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Asymmetric All-Pay Auctions with Incomplete Information: The Two-Player Case

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We prove existence and uniqueness of (Bayesian) equilibrium for a class of generally asymmetric all-pay auctions with incomplete information. Due to its importance in applications some prominence is given to the first-price all-pay auction, for which a detailed characterization of equilibrium and an approximation to equilibrium of its well-studied complete information version are supplied. Furthermore, we relate our uniqueness result to the well-known multiplicity of equilibria in the "war of attrition" (second-price all-pay auction), which emerges as a "limit" point of the class of two-player auction games considered. *Journal of Economic Literature* Classification Numbers: D44, C62, C72.

1. INTRODUCTION

In a first-price auction the bidder who submits the highest bid gets the object and pays the amount of her bid, while in a second-price auction she pays the amount of the second highest bid. Unsuccessful bidders pay nothing in either of these auctions. In contrast, in a first- or second-price *all-pay* auction all bidders have to pay their bids (except the highest bidder in the second-price all-pay auction, who pays the second highest bid), while the object is still awarded to the highest bidder exclusively. Hence active participation in a standard auction of the former type constitutes only a *conditional* commitment (of resources), namely conditioned on winning the auction, whereas active participation in an auction of the latter type constitutes an *unconditional* commitment. This, obviously, must have profound consequences for participants' behavior.

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All-pay auctions have received much less attention than standard auctions in theoretical work on auctions. Yet first-price all-pay auctions or equivalent models, which exhibit the characteristic "unconditional commitment mode," have recently been used to model a variety of economic and social instances of conflict and competition. Notably problems in the theory of rent seeking and lobbying (e.g., Tullock, 1980; Becker, 1983, 1985; Hillman and Samet, 1987; Hillman and Riley, 1989), but also in the theory of R & D—and patent competition (e.g., Dasgupta and Stiglitz, 1980; Fudenberg *et al.*, 1983; Harris and Vickers, 1985a,b; Leininger, 1991), the theory of contests and tournaments (e.g., Nalebuff and Stiglitz, 1983; Baye *et al.*, 1990b; Riley 1991), and the theory of arms races and other social phenomena exhibiting escalation (e.g., Shubik, 1971; O'Neill, 1986; Leininger, 1989) have been addressed with the help of models which can be interpreted as first-price all-pay auctions. The second-price all-pay auction or "war of attrition" plays an important role in theoretical biology (e.g., Maynard Smith, 1974; Bishop and Cannings, 1978; Riley, 1979, 1980), but also appears in economic theory, e.g., as a theory of market exit (Fudenberg and Tirole, 1986). The cited applications of the first-price all-pay auction all represent *complete* (or even *perfect*) *information* games, which for their respective domains of investigation constitutes a serious informational restriction.

The present paper models the first-price all-pay auction as a game of *incomplete information*, in which after nature's casting moves the two players may be of different types (not known to one another). This not only allows for more realistic informational assumptions in applications, but also moves the model closer to the realm of (standard) auction theory and the independent private values model. But note that we do *not* require that the beliefs of the players be symmetric, as is done in most parts of the received literature on auction theory. Some of the most notable auction theoretic results, such as the celebrated "revenue equivalence theorem" (Myerson, 1981), do not apply to asymmetric problems. Standard (first-price) auctions have only recently been analyzed with regard to existence, uniqueness, and computation of equilibrium under general asymmetry of beliefs (Maskin and Riley, 1991, 1992, Marshall *et al.*, 1994). Maskin and Riley (1992) prove existence of equilibrium for the n-bidder case and uniqueness of equilibrium for the two-bidder case of continuously distributed types.

