 0.93]$. Between 0.93 and 0.94 $b(v)$ jumps discontinuously from zero to approximately 0.87 and then increases continuously until $b(1) = 1$. This is the only function which satisfies Theorem 5.2; there are no equilibria with continuous bid functions. Figure 2 depicts its graph.

Example 4. This example illustrates two phenomena: pooling at intermediate levels and multiple (in fact, a continuum of) equilibria. For $a > 0$, let $H(x)$ be the following continuous c.d.f.:

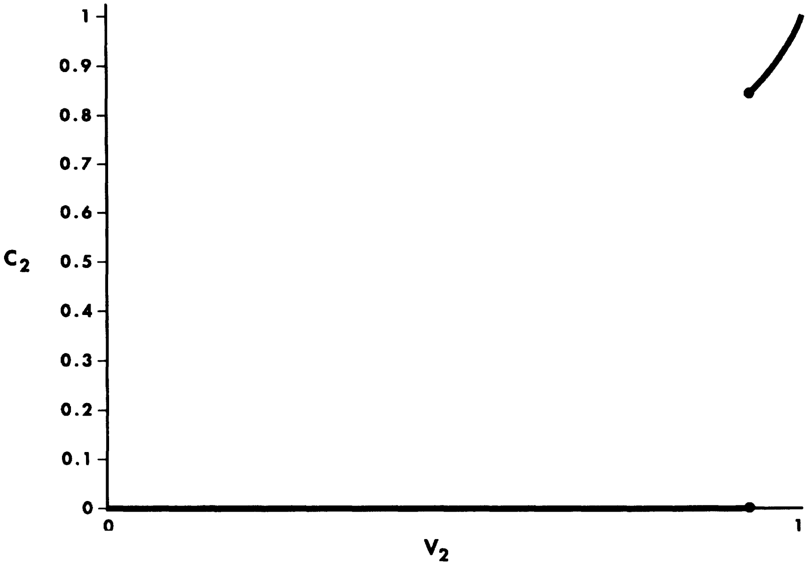

Figure 2

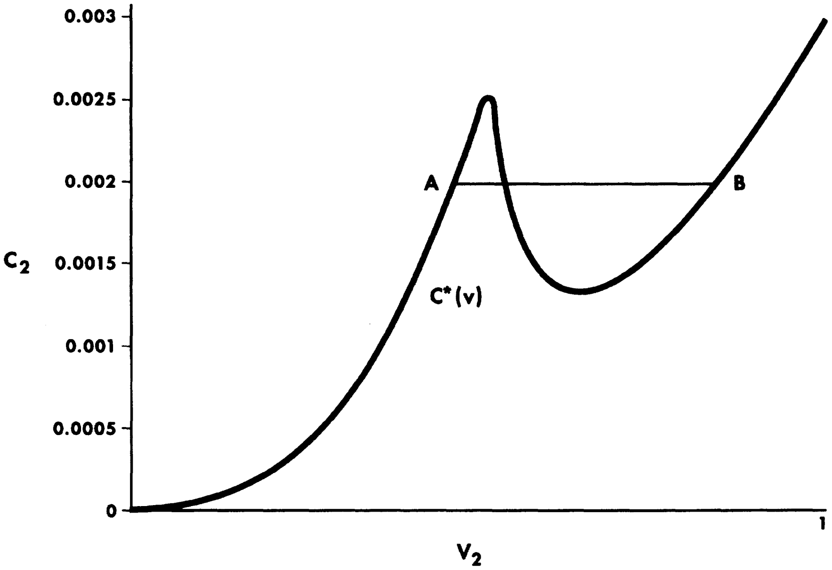

Figure 3

$$H(x)=\begin{cases}0 & \text{for } x\le 0\\ x^{1/3}-\frac{4x}{3} & \text{for } 0\le x\le 1/8\\ 1-\frac{2}{3a^2}\left(\frac{1}{8}+a-x\right)^2 & \text{for } 1/8\le x\le a+1/8\\ 1 & \text{for } a+1/8\le x\end{cases}$$

For this example the best response set $C^*(v)$ has a single value for almost every v in [0, 1]. But it does not define a monotonic function. For example, Figure 3 depicts its graph when $a = 0.115$.

Corollary 5.5 allows us to construct a continuum of equilibria from such a $C^*(v)$. For example one equilibrium follows the best response set C^* except in the interval AB in the graph, where c^* is constant. Any horizontal interval connecting points on $C^*(v)$, and starting strictly before the peak of the function, and ending strictly after the trough, will yield another solution.

6.1 A three-bid extension

All our examples of pooling thus far have had at least as many bidders as units. What if there are more units than bidders? Since there must be at least two bidders, this means at least three units, and Theorem 3.1 assures that any S-equilibrium must then be strictly monotonic. There can be no pooling.

However, suppose we extended the model to allow each bidder to make more than two bids. In particular, consider allowing each bidder to make *three* bids. Imagine that the number of units is greater than the number of bidders, but no greater than twice the number of bidders. Then it is possible that no third bid wins. When it was possible that no second bid won, then pooling was not unusual. Now that it is possible that no third bid wins, one might conjecture that there would be examples in which the third bid is always zero.

A general analysis of the multi purchase auction is beyond the scope of this paper; however we include a natural extension of the previous examples to the case of three bids per person, to show the possibility of a "two unit bid" equilibrium:

Example 5. Three units are being auctioned to two bidders, each bidder being allowed to make three bids. Each bidder's three valuations (v_1, v_2, v_3) are distributed in such a way that the following are the marginal distributions:

$$G_1(v_1) = v_1^2$$
$$G_2(v_2) = (2 - v_2)v_2$$

G_3 can be arbitrary, provided v_3 is with certainty no greater than v_2. In other words, v_1 and v_2 are distributed as the first and second draws from two independent uniform distributions; v_3 could be, for example, distributed uniformly on $[0, v_2]$. Since we already noted above that $b(v_1, v_2) = (v_1, v_2^2)$ is an equilibrium strategy for the case where the bidder makes only two bids, it suffices to show that, given the other bidder's strategy is $(v_1, v_2^2, 0)$, then a bid of $(v_1, v_2^2, 0)$ dominates (v_1, v_2^2, c) for any c.

