Multi-Item Auctions

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A collection of items is to be distributed among several bidders, and each bidder is to receive at most one item. Assuming that the bidders place some monetary value on each of the items, it has been shown that there is a unique vector of equilibrium prices that is optimal, in a suitable sense, for the bidders. In this paper we describe two dynamic auction mechanisms: one achieves this equilibrium and the other approximates it to any desired degree of accuracy.

I. Introduction

Recent studies by Demange (1982), Leonard (1983), and Demange and Gale (1985) have considered an allocation mechanism that turns out to be a generalization of the well-known "second-price" auction first described by Vickery (1961). Recall that in this auction the participants submit sealed bids for a single item, and the item is sold to the highest bidder at a price equal to the second highest bid. In order to describe the multi-item generalization of this mechanism it is conve-

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nient to think of the second-price scheme as an ordinary competitive equilibrium. At the chosen price the item is demanded by only one "consumer," the highest bidder, and, since there is only one item, this yields a balance of supply and demand. The important property of the second highest bid is that it is the *smallest* equilibrium price since for any smaller price at least two bidders would demand the item.

In the multi-item generalization it is assumed that each bidder is interested in acquiring at most one item, as might be the case if, for example, the auction was designed to assign individuals to positions, as considered by Leonard (1983). Each bidder is assumed to place a monetary value on each of the items, and, given a price vector, he will demand that item or those items that maximize his surplus, the difference between his valuation and the price of the item, assuming that this surplus is positive. A price vector then yields equilibrium if every bidder can be assigned an item in his demand set and no two bidders are assigned the same object. It is an interesting and by no means obvious fact that (*a*) the model always has an equilibrium, and (*b*) among the equilibrium price vectors there is a unique one, $\mathbf{p}$, that is smallest in the strong sense that $\mathbf{p}$ is at least as small in every component as any other equilibrium price vector. Thus, $\mathbf{p}$ is the "best" equilibrium price vector from the point of view of the bidders. This observation is due to Shapley and Shubik (1972).

The multi-item auction mechanism requires each bidder to submit a sealed bid listing his valuation of all the items. The auctioneer then allocates the items in accordance with the minimum price equilibrium. A main point of the papers cited in our first paragraph is that the important "incentive capability" of the single-item auction carries over to the multi-item case, meaning that submitting true valuation is a dominant strategy for the bidders. (More generally [Demange and Gale 1985], by jointly falsifying valuations, no subset of bidders can improve the outcome for all its members.)

The purpose of this note is to show that there is another familiar property of the single-item auction that generalizes to the multi-item case. Namely, instead of a one-shot sealed bid auction it is possible to achieve (approximately) the minimum equilibrium price allocation by "dynamic" or, as we shall call them, "progressive" auctions. These are natural generalizations of the familiar auctions that occur in practice, say, at Sothebys or Park-Bernet, in which the auctioneer systematically raises the price of an item until all but one of the bidders has dropped out. In these auctions the sale price will then be approximately the second highest bid since, presumably, the highest bidder will try to outbid the competition by as small an amount as possible.

We shall present here two different dynamic auction mechanisms for the multi-item case. The first will be the more structured of the two and will produce in a finite number of steps the exact minimum price equilibrium. The second mechanism will imitate almost exactly the usual free-for-all that occurs at real auctions and will lead to a final allocation that can be made as close as one wishes to the minimum price equilibrium. Both of these mechanisms have already appeared in the literature. The first is a variant of the so-called Hungarian method of Kuhn (1955) for solving the optimal assignment problem, while the second is a special case of the algorithm of Crawford and Knoer (1981). We will describe these mechanisms in the next section and prove their convergence properties in the final sections.