The few papers that address the first-price *all-pay* auction from an auction theoretic viewpoint either postulate complete information (see Baye *et al.* 1990a, for a summary) or invoke the assumption of *symmetric beliefs* if the information structure is incomplete (two brief examples occur in Weber, 1985, and Hillman and Riley, 1989). Both of these assumptions are absent from the present analysis. Completeness of information guarantees existence of equilibrium in mixed strategies, which is unique in the two-bidder case and generically unique in the n-bidder case (Hillman and Riley, 1989). Interestingly, biological interest has led to investigations of the "war of attrition" under both complete and incomplete information (Riley, 1980; Milgrom and Weber, 1985), which in the latter case also dispensed with the assumption of symmetric beliefs (Nalebuff and Riley, 1985). One immediate contribution of the paper therefore is to close the gap between the existing knowledge on solutions of the first- and second-price all-pay auctions under incomplete information.

Our main result proves existence and uniqueness of Bayesian equilibrium for a class of generally asymmetric all-pay auction games with incomplete information. This class of games includes the first-price all-pay auction, but not the second-price all-pay auction which arises only as a "limiting" case of it. This, however, suffices for an explanation of the drastically different solution behavior of the "war of attrition," which possesses infinitely many equilibria, in the light of the uniqueness result obtained here. This part of the analysis parallels the proceedings in Riley (1991), who investigates asymmetric contests with *perfect information*, when he compares solutions under discriminatory (i.e., first-price) and nondiscriminatory (i.e., second-price) pricing rules.

Given the prominence of the *first*-price all-pay auction in the cited applications and the more advanced state of knowledge regarding its theoretically important "alternative," the *second*-price all-pay auction, Section 2 starts out with the basic two-bidder case of the first-price all-pay auction as a Bayesian game. Section 3 contains a detailed equilibrium analysis and gives special emphasis to the discussion of examples with asymmetric priors. A by-product of these investigations is an approximation result, which comments on the relation between equilibria of incomplete information all-pay auction games (in pure strategies) and the (mixed strategy) equilibria of complete information games, if the former more closely approximate the latter. Section 4 states the main result (Theorem 2) by demonstrating that the solution theory developed in Section 3 for the first-price all-pay auction generalizes to an entire class of (all-pay) auctions, whose specific price rules were first considered by Güth and van Damme (1986) in the context of standard auctions. The second-price rule of the "war of attrition" emerges as a limiting case. This gives rise to an equilibrium selection argument for the "war of attrition."

2. THE FIRST-PRICE ALL-PAY AUCTION AS A BAYESIAN GAME

Consider two players, X and Y, who simultaneously bid for an indivisible object they value at V_X and V_Y, respectively. The informational structure of the auction game is such that each player knows her own valuation, but not the other player's. Player X, besides knowing V_X, holds information on V_Y in the form of a probability distribution (prior) $N(V_Y)$; player Y, symmetrically, knows V_Y and prior $M(V_X)$. M and N are assumed to be independent probability distributions. W.l.o.g. we assume that V_X and V_Y are distributed on $[0, 1]$, such that $\text{supp}(M) = \text{supp}(N) = [0, 1]$. If player X submits a (sealed) bid of x and player Y a bid of y, payoffs are determined by

$$U_X(x, y) = \begin{cases} V_X - x & \text{if } x > y \\ \frac{1}{2}V_X - x & \text{if } x = y \\ -x & \text{if } x < y \end{cases}$$

and

$$U_Y(x, y) = \begin{cases} V_Y - y & \text{if } y > x \\ \frac{1}{2}V_Y - y & \text{if } y = x, \\ -y & \text{if } y < x \end{cases}$$

i.e., all bids are sunk; even the loser has to pay her bid. We can think of this game of incomplete information as a two-stage game (with no proper subgames): At stage I "Nature" independently chooses V_X according to $M(V_X)$ from [0, 1] and V_Y according to $N(V_Y)$ from [0, 1]. At stage II player X and Y simultaneously choose bids x and y by making optimal use of their respective information on the valuations determined at stage I.