Let u_1 and u_2 denote the other bidder's first and second values, and assume the strategies are as described above. Then a bidder's expected profit is

$$
\begin{aligned}
U = & \int_{u_2^2>v_2^2} (v_1 - v_2^2)\, dG(u_1,u_2) + \int_{v_2^2>u_2^2>c} (v_1 + v_2 - 2u_2^2)\, dG(u_1,u_2) \\
& + \int_{u_1>c>u_2^2} (v_1 + v_2 - 2c)\, dG(u_1,u_2) + \int_{u_1<c} (v_1 + v_2 + v_3 - 3u_1) dG(u_1,u_2) \\
= & \; v_1 - v_2^2 + \int_{v_2>u_2} (v_2 + v_2^2 - 2u_2^2)\, dG_2(u_2) + \int_{u_2^2<c} (2u_2^2 - 2c)\, dG_2(u_2) \\
& + \int_{u_1<c} (v_3 + 2c - 3u_1)\, dG_1(u_1)
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial U}{dc} & = -\,2G_2(\sqrt{c}) + (v_3 - c)g_1(c) + 2G_1(c) \\
& = -\,2(2 - \sqrt{c})\sqrt{c} + 2(v_3 - c)c + 2c^2 \\
& = -\,4\sqrt{c} + 2c + 2v_3 c \leq 0
\end{aligned}
$$

since $0 \leq c \leq v_3 \leq 1$, with strict inequality if $c < 1$. Therefore it is best for $c = 0$.

7 Discussion

This paper has provided characterizations of equilibria in uniform price auctions under private values, where each bidder makes two bids. We have shown conditions under which the auction yields strictly monotonic equilibria, and conditions under which there is or is not pooling of bids at zero. We have also provided an example in which there is a continuum of continuous equilibria, none of them strictly monotonic, and an example in which the only equilibria are discontinuous.

As the three-bid extension shows, zero bids are likely to be common in uniform-price auctions under private values in cases beyond those examined here. The crucial factor for the generalization appears to be the relation between the total number of bids (the number of players times the number of bids each will make) and the number of objects. If each individual can make k bids, and the number of objects is no more than $k(N + 1)$, it is likely that there will be equilibria with zero bids on the final objects. In particular, in applications in which the goods are infinitely divisible, it is particularly easy to find perfect equilibria in which zero is the equilibrium price (see [6]; for early examinations of auctions with infinitely divisible goods see [13], [21]). Back and Zender [3] also find zero bids in equilibrium; however in their case it is somewhat less striking: it is clear that in a common values auction with no diminishing returns to purchases, there is no reason for an auctioneer to divide his lots in the first place. In their environment, the revenue the auctioneer obtains in a multi-unit auction can only be less than the revenue obtained by selling all units as a single lot; multi-unit auctions simply give the opportunity for implicit collusion among the buyers. In a private values auction, on the other hand, it is clear that selling all units in a single lot will frequently give less revenue than dividing them for sale.

Noussair [16] provides the first order necessary condition (2.4) in his examination of a special class of symmetric equilibria in which strategies are assumed to be continuous, differentiable, monotonic functions. His class is a subset of our S-equilibria. It omits the equilibria to Examples 2 and 3 of the previous section; in those examples his class is empty.

Our results demonstrate that auctions in which individuals can receive more than one unit differ dramatically from auctions in which each individual receives at most one unit. In the common forms of single-unit auctions or multi-unit auctions in which each individual wins at most one unit, objects go the bidders with the highest valuation. Except for the effect of reservation prices, these auctions result in efficient outcomes. Efficiency underlies the revenue equivalence results for such auctions.

In the auction we have described here, however, the assignment of objects to bidders is not efficient. In particular, in zero bid equilibria each bidder wins at most one object even if his value for a second object exceeds the value to other bidders. As we change auction rules, the assignment of goods varies. In the multi-unit case, the standard forms of auctions are no longer revenue equivalent.[9]

Pronouncements on the relative merits of various forms of auctions cannot be justified by reference to single-unit models. Policy prescriptions require a systematic examination of multi-unit models. This article should be regarded as a first step in such a process; for the corresponding results in the "pay-your-bid" auction, see [8].

8 Appendix

The first section of the appendix deals with two technical issues that arise in the analysis of multiunit auctions – namely, the treatment of ties and the definition of perfect equilibrium. The final section of the appendix provides a general analysis of characteristics of asymmetric equilibria; the theorems on symmetric equilibria in the text are corollaries of these more general results.

The need for care in defining perfect equilibria is clear in a game in which the strategy space is a function space. The need for care in addressing ties may be less apparent. It is conventional in analyzing auctions simply to assume that the distributions of players' bids is non-atomic, so that ties are zero-probability events. In single unit auctions this is immaterial; however, as we have seen, there are examples of multi-unit auctions in which, even though the distribution of valuations is non-atomic, the only symmetric equilibria have pooling of bids. Thus in the multi-unit context it is not innocuous to rule out ties a priori.

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[9] In [6] we examine the limits on the degree of inefficiency possible in uniform price auctions. For an investigation of arrangements which perform better than uniform-price auctions, see [12]. For an implementation of an efficient auction see [1].