Aside from the theoretical interest, it may be of some practical use to have available the progressive as well as the sealed bid mechanisms. In particular, the sealed bid mechanism operates on the assumption that a bidder's utility is of the simple linear surplus type already described, but this is a rather special case. For example, if a worker must choose between taking job A at salary α or job B at salary β, it seems rather unlikely that he would make the decision by comparing the difference between the salaries offered and the minimum salary he would accept for each job. The dynamic mechanisms allow for a much wider range of preferences for the bidders. It is also known (Crawford and Knoer 1981; Demange and Gale 1985) that even for quite general preferences there is a minimum equilibrium price. The results given here suggest that for small enough bid increases our progressive auctions will lead to prices approximating the minimum price equilibrium. However, our results are given here only for the linear surplus case. The analysis for more general preferences remains a problem for future investigation.

II. The Progressive Auction Mechanisms

In actual auctions the seller of an item as well as the bidders plays a role in determining the outcome, in that the seller usually specifies some reservation price, this being the minimum price he will accept for his item. In our model it is therefore assumed that for each item α there is such a minimum sale price $s(\alpha)$. Further, in general, at the minimum equilibrium price some items may remain unsold. This leads to the following additional requirement for a price equilibrium: If item α is unsold, then its price $p(\alpha)$ must equal the sale price $s(\alpha)$. Clearly this condition is required for equilibrium since, if $p(\alpha)$ exceeds $s(\alpha)$, there would be excess supply and the seller of item α would want to lower its price.

In both the mechanisms to be described we start with an initial price vector $\mathbf{p}_0$ announced by the auctioneer and equal to the vector of sales prices $\mathbf{s}$. We first describe what we will call the "exact auction mechanism." We suppose that all prices and valuations are integers. The unit of price could be, say, dollars or hundreds of dollars depending on the type of item being auctioned. Each bidder now announces which item or items he wants to buy at the initial prices. If it is possible to assign each item to a bidder who demands it, one already has the desired equilibrium. If no such assignment exists, the procedure depends on a celebrated theorem of combinatorics due to Hall (1935). We will say that a set of items is overdemanded if the number of bidders demanding only items in this set is greater than the number of items in the set. Clearly a necessary condition for the existence of an equilibrium assignment is that there be no overdemanded set. Hall's theorem asserts that this condition is also sufficient. A short proof of this result can be found, for example, in Gale (1960), which also describes an efficient algorithm that finds either an equilibrium assignment or an overdemanded set. In fact, the algorithm actually locates a minimal overdemanded set, that is, an overdemanded set with the property that none of its proper subsets is overdemanded. Now the auctioneer (probably with the aid of a computer) locates such a minimal overdemanded set and then raises the price of each item in the set by one unit. He then again elicits the demands of the bidders in some systematic way. For example, he might announce each item in turn and ask which bidders are interested in buying it at its current price. (Note that a bidder expresses interest in more than one item if and only if they both maximize his surplus at the given prices.) After the new bids are announced, the auctioneer again finds either a complete assignment or a new minimal overdemanded set, whose prices he again raises. Now it is clear that this second alternative cannot occur indefinitely because, as soon as the price of an item becomes sufficiently large, say, higher than any of the bidders' valuations for it, the item cannot belong to any overdemanded set. It follows that for some set of prices there will be an equilibrium assignment. We have thus, in fact, proved the existence of equilibrium. Of course, this proof depended on using Hall's theorem. What is not so obvious is that the prices obtained in this way are the minimum equilibrium prices. This will be proved in the next section.

We remark that, in order for the mechanism to work and lead to the minimum price equilibrium, it is necessary for the bidders to be quite precise in their responses to changing prices. Namely, we must assume that at each stage of the auction each bidder demands *all* items that maximize his surplus at the current prices. So, for example, if a bidder demands only one item at some step he must also demand that item, and possibly others, at the next step. He cannot switch in one step from one item to a different one. This is a consequence of the requirement that his valuations are integral numbers of units. For this reason the exact auction mechanism would seem difficult to implement in realistic situations as contrasted with the method we describe next.