The private information regarding the value of V_X (resp. V_Y) a player is holding at the beginning of stage II is henceforth referred to as her "type." As usual, a (pure) strategy for a player in a Bayesian game is defined to be a function from her set of types (here, feasible valuations) into her set of actions (here, feasible bids):

$$\begin{array}{ll} \text{Strategy for } X\text{: } \alpha\text{: } [0, 1] \to \mathbb{R}_+ & V_X \mapsto \alpha(V_X) = x \\ \text{Strategy for } Y\text{: } \beta\text{: } [0, 1] \to \mathbb{R}_+ & V_Y \mapsto \beta(V_Y) = y. \end{array}$$

For the moment we assume $\alpha(\cdot)$ and $\beta(\cdot)$ to be monotonically (strictly) increasing, a requirement which will turn out to be satisfied *automatically* in equilibrium.

Equilibrium. A pair of strategies (α^*, β^*) constitutes a Bayesian (Nash) equilibrium if for all V_X and for all V_Y, the following conditions hold:

(a) $E_{V_Y} U_X(\alpha^*(V_X), \beta^*(V_Y)) \geq E_{V_Y} U_X(x, \beta^*(V_Y))$ for all $x \in \mathbb{R}_+$;
(b) $E_{V_X} U_Y(\alpha^*(V_X), \beta^*(V_Y)) \geq E_{V_X} U_Y(\alpha^*(V_X), y)$ for all $y \in \mathbb{R}_+$.

Conditions (a) and (b) require that a strategy specify an action for each type of a player, such that each type is maximizing her expected utility when she knows her type and the distribution of the other player's type, where E_{V_X} (resp. E_{V_Y}) stands for expectation operators w.r.t. $M(V_X)$ (resp. $N(V_Y)$). To be more precise, a player has to optimize against the distribution of actions resulting from her opponent's type distribution and employed strategy. So X is faced with an expected bid distribution $N(\beta^{*-1}(y))$, if V_Y has c.d.f. $N(V_Y)$ and Y uses strategy β^*; and Y, similarly, is faced with bid distribution $M(\alpha^{*-1}(x))$.

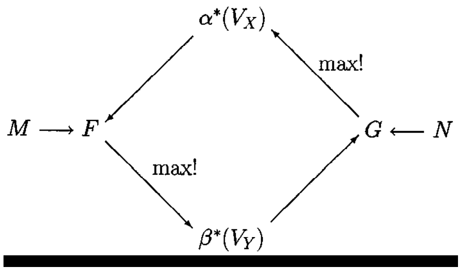

FIG. 1. Bayesian Equilibrium

Let us define $F(x) := M(\alpha^{*-1}(x))$ and $G(y) := N(\beta^{*-1}(y))$. Figure 1 then shows the interaction of decisions and expectations in Bayesian equilibrium: equilibrium is reached if the arrows (indicating causal influence) relate the functions to one another in a consistent, self-confirming way. The inner circle of the diagram then "commutes." Hence equilibrium conditions can be stated either "vertically" in terms of strategies α^* and β^* or "horizontally" in terms of bid distributions F and G.

3. EQUILIBRIUM ANALYSIS

Let $M(V_X)$ and $N(V_Y)$ denote the c.d.f.'s of arbitrary priors on [0, 1]. We assume that $M(0) = N(0) = 0$ and that M and N possess positive densities $m(V_X)$ and $n(V_Y)$. Suppose further that player X uses strategy $\alpha(V_X)$ and player Y uses strategy $\beta(V_Y)$.

We now provide some general elementary results for equilibrium bidding distributions (F and G) and equilibrium bidding strategies (α and β) of the first-price all-pay auction with private values. The derivation of similar statements is common in the auction theory literature, the earliest example is probably the work of Griesmer *et al*. (1967). We relegate all proofs to the Appendix.

LEMMA 1 (Common Support). supp(F) = supp(G).

LEMMA 2 (No Atoms). *F is continuous on* $[0, \beta(1)]$ *with* $\beta(1) \leq 1$, *G is continuous on* $[0, \alpha(1)]$ *with* $\alpha(1) \leq 1$.