8.1 A formal treatment of the game

The appendix will continue to make use of the following conventions, introduced in the text: For any function $\gamma(x)$ of a real number x, $\gamma(x-)$ denotes $\lim_{\hat{x}\uparrow x}\gamma(\hat{x})$ and $\gamma(x+)$ denotes $\lim_{\hat{x}\downarrow x}\gamma(\hat{x})$. The inverse γ^{-1} of a function γ is defined as follows:

$$\gamma^{-1}(c) \equiv \sup\{x|\gamma(x) < c\}.$$

Let $V = \{(v_1, v_2)|\bar{v} > v_1 \geq v_2 > 0\}$. Let the interval $\mathbf{B} = [\underline{b}, \bar{b}]$ denote the set of "permitted bids." We assume $\mathbf{B} \supseteq [0, \bar{v}]$. A *pure strategy* for player n is a Lebesgue measurable mapping from the player's pair of valuations to a pair of bids $\mathbf{c^n} = (c_1^n, c_2^n)$ chosen from the following set $\mathbf{S}$:

$$\mathbf{S} = \{(c_1^n, c_2^n) \in \mathbf{B} \times \mathbf{B},\ c_1^n \geq c_2^n\}$$

A pure strategy $\mathbf{c}$ has a *totally mixed outcome* if for any open subset $\mathbf{C}$ of $\mathbf{S}$ (relative to $\mathbf{S}$),

$$\int_{\mathbf{c}(v_1,v_2)\in\mathbf{C}} dG(v_1, v_2) > 0\ .$$

in other words, roughly speaking, any pair of permitted bids is a possibility. A *totally mixed strategy* is a mixed strategy which places positive probability on a pure strategy (or a set of pure strategies) with totally mixed outcomes.[10]

Definition. A strategy profile $(\mathbf{c_1}, \ldots, \mathbf{c_{n+1}})$ is a *perfect equilibrium* iff there exists a sequence of profiles of totally mixed strategies converging (in distribution of outcomes) to $(\mathbf{c_1}, \ldots, \mathbf{c_{n+1}})$ such that for each (v_1, v_2) in V, $\mathbf{c_i}(v_1, v_2)$ is a best response to all profiles in the sequence.

8.1.1 Ties – A general formulation

A formal analysis must begin with a clear specification of the game's rules and expected payoffs in the presence of ties. For concreteness we assume that each of the tied bids is equally likely to receive the good.[11]

For $2N \geq l \geq k \geq 1$, let $P_{k,l}(c)$ denote the probability that the $k-1$th highest of opponents' bids is greater than c, the $l+1$th highest of opponents' bids is less than c and all bids in between are equal to c. Each player's strategy generates a joint distribution of a pair of bids. Let $J_{(n)}(b_1, b_2)$ represent the joint distribution of the two bids for player n. Let $Q_{(n)}(c; i, j)$ denote the probability that exactly i of player n's bids are above c and exactly

---

[10] We consider mixing in the distributions to avoid the technical complications of mixed strategies for a continuum of types; see [15]. Because players' preferences are independent, we do not need to consider more general mixtures placing positive probability on general classes of distributional strategies. Allowing for correlation between individuals' preferences will add significant complications; see [9, pp. 236-237].

[11] There are other possibilities, even in symmetric games: for example, we could suppose that tied bids have a higher priority if the bid is the higher of the two bids from an individual. For all our results the actual rules for breaking the tie are immaterial, so long as no individual bid is guaranteed to receive the good in the event of a tie.

i bids are below c (so $Q_{(n)} = 0$ unless $0 \le i + j \le 2$). Given $J_{(n)}$, $Q_{(n)}$ is readily calculated; if $J_{(n)}$ is non-atomic, then $Q_{(n)} = 0$ unless $i + j = 2$. Then

$$P_{k,l}(c) = \sum_{\substack{i_1,j_1 \\ \sum_n i_n = k-1; \sum_n j_n = 2N-l}} \cdots \sum_{i_n,j_n} \prod_n Q_{(n)}(c, i_n, j_n) \ .$$

If the individuals' distributions are identical, then the formula simplifies to

$$\sum_{\substack{n_1,..,n_6 \\ n_1+n_2+n_3+n_4+n_5+n_6=N \\ n_3+n_4+2n_5=k-1 \\ n_2+n_4+2n_6=2N-l}} \frac{N!}{n_1!n_2!n_3!n_4!n_5!n_6!} \begin{matrix}[Q(c,0,0)^{n_1} Q(c,0,1)^{n_2} Q(c,1,0)^{n_3} \\ Q(c,1,1)^{n_4} Q(c,2,0)^{n_5} Q(c,0,2)^{n_6}]\end{matrix}$$

Note $P_{k,l}(c) = 0$ if there is zero probability that any opponent's bid is exactly c.

Theorem 8.1. If $c_1 > c_2$, then the player's expected payoff U is

$$\begin{aligned} &\iint_{\{(b_1,b_2)|b_2<c_1\}} [v_1 - \max\{c_2, b_2\}]\ dF(b_1, b_2) \\ &+ \iint_{\{(b_1,b_2)|b_1<c_2\}} [v_2 - 2b_1 + c_2]\ dF(b_1, b_2) \\ &+ (v_1 - c_1) \sum_{l=M}^{2N} \sum_{k=1}^{M} \frac{(M-k+1)P_{k,l}(c_1)}{l-k+2} \\ &+ (v_2 - c_2) \sum_{l=M-1}^{2N} \sum_{k=1}^{M-1} \frac{(M-k)P_{k,l}(c_2)}{l-k+2} \ . \end{aligned} \tag{8.1}$$

If $c_1 = c_2$ then U is

$$\begin{aligned} &\iint_{\{(b_1,b_2)|b_2<c_2\}} [v_1 - \max\{c_2, b_2\}]\ dF(b_1, b_2) + \\ &\iint_{\{(b_1,b_2)|b_1<c_2\}} [v_2 - 2b_1 + c_2]\ dF(b_1, b_2) \\ &+ (v_1 - c_2) \sum_{l=M}^{2N} \sum_{k=1}^{M} \frac{(2l-k-M+4)(M-k+1)P_{k,l}(c_2)}{(l-k+3)(l-k+2)} \\ &+ (v_2 - c_2) \sum_{l=M-1}^{2N} \sum_{k=1}^{M-1} \frac{(M-k)(M-k+1)P_{k,l}(c_2)}{(l-k+3)(l-k+2)} . \end{aligned} \tag{8.2}$$