In the "approximate auction mechanism" the auctioneer again announces an initial sales price. At this point any bidder may bid for any item. When he does so he is said to be committed to that item, which means he commits himself to possibly buying the item at the announced price. The item is said to be (tentatively) assigned to that bidder. At a general stage of the auction some subset of bidders will be committed to some subset of items at some set of prices. At this point any uncommitted bidder may (i) bid for some unassigned item, in which case he becomes committed to it at its initial price; (ii) he may bid for an assigned item, in which case he becomes committed to that item, its price increases by some fixed amount δ, and the bidder to whom it was assigned becomes uncommitted; or (iii) he may drop out of the bidding. If one wishes to structure this procedure the auctioneer could call on the uncommitted bidders, say in alphabetical order, requiring them to choose one of the three alternatives listed above. The auction terminates when there are no more uncommitted bidders, at which point each committed bidder buys the item assigned to him at its current price.

One can imagine that this sort of auction would be appealing to the bidders for it does not require them to decide in advance exactly what their bidding behavior will be. Instead, at each stage a bidder can make use of present and past stages of the auction to decide his next bid. Of course, the outcome of this auction may depend on the order in which people bid. However, our result for this case shows that this variation is limited. More precisely, if people behave in accordance with linear valuations we show that the final prices will differ from the minimum equilibrium price by at most $k\delta$ units, where k is the minimum of the number of items and bidders. Thus, by making δ (the unit by which bids are increased) sufficiently small, one can come arbitrarily close to the minimum equilibrium price.

In analyzing the outcome of these two auction mechanisms in the following sections we will always suppose that the bidders behave in accordance with the linear surplus utility functions described in the Introduction. That is, in the case of the exact mechanism each bidder announces (honestly) at each stage the item or items whose value to him exceeds its current price by the largest amount. In the approximate mechanism he computes the difference between his value of an item and its initial price if it is unassigned or its price plus δ if it is assigned, and he chooses an item for which this difference (his surplus) is a maximum. (We will not consider here possible "manipulative" behavior in which a bidder at some stage may choose an item that is not his most preferred. Such behavior for the sealed bid auction is the main subject of Leonard [1983] and Demange and Gale [1985].)

In the next section our results are presented in the form of four theorems. The accompanying proofs are intended for readers who may wish to pursue these matters in more detail.

III. Convergence of the Exact Auction Mechanism

Let B be the set of bidders and let I be the set of items. For each item α in I there is a sale price $s(\alpha)$. The value of item α to bidder b is $v_{b\alpha} \geq 0$. A feasible price vector $\mathbf{p}$ is a function from I to R^+ such that $p(\alpha) \geq s(\alpha)$. It is convenient to assume that there is a null item α_0 whose value is zero to all bidders and whose price is always zero. The demand set of b at price $\mathbf{p}$ is defined by

$$D_b(\mathbf{p}) = \{\alpha | v_{b\alpha} - p(\alpha) = \max_{\beta \in I} [v_{b\beta} - p(\beta)]\}.$$

The price $\mathbf{p}$ is called competitive if there is an assignment μ from B to I such that $\mu(b) \in D_b(\mathbf{p})$, and if $b' \neq b$ and $\mu(b) = \mu(b')$, then $\mu(b) = \alpha_0$. The pair $(\mathbf{p}, \mu)$ is an equilibrium if this condition is satisfied and also the following condition: if $\alpha \notin \mu(B)$, then $p(\alpha) = s(\alpha)$.

We will show that the exact auction mechanism converges to the minimum competitive price $\mathbf{p}$. Finally, we show that μ can then be chosen so that $(\mathbf{p}, \mu)$ is an equilibrium.

THEOREM 1. Let $\mathbf{p}$ be the price vector obtained from the exact auction mechanism and let $\mathbf{q}$ be any other competitive price. Then $\mathbf{p} \leq \mathbf{q}$.

Proof. By contradiction, suppose $\mathbf{p} \not\leq \mathbf{q}$. Now at stage $t = 0$ of the auction we have $\mathbf{p}_0 = 0$ so $\mathbf{p}_0 \leq \mathbf{q}$. Let t be the last stage of the auction at which $\mathbf{p}_t \leq \mathbf{q}$, $S_1 = \{\alpha | p_{t+1}(\alpha) > q(\alpha)\}$, and S be the overdemanded set whose prices are raised at stage $t + 1$. Thus $S = \{\alpha | p_{t+1}(\alpha) > p_t(\alpha)\}$, so $S_1 \subset S$. We will show that $S - S_1$ is nonempty and overdemanded; hence S is not a minimal overdemanded set, contrary to the rules of the auction.