LEMMA 3 (Monotonicity). *Let* $V_X > V_X'$, $v = \alpha(V_X)$, *and* $v' = \alpha(V_X')$. *Then* $G(v) \geq G(v')$ *holds.*

Note that this—by Lemma 2—implies that $\alpha(V_X) \geq \alpha(V_X')$ with strict inequality if $\alpha(V_X) > 0$. The obvious analog holds for types of player Y.

LEMMA 4 (Full Support).

$$\text{supp}(F) = [0, \max_{V_X \in [0,1]} \alpha(V_X)],$$

$$\text{supp}(G) = [0, \max_{V_Y \in [0,1]} \beta(V_Y)].$$

Note that Lemma 1 combined with Lemmas 3 and 4 implies that $\alpha(1) = \beta(1)$; i.e., the two types with, respectively, the highest valuation must submit the *same* bid.

LEMMA 5. *If* $M(0) = N(0) = 0$, *then* $\min\{G(0), F(0)\} = 0$.

Lemma 5 means that at most one player can have an atom at 0 in her bid distribution.

Denote the expected payoff for player X (resp. Y) from bidding x (resp. y) against strategy β (resp. α) when the valuation is V_X (resp. V_Y) by $\Pi_X(x, \beta \mid V_X)$ (resp. $\Pi_Y(y, \alpha \mid V_Y)$. The above lemmas then imply that we can write

$$\Pi_X(x, \beta \mid V_X) = V_X \cdot N(\beta^{-1}(x)) - x$$

and

$$\Pi_Y(y, \alpha \mid V_Y) = V_Y \cdot M(\alpha^{-1}(y)) - y.$$

Note that $N(\beta^{-1})$ resp. $M(\alpha^{-1})$ represents the bid distributions G resp. F of Fig. 1. Maximization of Π_X and Π_Y w.r.t. x and y then yields equilibrium first-order conditions

$$\begin{aligned} V_X \cdot N'(\beta^{-1}(x))(\beta^{-1})'(x) &= 1 \\ V_Y \cdot M'(\alpha^{-1}(y))(\alpha^{-1})'(y) &= 1. \end{aligned} \tag{FOC}$$

(FOC) has an intuitive explanation. Any type V_X, say, must not be able to increase her payoff by bidding $x+dx$ instead of x, when $x = \alpha(V_X)$ and $y = \beta(V_Y)$. Such a change in her bid would increase cost by $1 \cdot dx$ with certainty (because of the all-pay rule). On the other hand, it would yield a gain of $V_X = \alpha^{-1}(x)$ *if* the other player's bid, y, were in $[x, x+dx)$, because of the first-price rule. The probability that Y's bid indeed is in $[x, x + dx)$, however, is exactly $N'(\beta^{-1}(x))(\beta^{-1})'(x) \cdot dx = G'(x)dx$. Equating cost and gains yields the above first-order conditions. (FOC) is readily turned into a condition involving strategies *only*. Equilibrium *strategies* must satisfy

$$\begin{aligned} N'(\beta^{-1}(x)) \cdot (\beta^{-1})'(x) &= \frac{1}{\alpha^{-1}(x)} \\ M'(\alpha^{-1}(y)) \cdot (\alpha^{-1})'(y) &= \frac{1}{\beta^{-1}(y)}; \end{aligned} \tag{E}$$

i.e., the relation between equilibrium strategies and equilibrium bid distributions (Fig. 1) is

$$\alpha^{-1}(x) = \frac{1}{G'(x)} \qquad \text{and} \qquad \beta^{-1}(y) = \frac{1}{F'(y)}. \tag{S-D}$$

Equivalently, equilibrium *bid* distributions F, G must solve

$$\begin{aligned} F(x) &= M\left(\frac{1}{G'(x)}\right) \\ G(y) &= N\left(\frac{1}{F'(y)}\right). \end{aligned} \tag{E$'$}$$