Proof. Theorem 2.1 establishes the integral portions of the payoff formulas, which are applicable as long as ties are not of positive probability. The bidder wins his first unit outright if and only if $c_1 > b_2$; he wins his second unit outright if and only if $c_2 > b_1$. The second summation in (8.1) is the expected payoff when $c_1 > b_1 = c_2$; in this case there are $l - k + 1$ bids by opponents competing with the bidder's lower bid for $M - k$ unallocated goods. The first summation is the expected payoff when $c_1 = b_2 > c_2$; in this case $l - k + 1$ bids by opponents compete with the bidder's higher bid for $M - k + 1$ unallocated goods.

Next we consider the expression when the two bids are equal (8.2). If $b_1 = b_2 = c_1 = c_2$, a tie will give the bidder *two* chances at obtaining the goods. In this case there will be $M - k + 1$ unallocated goods, and $l - k + 1$ competitors; the chance of obtaining two units is $\frac{M-k+1}{l-k+3}\frac{M-k}{l-k+2}$; the chance of obtaining at least one unit is $\frac{M-k+1}{l-k+3} + \left(1 - \frac{M-k+1}{l-k+3}\right)\frac{M-k+1}{l-k+2}$. This is the interpretation of the summations when $k < M$ and $l > M - 1$. If $b_1 = c_1 = c_2 > b_2$, i.e., if $l = M - 1$, then one of the two units is guaranteed, and its value is already included in the integral, which covers situations where $c_1 > b_2$. In this case the chance to gain the second unit as well is the only part to be included in the summation; it is for this reason that the limit of summation on the term in v_1 is $l = M$, while the limit of summation on the term in v_2 is $l = M - 1$. Finally, if $b_1 > c_1 = c_2 = b_2$, i.e., if $k = M$, there is only one unit in dispute, and the probability of receiving the second unit reduces to zero, hence the limit of summation $k = M - 1$ in the term in v_2.

The summations in the two expressions are zero unless some $P_{k,l}(c) > 0$ for suitable k, l – which again implies that there is a positive probability that the $M - 1$th highest bid equals c_2 or the Mth highest bid equals c_1. □

8.1.2 S-Equilibria and perfection

The main theorem of this subsection shows that S-strategies dominate all other strategies. From this theorem it follows immediately that every perfect equilibrium is an S-equilibrium (Theorem 2.3 of the text). The main theorem of this subsection differs from the lemma on dominance in the text (Lemma 2.2) in two ways: it shows that dominance is strict in the presence of trembles (necessary for perfection) and it addresses the possibility of ties in its calculations. It is the latter difference which causes most of the difficulty. In the absence of ties strict dominance follows immediately from the following four properties (analogues of the facts used in Lemma 2.2):

1. For $c_1 > \max\{c_2, v_1\}$ (2.1) is decreasing in c_1.
2. For $v_1 > c_1 \geq c_2$, (2.1) is increasing in c_1.
3. The formula (2.1) is larger when c_2 is zero than when it is negative.
4. For $c_2 > v_2$ (2.1) is decreasing in c_2.

All of these properties are strict if the distribution of opponents' bids is totally mixed. (The last of the results in its strict form is due to the fact that (2.1) is the integral of Ω, which decreases discontinuously when c_2 passes b_1. Thus if there is a positive probability that b_1 is in the interval over which c_2 is changing, U decreases strictly.)

However, across discontinuities the behavior of the function U depends on the summation terms in formulas (8.1-8.2). The following lemma provides characterization of the behavior of this function across discontinuities.

Lemma 8.2. (a) Suppose $c_2 > c_1$.

$$\text{If } c_1 < v_1, \text{ then } U(c_1-,c_2) \le U(c_1,c_2) \le U(c_1+,c_2)\ . \tag{8.3}$$

$$\text{If } c_2 < v_2, \text{ then } U(c_1,c_2-) \le U(c_1,c_2) \le U(c_1,c_2+)\ . \tag{8.4}$$

If $c_1 > v_1$ or $c_2 > v_2$ the inequalities reverse. If $c_1 = v_1$ or $c_2 = v_2$ the expressions are equalities.

(b) If $c_1 > v_1 > c > v_2 \ge c_2$ then

$$\lim\nolimits_{\epsilon\downarrow 0} U(c+\epsilon, c-\epsilon) \ge U(c,c) \tag{8.5}$$

$$\lim\nolimits_{\epsilon\downarrow 0} U(c_1+\epsilon, c_1+\epsilon) \le U(c_1,c_1) \le \lim\nolimits_{\epsilon\downarrow 0} U(c_1-\epsilon, c_1-\epsilon) \tag{8.6}$$

$$\lim\nolimits_{\epsilon\downarrow 0} U(v_1+\epsilon, v_1+\epsilon) \le U(v_1,v_1) \tag{8.7}$$

$$U(c_1+,c_1) \le U(c_1,c_1) \tag{8.8}$$

$$U(c_2+,c_2) \ge U(c_2,c_2)\ . \tag{8.9}$$

Part (a) says that when $c_1 \ne c_2$, profits increase in c_1 across any discontinuity if $(v_1 - c_1)$ is positive; they decrease across any discontinuity if it is negative. If $v_1 = c_1$, there is no discontinuity. The interpretation is identical for discontinuities in c_2.