Define $T = \{b | D_b(\mathbf{p}_t) \subset S\}$. Since S is overdemanded, this means exactly that

$$|T| > |S|. \tag{1}$$

Define $T_1 = \{b | b \in T \text{ and } D_b(\mathbf{p}_t) \cap S_1 \neq \varnothing\}$. We claim that $D_b(\mathbf{q}) \subset S_1$ for all b in T_1. Indeed, choose α in $S_1 \cap D_b(\mathbf{p}_t)$. If $\beta \notin S$, then b prefers α to β at price $\mathbf{p}_t$ because $b \in T$. But $p_t(\beta) \leq q(\beta)$ and $p_t(\alpha) = q(\beta)$, so b prefers α to β at price $\mathbf{q}$. On the other hand, if $\beta \in S - S_1$, then b likes

α at least as well as β at price $\mathbf{p}_t$. But $p_t(\beta) < p_{t+1}(\beta) \leq q(\beta)$—and, again, $p_t(\alpha) = q(\alpha)$—so b prefers α to β at price $\mathbf{q}$, as claimed. Now since $\mathbf{q}$ is competitive, there are no overdemanded sets at price $\mathbf{q}$, so

$$|T_1| \leq |S_1|. \tag{2}$$

Now $T - T_1 = \{b | b \in T \text{ and } D_b(\mathbf{p}_t) \in S - S_1\}$. But from (1) and (2), $|T - T_1| > |S - S_1|$, so $S - S_1$ is overdemanded, giving the desired contradiction. Q.E.D.

THEOREM 2. If $\mathbf{p}$ is the minimum competitive price then there is an assignment μ^* such that $(\mathbf{p}, \mu^*)$ is an equilibrium.

Proof. Let μ be an assignment corresponding to $\mathbf{p}$. We call an item α overpriced if it is not assigned but $p(\alpha) > s(\alpha)$. If $(\mathbf{p}, \mu)$ is not an equilibrium, there is at least one overpriced item. We will give a procedure for altering μ so as to eliminate overpriced items. For this purpose we construct a directed graph whose vertices are $B \cup I$. There are two types of arcs. If $\mu(b) = \alpha$ there is an arc from b to α. If $\alpha \in D_b(\mathbf{p})$ there is an arc from α to b. Now let α_1 be an overpriced item. Then $\alpha_1 \in D_b(\mathbf{p})$ for some b, for if not one could decrease $p(\alpha)$ and still have competitive prices. Let $\tilde{B} \cup \tilde{I}$ be all vertices that can be reached by a directed path starting from α_1.

Case 1.—Set $\tilde{B}$ contains an unassigned bidder, b. Let $(\alpha_1 b_1, \alpha_2 b_2, \ldots, \alpha_k, b)$ be the path from α to b. Then we may change μ by assigning b_1 to α_1, b_2 to $\alpha_2, \ldots, b$ to α_k. The assignment is still competitive and α_1 is no longer overpriced, so the number of overpriced items has been reduced.

Case 2.—All b in $\tilde{B}$ are assigned. Then we claim that there must be some α in I such that $p(\alpha) = s(\alpha)$, for suppose not. By definition of $\tilde{B} \cup \tilde{I}$ we know that if $b \notin \tilde{B}$ then b does not demand any item in $\tilde{I}$. Therefore, we can decrease the price of each item in I by some positive t and still have competitiveness, contradicting the minimality of $\mathbf{p}$. So choose $\alpha \in \tilde{I}$ such that $p(\alpha) = s(\alpha)$ and let $(\alpha_1, b_1, \alpha_2, b_2, \ldots, b_k, \alpha)$ be the path from α_1 to α. Again change μ by assigning b_i to α_i for all i, leaving α unassigned. Again the number of overpriced items has been reduced. Q.E.D.