Thus, in a *symmetric* situation where $F = G$ (and, of course, $M = N$)

$$F(x) = M\left(\frac{1}{F'(x)}\right) \qquad \text{for all } x \tag{E$''$}$$

must hold. For example, with uniform priors we have $M(V) = N(V) = V$, and (E$''$) becomes

$$F(x) \cdot F'(x) = 1 \qquad \text{for all } x \quad (\text{and } F(0) = 0).$$

Try $F(x) = a \cdot x^r$, which implies $F'(x) = a \cdot r \cdot x^{r-1}$. Then $(a \cdot x^r) \cdot (a \cdot r \cdot x^{r-1}) = 1$ implies $2r - 1 = 0$ and $a^2 \cdot r = 1$. It follows that $r = \frac{1}{2}$ and $a = \sqrt{2}$. Thus, $F^*(x) = \sqrt{2} \cdot x^{1/2} = (2x)^{1/2}$ solves (E$''$). Inverting F^* yields the symmetric Bayesian equilibrium strategies $\alpha^*(V_X) = \frac{1}{2}V_X^2$ and $\beta^*(V_Y) = \frac{1}{2}V_Y^2$. In this equilibrium each type of player always bids *less* than her true valuation.[1]

We are now ready to prove:

THEOREM 1. *Let V_X and V_Y be independently distributed on* $[0, 1]$ *according to M and N, whose densities are assumed to be continuously differentiable and positive on* $(0, 1)$. *Then there exists a unique Bayesian equilibrium of the first-price all-pay auction.*

Proof. We will use the first-order conditions (FOC) and Lemmas 1–5, which are necessary conditions for equilibrium, to derive unique equilibrium bidding functions.

Consider the mapping k defined by $k\colon [0, 1] \to [0, 1]$, $k(V_X) = \beta^{-1}(\alpha(V_X))$, which maps any type V_X of Player X who bids $\alpha(V_X))$ into that type of Player Y who makes the same bid; $V_Y = \beta^{-1}(\alpha(V_X))$ as $\beta(\beta^{-1}(\alpha(V_X))) = \alpha(V_X)$.

[1] There also exist asymmetric solutions to (E$''$); e.g., it is readily verified that $F(x) = s \cdot x^{1/2}$ and $G(y) = (s/2) \cdot y^{1/2}$, $s \neq 0$, solve (E$''$). However, the implied strategies $\alpha(V_X) = (1/s^2) \cdot V_X^2$ and $\beta(V_Y) = (s^2/4) \cdot V_Y^2$ are *only* equilibrium strategies, if $1/s^2 = s^2/4$, i.e., $s = \sqrt{2}$ (which is the symmetric case!). This follows from the "Common Support Lemma."

The preceding Lemmas 1–5 precisely imply that k is well-defined on (0, 1] and maps [0, 1] *onto* [0, 1]. Moreover, because of the Monotonicity lemma, k must be strictly increasing except possibly on $k^{-1}(0)$, i.e., if F has an atom at 0 (recall that *at most* one player can have an atom at 0 of her bid distribution). This means that on $[0, 1]\backslash\{k^{-1}(0)\} = (a, 1]$, with $a \geq 0$, we have that $k' = dk/dv$ is defined and

$$k'(v) = (\beta^{-1})'(\alpha(v)) \cdot \alpha'(v) \tag{$*$}$$

must hold (as $(\beta^{-1})'$ and α' are defined from (FOC)).