Proof. (a) If $c_2 > c_1$ then at a discontinuity,

$$\begin{aligned} &\operatorname{sgn}\{U(c_1,c_2) - U(c_1-,c_2)\} \\ &\quad = \operatorname{sgn}\left\{(v_1 - c_1)\sum\nolimits_{l=M}^{2N}\sum\nolimits_{k=1}^{M}\frac{(M-k+1)P_{k,l}(c_1)}{l-k+2}\right\} = \operatorname{sgn}\{v_1 - c_1\} \end{aligned}$$

and

$$\begin{aligned} &\operatorname{sgn}\{U(c_1+,c_2) - U(c_1,c_2)\} \\ &\quad = \operatorname{sgn}\left\{(v_1 - c_1)\left[\Pr\{b_2 = c_1\} - \sum\nolimits_{l=M}^{2N}\sum\nolimits_{k=1}^{M}\frac{(M-k+1)P_{k,l}(c_1)}{l-k+2}\right]\right\} \\ &\quad = \operatorname{sgn}\{v_1 - c_1\}\ . \end{aligned}$$

The jump in the integral is $(v_1 - c_1)\Pr\{b_2 = c_1\}$; the discontinuity in the integral occurs to the right of c_1. The summation equals zero except at c_1, where it is less than $(v_1 - c_1)\Pr\{b_2 = c_1\}$ in absolute value. (To see this, note that the sum of all the $P_{k,l}(c)$ in the summation is equal to $\Pr\{b_2 = c_1\}$; in the summation each is multiplied by a coefficient which is no greater than one.) The argument is identical for $\operatorname{sgn}\{U(c_1,c_2) - U(c_1,c_2-)\}$ and $\operatorname{sgn}\{U(c_1,c_2+) - U(c_1,c_2)\}$.

(b) At a discontinuity,

$$\begin{aligned}
&\lim_{\epsilon\downarrow 0} U(c+\epsilon, c-\epsilon) - U(c,c) \\
&\quad = \Pr\{b_2 = c\}(v_1 - c) - (v_1 - c) \\
&\qquad \sum_{l=M}^{2N}\sum_{k=1}^{M} \frac{(2l-k-M+4)(M-k+1)P_{k,l}(c_2)}{(l-k+3)(l-k+2)} \\
&\qquad - (v_2 - c)\sum_{l=M-1}^{2N}\sum_{k=1}^{M-1} \frac{(M-k)(M-k+1)P_{k,l}(c_2)}{(l-k+3)(l-k+2)} \\
&\quad \geq \Pr\{b_2 = c\}(v_1 - c) - \sum_{l=M}^{2N}\sum_{k=1}^{M}(v_1 - c) \\
&\qquad \frac{(2l-k-M+4)(M-k+1)P_{k,l}(c_2)}{(l-k+3)(l-k+2)} \geq 0\ .
\end{aligned}$$

if $v_1 > c > v_2$.

$$\begin{aligned}
&\lim_{\epsilon\downarrow 0} U(c_1+\epsilon, c_1+\epsilon) \\
&\quad = \Pr\{b_2 = c_1\}(v_1 - c_1) + \Pr\{b_1 = c_1\}(v_2 - c_1) \\
&\qquad - (v_1 - c_1)\sum_{l=M}^{2N}\sum_{k=1}^{M} \frac{(2l-k-M+4)(M-k+1)P_{k,l}(c_2)}{(l-k+3)(l-k+2)} \\
&\qquad - (v_2 - c_1)\sum_{l=M-1}^{2N}\sum_{k=1}^{M-1} \frac{(M-k)(M-k+1)P_{k,l}(c_2)}{(l-k+3)(l-k+2)} \leq 0
\end{aligned}$$

and

$$\begin{aligned}
&U(c_1, c_1) - \lim_{\epsilon\downarrow 0} U(c_1-\epsilon, c_1-\epsilon) \\
&\quad = (v_1 - c_1)\sum_{l=M}^{2N}\sum_{k=1}^{M} \frac{(2l-k-M+4)(M-k+1)P_{k,l}(c_2)}{(l-k+3)(l-k+2)} \\
&\qquad + (v_2 - c_1)\sum_{l=M-1}^{2N}\sum_{k=1}^{M-1} \frac{(M-k)(M-k+1)P_{k,l}(c_2)}{(l-k+3)(l-k+2)} \leq 0
\end{aligned}$$

if $c_1 > v_1$. Finally,

$$\begin{aligned}
&\lim_{\epsilon\downarrow 0} U(c+\epsilon, c) - U(c,c) = \\
&\quad \Pr\{b_2 = c\}(v_1 - c) + (v_2 - c)\sum_{l=M-1}^{2N}\sum_{k=1}^{M-1} \frac{(M-k)P_{k,l}(c)}{l-k+2} \\
&\qquad - (v_1 - c)\sum_{l=M}^{2N}\sum_{k=1}^{M} \frac{(2l-k-M+4)(M-k+1)P_{k,l}(c)}{(l-k+3)(l-k+2)} \\
&\qquad - (v_2 - c)\sum_{l=M-1}^{2N}\sum_{k=1}^{M-1} \frac{(M-k)(M-k+1)P_{k,l}(c)}{(l-k+3)(l-k+2)}
\end{aligned}$$

which is non-positive (non-negative) if both $(v_1 - c)$ and $(v_2 - c)$ are non-positive (non-negative). □

We are now able to prove the main theorem of this subsection:

Theorem 8.3. Any strategy is weakly dominated by a strategy in set S. Furthermore, if strategies of opponents are limited to classes in which bids occur in every feasible interval with positive probability (for example, strategy sets in which "trembles" in bidding occur), then the dominance is strict.

Proof. Within each step of the proof, we use the term "dominant" to mean *strictly dominant* if the distribution of opponents' bids is totally mixed and *weakly dominant* otherwise.

Step 1. If c_1 *or* $c_2 < 0$, *then the bid* (c_1, c_2) *is dominated by the following bid:* $(\max\{c_1, 0\}, \max\{c_2, 0\})$.

By property 3.