IV. Convergence of the Approximate Auction Mechanism

In this section we show that the final price of an item using the approximate auction mechanism will differ from the minimum equilibrium price by at most $k\delta$, where $k = \min(|I|, |B|)$. We will prove this in two parts. If $\mathbf{p}$ is the minimum equilibrium price and $\hat{\mathbf{p}}$ is the final auction price, we first show that $\hat{p}(\alpha)$ can exceed $p(\alpha)$ by at most $k\delta$. This is a consequence of the following theorem.

THEOREM 3. No bidder bids for an item α if its price at time t is $p_t(\alpha) \geq p(\alpha) + k\delta$.

We need some preliminary results. Let us call an item α expensive at time t if $p_t(\alpha) > p(\alpha)$. Let μ be any assignment corresponding to $\mathbf{p}$.

LEMMA 1. If b bids for an expensive item α, then he is assigned by μ.

Proof. If b were unassigned, then $v_{b\alpha} - p(\alpha) \leq 0$. Thus, $v_{b\alpha} - p_t(\alpha) \leq 0$ since α is expensive. Therefore, b would not bid for α at time t. Q.E.D.

LEMMA 2. If $\mu(b) = \alpha$ and b bids for β at time t, then $p_t(\alpha) - p(\alpha) \geq p_t(\beta) - p(\beta)$.

Proof. We have $v_{b\beta} - p_t(\beta) \geq v_{b\alpha} - p_t(\alpha)$ since b bids for β but also $v_{b\alpha} - p(\alpha) \geq v_{b\beta} - p(\beta)$ since $\mu(b) = \alpha$, and adding these inequalities gives the asserted result. Q.E.D.

To prove theorem 3, suppose b_1 bids for α when $p_t(\alpha) \geq p(\alpha) + k\delta$. Then α is expensive, so from lemma 1, b_1 is assigned under μ to some item α_1 (possibly $\alpha_1 = \alpha$). From lemma 2,

$$p_t(\alpha_1) - p(\alpha_1) \geq p_t(\alpha) - p(\alpha) \geq k\delta. \tag{3}$$

So $p_t(\alpha_1) \geq p(\alpha_1) + k\delta > p(\alpha_1) \geq s(\alpha_1)$. Thus some bidder b_2 is assigned to α_1 at time t. Then $k > 1$, for if not, $\alpha = \alpha_1$ and α_1 would be expensive. Therefore, b_2 should be matched to some item by μ, which is a contradiction since $b_1 \neq b_2$. Therefore, b_2 must have bid for α_1 at price $p_t(\alpha_1) - \delta \geq p(\alpha_1) + (k - 1)\delta > p(\alpha_1)$, so α_1 was expensive. By lemma 1, b_2 is assigned under μ to some α_2. Since b_1 and b_2 are both assigned, we have $k \geq 2$. Again from lemma 2, $p_t(\alpha_2) - p(\alpha_2) \geq p_t(\alpha_1) - \delta - p(\alpha_1) \geq (k - 1)\delta$, so $p_t(\alpha_2) \geq p(\alpha_2) + (k - 1)\delta > p(\alpha_2) \geq s(\alpha_2)$. So some b_3 is committed to α_2 at price $p_t(\alpha_2)$. Then $k > 2$, for if not $\alpha \in \{\alpha_1, \alpha_2\}$ and α_2 would be expensive, and so b_3 should be matched by μ, which is a contradiction. So b_3 must have bid for α_2 at price $p_t(\alpha_2) - \delta \geq p(\alpha_2) + (k - 2)\delta > p(\alpha_2)$, so α_2 was expensive, so by lemma 1, b_3 is assigned by μ to some α_3, so $k \geq 3$. It is clear that this process can never terminate. Thus k is unbounded, which is impossible. Q.E.D.

Now let $\hat{\mathbf{p}}$ be the final price for an approximate auction. We must show that no price will be very much lower than the minimum equilibrium price $\mathbf{p}$.