Step 2. If $c_1 > \max\{c_2, v_1\}$ *then the bid* (c_1, c_2) *is dominated by the following bid:* $(\max\{c_2, v_1\}, c_2)$.

By property 1, U is decreasing in c_1 whenever it is continuous in this region; and by Lemma 8.2(a) and inequality (8.8), U is decreasing over discontinuities in this region. Finally, by Lemma 8.2(a), U is continuous in c_1 at v_1. Thus, $(\max\{c_2, v_1\}, c_2)$ dominates (c_1, c_2).

Step 3. If $c_1 > v_1$ *the bid* (c_1, c_1)*is dominated by the bid* (v_1, v_1).

If $c_1 = c_2$ formula (2.1) can be rewritten as

$$\iint_{\{(b_1,b_2)|b_2 \le c_1\}} [v_1 + v_2 - 2b_1]\, dF(b_1, b_2) + \iint_{\{(b_1,b_2)|b_2 \le c_1 \le b_1\}} [v_1 - c_1]\, dF(b_1, b_2) \ .$$

which is weakly decreasing in c_1 for c_1 greater than v_1. So if U is continuous in this interval, it is weakly dominating to decrease both bids (and, again, strictly dominating if there is a positive probability that a competing bid lies between the old and the new bids). Moreover, by inequalities (8.6–8.7), U increases as c_1 decreases across discontinuities in this region, up to and including any discontinuities at (v_1, v_1). Thus (v_1, v_1) dominates (c_1, c_1).

Step 4. For $c_1 < v_1$ *every bid* (c_1, c_1) *is dominated by a nearby bid* (c_1, c_2) *with* $c_1 > c_2$.

For $c_1 > v_2$, we will consider a nearby bid $(c_1 + \epsilon, c_1 - \epsilon)$. If (c_1, c_1) is a point of continuity, this claim follows because of properties 1, 2, and 4. If (c_1, c_1) is a point of discontinuity, it follows from inequality (8.5). For $c_1 \le v_2$, we will consider a nearby bid $(c_1 + \epsilon, c_1)$. If (c_1, c_1) is a point of continuity, this claim follows because of property 2. If it is a point of discontinuity, it follows from inequality (8.9).

Step 5. For all $c_2 < c_1 < v_1$, (c_1, c_2) *is dominated by* (v_1, c_2).

Follows from property 2 for continuous points and Lemma 8.2(a) for discontinuous parts.

So far we have shown that all other bids are dominated by bids of the form (v_1, c_2), for $c_2 \ge 0$. The final step is to show that if $c_2 > v_2$, this bid is in turn dominated by (v_1, v_2). On continuous intervals, this follows from property 4. Over discontinuities, it follows from Lemma 8.2(a). □

8.2 Characterizing S-equilibria

8.2.1 Preliminary results

As a matter of convention, we take the marginal distributions $F_1(\cdot)$ and $F_2(\cdot)$ of opponents' bids to be *left* continuous. In other words, $F_i(c+) - F_i(c) = \Pr\{b_i = c\}$. We will use the symbolism $\int_a^b$ to denote the Stieltjes integral on $[a, b)$. The following theorem extends the formula for U in S-strategies (Theorem 2.4) to cover the possibility of ties.

Theorem 8.4. If the player has valuations (v_1, v_2) and is bidding (v_1, c_2) where $v_2 \geq c_2 \geq 0$, then the expected payoff $U(c_2, v_1, v_2)$ takes the following form:

$$U = \int_{c_2}^{v_1} F_2(b_2)\, db_2 + 2\int_0^{c_2} F_1(b_1)\, db_1 + (v_2 - c_2)[F_1(c_2) + \phi(c_2)(F_1(c_2+) - F_1(c_2))] \tag{8.10}$$

where ϕ is independent of $(v_1, v_2), 0 < \phi < 1$.

Proof. When $v_1 = c_1$, the equation (2.1) reduces to the expression (8.10), omitting the final term. If $c_2 > c_1$, then $(v_2 - c_2)\phi(c_2)\Pr\{b_1 = c_2\}$ must be the summation terms of formula (8.1). The first summation is multiplied by $(v_1 - c_1) = 0$; in the second, the coefficient on $(v_2 - c_2)$ is positive, independent of (v_1, v_2), and less than $\Pr\{b_1 = c_2\}$. If $c_1 = c_2$ then both bids must equal $v_1 = v_2$, and all summation terms in (8.2) vanish. □

A corollary of this result places strict limitations on the occurrence of pooling in S-equilibria.

Corollary 8.5. If $c_2 < v_2$ and $\Pr\{b_1 = c_2\} > 0$, then (v_1, c_2) is not a best response given valuations (v_1, v_2).

Proof. Under these conditions, $U(c_2+, v_1, v_2) > U(c_2, v_1, v_2)$. □

The following two lemmas underlie many of the characteristics of S-equilibria. The first generates conditions under which shading the lower bid is desirable. The second puts severe restrictions on non-monotonic best responses.

Lemma 8.6. If $F_1(v_2) < F_2(v_2)$, then there exists $c_2 < v_2$ such that $U(c_2, v_1, v_2) > U(v_2, v_1, v_2)$.

Proof. Direct calculation shows that

$$\begin{aligned} U(v_2, v_1, v_2) - U(c_2, v_1, v_2) &= -\int_{c_2}^{v_2} F_2(b_2)\, db_2 + 2\int_{c_2}^{v_2} F_1(b_1)\, db_1 - (v_2 - c_2) \\ &\quad \times [F_1(c_2) + \phi(c_2)(F_1(c_2+) - F_1(c_2))] \\ &\leq -\int_{c_2}^{v_2} F_2(b_2)\, db_2 \\ &\quad + 2\int_{c_2}^{v_2} F_1(b_1)\, db_1 - (v_2 - c_2)F_1(c_2) \end{aligned}$$

Since F_1 and F_2 are left continuous, this last expression is left continuous and left differentiable at v_2. Its derivative with respect to c_2, evaluated at v_2, is

$$F_2(v_2) - F_1(v_2)$$

Thus for c_2 sufficiently close to v_2, as long as $F_2(v_2) > F_1(v_2)$, this final expression is negative. (If U is differentiable in c_2 at $c_2 = v_2$, then the result follows immediately from formula 2.3.) ☐

Lemma 8.7. Given a distribution of opponents' bids, let bids (v_1, c_2) be a best response when the player's valuations are (v_1, v_2) and let (v_1', c_2') be a best response when valuations are (v_1', v_2'). Suppose $v_2' > v_2$. Then $c_2' < c_2$ implies that $F_1(c_2+) = F_1(c_2') = F_2(c_2+) = F_2(c_2')$

In other words, the probability that either b_1 or b_2 lies in the interval $[c_2', c_2]$ is zero.

Proof. Define $\hat{F}_1(c) = F_1(c) + \phi(c)(F_1(c+) - F_1(c))$. We know that

$$U(c_2, v_1, v_2) - U(c_2', v_1, v_2) \geq 0$$
$$U(c_2, v_1', v_2') - U(c_2', v_1', v_2') \leq 0$$

or, substituting formula (8.10)

$$-\int_{c_2'}^{c_2} F_2(b)\,db + 2\int_{c_2'}^{c_2} F_1(b)\,db + (v_2 - c_2)\hat{F}_1(c_2) - (v_2 - c_2')\hat{F}_1(c_2') \geq 0 \tag{8.11}$$

$$-\int_{c_2'}^{c_2} F_2(b)\,db + 2\int_{c_2'}^{c_2} F_1(b)\,db + (v_2' - c_2)\hat{F}_1(c_2) - (v_2' - c_2')\hat{F}_1(c_2') \leq 0 \tag{8.12}$$

Combining, we have

$$(v_2' - v_2)(\hat{F}_1(c_2') - \hat{F}_1(c_2)) \geq 0$$

or

$$\hat{F}_1(c_2') \geq \hat{F}_1(c_2) \ .$$

Since $c_2' < c_2$, we must have in fact

$$\hat{F}_1(c_2') = \hat{F}_1(c_2) \ .$$

In turn this implies that $\phi(c_2') = \phi(c_2) = 0$ and

$$F_1(c_2'+) = F_1(c_2) \ ,$$

for if either $F_1(c_2'+) > F_1(c_2')$ or $F_1(c_2+) > F_1(c_2)$, then it would have to be the case that $\hat{F}_1(c_2') > \hat{F}_1(c_2)$.

Substituting this information into the two inequalities (8.11–8.12) and comparing the two yields the following equation:

$$-\int_{c_2'}^{c_2} F_2(b)\,db + 2\int_{c_2'}^{c_2} F_1(b)\,db - (c_2 - c_2')F_1(c_2) = 0$$

or since F_1 must be constant on the interval (c'_2, c_2) and its integral over that interval equals $(c_2 - c'_2)F_1(c_2)$

$$-\int_{c'_2}^{c_2} F_2(b)\,db + (c_2 - c'_2)F_1(c_2) = 0\ .$$

and, since $F_2 \geq F_1$ at all levels, this in turn implies that F_2 must be constant on the interval and equal to the constant value of F_1. □

8.2.2 Results for asymmetric equilibria

In this section we describe some characteristics of S-equilibria, consequences of results 8.6 and 8.7. The results in this section apply to all equilibria. Some of the results require the following mild restriction on the distribution G:

Condition C: For every positive valuation v, there is a number $\epsilon > 0$ such that for any valuation $v_1, \Pr\{v_2 < v|v_1\} > \epsilon$.

In other words, there is some chance that v_2 is arbitrarily small, no matter what v_1 is.[12] If G is based on independent draws, condition C holds automatically.

Theorem 8.8. In any S-equilibrium, if the number of bidders is at least as great as the number of units for sale, then all bids are weakly increasing functions of the valuations.

Proof. Recall that the distribution of a bidder's higher valuation is assumed to be strictly increasing throughout the interval $(0, \bar{v})$. From the point of view of agent 1, as long as there are at least $M - 1$ bidders competing with him, and each of these competitors has a positive probability that his high bid lies in any interval within $(0, \bar{v})$, then there can be no two distinct values c'_2, c_2, lying in the interval, such that $F_1(c'_2+) = F_1(c_2)$. Thus by result 8.7, bids must be monotonic in valuation. □

Theorem 8.9. If condition C holds, and if the number of bidders is at least as great as the number of units for sale, then in any S-equilibrium for all $v_2 \in (0, \bar{v})$, $c_2 < v_2$.

Proof. By result 8.6, if $c_2 = v_2$ then $F_1(v_2) = F_2(v_2)$. In other words, there is zero probability that exactly $M - 1$ bids are equal to or above v_2. There is a positive probability that any bidder's top bid is above or below v_2, and a positive probability that his lower valuation (and therefore his lower bid) is below v_2. Therefore, as long as $N + 1 \geq M$, there is a positive probability that exactly $M - 1$ top bids are above v_2 and the remaining bids are below. □

The next theorem is the analogue of results 8.8–8.9 when the number of bidders is less than the number of units for sale.

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[12] For some results, the uniformity of the bound ϵ can probably be relaxed.

Theorem 8.10. Suppose the distribution G satisfies condition C.

1) Suppose a player bids (v_1, c_2) given valuations (v_1, v_2) and (v_1', c_2') given valuations (v_1', v_2'), where $v_2' > v_2$ and $c_2' < c_2$. Then with probability 1, $b_1 < c_2'$ and the player is indifferent between bidding c_2' and c_2.

2) Suppose a player whose valuation is (v_1, v_2) bids (v_1, v_2). Then with probability 1, $b_1 < v_2$.