THEOREM 4. For any item α, $\hat{p}(\alpha) \geq p(\alpha) - k\delta$, where $k = \min(|I|, |B|)$.

Proof. We will show that if there is some α_1 such that $\hat{p}(\alpha_1) < p(\alpha_1) - k\delta$ then there must be more than k items α such that $p(\alpha) > s(\alpha)$. This would contradict equilibrium since at most k items can be assigned. By assumption, $p(\alpha_1) \geq \hat{p}(\alpha_1) + (k + 1)\delta > s(\alpha_1)$, so α_1 is assigned under μ, say to b_1. Now there must be some other bidder b_1' who demands α_1 at price $\mathbf{p}$ for if not one could decrease $p(\alpha_1)$ and still have equilibrium. It follows that b_1' is committed at $\hat{\mathbf{p}}$ to some item α_2 and $\hat{p}(\alpha_2) \leq p(\alpha_2) - k\delta$; namely,

$$v_{b_1'\alpha_2} - \hat{p}(\alpha_2) \geq v_{b_1'\alpha_1} - \hat{p}(\alpha_1) - \delta$$

and

$$v_{b_1'\alpha_1} - p(\alpha_1) \geq v_{b_1'\alpha_2} - p(\alpha_2).$$

Adding gives

$$p(\alpha_2) - \hat{p}(\alpha_2) \geq p(\alpha_1) - \hat{p}(\alpha_1) - \delta \geq k\delta.$$

It follows that $p(\alpha_2) \geq \hat{p}(\alpha_2) + k\delta > s(\alpha_2)$, so α_2 is matched at equilibrium, say to b_2. Now again there must be some third bidder b_2' who demands either α_1 or α_2 for if not $p(\alpha_1)$ and $p(\alpha_2)$ could be decreased. Also, as in the previous step, b_2' must be committed to some item α_3 whose price $\hat{p}(\alpha_3)$ satisfies $p(\alpha_3) - \hat{p}(\alpha_3) \geq (k - 1)\delta$. Again $p(\alpha_3) \geq s(\alpha_3) + (k - 1)\delta$, so α_3 is assigned at prices $\mathbf{p}$ to some b_3, and so forth. Continuing in this way we get $k + 1$ assigned items, which is impossible. Q.E.D.

V. A More General Model

The multi-item model treated here is "unsymmetrical" in that each seller specifies only one number, his reservation price, while buyers specify $|I|$ numbers, their valuations for each of the items. There are, however, economically natural situations in which sellers "discriminate," specifying different reservation prices to different buyers. In fact, in the job assignment problem this would be the expected situation. Here the sellers are workers who are selling their services to employers. Clearly, the minimum salary a worker would accept might vary depending on the job; for example, the more disagreeable the job, the higher the minimum acceptable salary.

How would this greater generality affect our auction mechanisms? Not very much. The only real difference would be that at each stage the auctioneer would specify not a vector of current prices but a matrix (s_{wj}) of current salaries, where s_{wj} is the salary w would demand if he were to accept job j. Given such a matrix it is clear how employers would behave in order to maximize their profits. Employer j would choose that worker or those workers for whom $v_{wj} - s_{wj}$ was a maximum, and overdemanded sets of workers, uncommitted employers, and so forth could be defined exactly as before. The price-raising mechanism would also operate as before so that if w belonged to an overdemanded set then s_{wj} would increase by one for all j. Similarly in the approximate mechanism, if an uncommitted employer bids for a tentatively assigned worker his salaries s_{wj} all increase by δ.

Actually there is a rather simple transformation that reduces this seemingly more general case to the one already treated. If a_{wj} is the minimum salary w would accept for job j and b_{wj} is the maximum that employer j would pay w, then let $c_{wj} = b_{wj} - a_{wj}$ and consider the multi-item auction with valuation matrix (c_{wj}) and reservation prices zero. Now if $\mathbf{p}$ is any price vector for this model the corresponding salary matrix for the general model is given by $s_{wj} = p_j + a_{wj}$. We leave it to the reader to verify that this transformation indeed establishes the equivalence of the two models.

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