Proof. The previous theorems already handle the case of $N + 1 \geq M$; therefore assume, without loss of generality, $M > N$. Suppose there is a non-monotonicity of bids such that $c_2' < c_2$ when the corresponding valuations are such that $v_2' > v_2$. We will show that all opponents' lower bids lie below c_2' with certainty.

Consider the event $b_1 \geq c_2'$; call this event E_0 and suppose $\Pr\{E_0\} > 0$. Let E_z denote the event $(b_1 \geq c_2'$ and $b_2 < c_2')$. From the theorem of the previous section, we know that $F_1(c_2'+) = F_2(c_2')$. This means that $\Pr\{E_z\} = 0$. In other words, when at least $M - 1$ bids are above c_2 the Mth highest bid must be as well. We will generate a finite sequence of events $E_0, E_1, \ldots, E_t$ in which $\Pr\{E_{i+1}\}/\Pr\{E_i\} > 0$, and $E_t \subseteq E_z$, thereby contradicting the assumption that $\Pr\{E_0\} > 0$. To do so we exploit the independence across bidders, and drop one bid at a time below c_2', until we have exactly $M - 1$ bids above. Start with the lowest bid above c_2'. If this bid is a lower bid for that individual, then consider the event in which all other bids are unchanged but this bid is reduced below c_2'. We know that there is a positive probability ϵ that, given his higher valuation, his lower valuation lies below c_2' (and therefore, so does his lower bid). Thus $\Pr\{E_1\}/\Pr\{E_0\} > \epsilon$. On the other hand, if the bid is a higher bid for that individual, then his lower bid must already be below c_2'. Thus $\Pr\{E_1\}/\Pr\{E_0\} > G_1(c_2'-) > 0$. Proceed in this way dropping bids one by one, until exactly $M - 1$ bids remain. Therefore we conclude that $\Pr\{b_1 \geq c_2'\} = 0$. In other words, at least $2N - M + 1$ of the competitors make second bids lying below c_2' with probability 1.

The proof for part 2 is analogous, and omitted. □

Part 1 of Theorem 8.10 says, in short, that the only time strategies in S-equilibria are not monotonic is when the region in which there is not monotonicity is strategically irrelevant to the player making the bids. This means that for every such equilibrium there is a closely related equilibrium which is monotonic:

Corollary 8.11. Suppose G satisfies condition C. If we are given an S-equilibrium in which some players' strategies are not monotonic in valuations, then there is another S-equilibrium in which all players' strategies are monotonic in valuations and in which each player's distribution of bids is the same as the distribution of bids in the original S-equilibrium.

Proof. Since the player is indifferent between any bids in the non-monotonic region, they can always be rearranged such that bids are monotonic functions of v_2 and the distribution of the second bid is unchanged. Such a rearrangement will have no effect on the distribution of F_1 and F_2 as observed by competitors and therefore will have no effect on their bids (it may have an effect on the exact valuation of ϕ since it may change the joint distribution of (v_1, c_2) but it will not change the set of values of c_2 such that pooling occurs, and therefore has no effect on opponents strategies). $\square$

We can find all S-equilibria by finding the monotone equilibria and then allowing for permutations of bids in the region above which b_1 never lies. Strictly speaking, it is not the case that all S-equilibria involve strategies in which the bidder's second bid depends only on the second value, but as the following theorem demonstrates, there is no loss of generality in restricting attention to such equilibria (cf. [16]). Again, once we have found this class of equilibria, we can readily generate the remaining equilibria.

Theorem 8.12. Suppose G satisfies condition C. For each weakly monotone S-equilibrium there is another equilibrium a.e. equivalent in which the player's second bid is independent of the valuation placed on his first unit.

Proof. This follows from two observations: 1) The set of choices for c_2 which maximize the bidder's expected utility is independent of the value of first unit v_1. 2) The set of v_2's such that there is more than one bid c_2 which maximizes profit is a set of measure zero. $\square$

8.2.3 Results for symmetric equilibria

The results in section 3 of the text are corollaries of the more general results provided in the previous section of this appendix. Their proofs follow.

Proof (Theorem 3.1). From Theorem 8.8, we know that $N + 1 \geq M$, symmetric equilibria are weakly monotone. In the case of $M > N$ Theorem 8.10 says that if the bid of player 1 is not monotone in some region, some other players are banned from making second bids in that region. Thus all symmetric equilibria are weakly monotone. Corollary 8.5 demonstrates that it is never an equilibrium for a bidders to make a lower bid c_2 which has a positive probability of equaling the $M - 1$th highest bid by other players and which is lower than the valuation v_2. Suppose in a symmetric equilibrium c_2, is identical across several valuations v_2. Then, in a symmetric equilibrium, as long the number of goods at auction is as at least three and no more than $2N - 1$ and either 1) the bid is greater than zero or 2) there are more goods than bidders, there is a positive probability that this bid is the lowest winning bid. Since this bid is occurring for several different valuations, it must be the case that for the highest of these valuations the value placed on the good by the bidder is strictly greater than the bid. In this case, an infinitesimal increase in the bid carries an infinitesimal cost in the possible increase in the price paid for the first unit, and a discrete gain in the possibility of winning the good outright. $\square$

Proof (Theorem 3.2). We have proved the result more generally in the asymmetric case where $N + 1 \geq M$. Therefore assume that $M > N$. By result

8.6, decreasing the bid slightly below the valuation v_2 increases the expected payoff as long as

$$F_1(v_2) < F_2(v_2) \ . \tag{8.13}$$

This is the case unless there is zero probability that exactly $M-1$ of the opponents' bids are above v_2. For interior values of v_2, there is a positive probability that any first bid is below or above, and positive probability that any second bid is below. As long as second bids are a strictly increasing function of second valuations on the positive portion of the domain, this is true for second bids as well. Therefore condition (8.13) holds. □

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