Now, the first equality of (FOC) can be rewritten as

$$(\beta^{-1})'(\alpha(v)) = \frac{1}{v \cdot N'(\beta^{-1}(\alpha(v)))} = \frac{1}{v \cdot N'(k(v))},$$

whereas the second condition of (FOC), upon observing that $\alpha'(v) = 1/(\alpha^{-1})'(\alpha(v))$, becomes

$$(\alpha^{-1})'(v) = \beta^{-1}(\alpha(v)) \cdot M'(\alpha^{-1}(\alpha(v))) = k(v) \cdot M'(v).$$

Thus, using ($*$), a necessary condition for equilibrium is that

$$k'(v) = \frac{k(v) \cdot M'(v)}{v \cdot N'(k(v))} \tag{1}$$

and

$$\alpha'(v) = k(v) \cdot M'(v) \tag{2}$$

hold. However, (1) is an ordinary first-order differential equation, which, under the hypotheses on M and N, admits a *unique* solution $k(v)$ satisfying the necessary (Lemmas 1, 3, and 4) boundary condition $k(1) = 1$ (see, e.g., Brauer and Nohel, 1973, Theorem 1, p. 331). Then (2) yields the unique equilibrium strategies

$$\alpha^*(V_X) = \int_{\max\{k^{-1}(0)\}}^{V_X} k(v) \cdot M'(v)\, dv$$

and

$$\beta^*(V_Y) = \alpha^*(k^{-1}(V_Y)) \qquad \text{(by definition of } k\text{)},$$

where $\alpha^*(V_X) = 0$ if and only if $V_X \in k^{-1}(0)$ (by Lemma 5). ■

The mapping k, which reduces to the identity mapping in the symmetric case, is also well-defined in other auction games. It is suggested as a potentially very efficient vehicle in existence proofs under asymmetry of beliefs.[2]

[2] Maskin and Riley (1991, 1992) obtain their existence results in a comparatively more laborious way. After establishing auxiliary lemmas, which are analogous to our Lemmas 1–5, they manipulate the resulting system of first-order conditions directly with the help of variable transformations. They obtain unqiueness for the two-player case by performing an additional sensitivity analysis with respect to boundary values.

The content of Theorem 1 calls for a brief comment on the *symmetric* case: Specialization of Theorem 1 to the case of identical priors yields the verification of a conjecture advanced by Weber (1985), namely that the unique *symmetric* equilibrium found by Weber in this case is the only equilibrium: since in *any* symmetric game with a unique equilibrium, that equilibrium must be symmetric, we have

COROLLARY 1. $M = N \Rightarrow \alpha^* = \beta^*$.

Furthermore, under symmetry, integration of (2) yields

$$\alpha^*(V_X) = \int_0^{V_X} V \cdot m(V)\,dV = \frac{\int_0^{V_X} V \cdot m(V)\,dV}{\int_0^{V_X} m(V)\,dV} \cdot \int_0^{V_X} m(V)\,dV$$
$$= E(V \mid V \leq V_X) \cdot M(V_X),$$

which allows for a direct comparison with the *standard* first-price auction with incomplete information. If only the winner pays her bid, the equilibrium bid of the type V_X in the symmetric case is given by $E(V \mid V \leq V_X)$, the expected valuation of her opponent's type *conditional* on holding the higher valuation. In the all-pay case this bid is seen to be weighted with the probability of this event occuring; i.e., $M(V_X) = \text{Prob}(V_Y \leq V_X)$ as M is also the c.d.f. of V_Y. This is a reflection of the fact that losing in the all-pay auction is costly, while it is costless in the standard auction. Consequently, individual bids in the former are on average *lower* than in the latter. This, of course, does not mean that the latter extracts higher expenditures from the bidders, since there only one of the two bids has to be paid for. In fact, it is a straightforward consequence of the "revenue equivalence theorem" (Myerson, 1981) that the two auctions are revenue equivalent and they both yield expected revenue of $E\{\min(V_X, V_Y)\}$.

In order to further exploit the closed-form nature of our equilibrium condition (2), we now consider a numerical example in some detail.

Let $M(V_X) = V_X^m$ and $N(V_Y) = V_Y^n$ with $m, n \in \mathbb{R}_+$.[3]

(i) For $m = n$ the corollary implies $k(v) = v$. Equation (2) then gives $\alpha_n^*(V_X) = n \cdot \int_0^{V_X} v^n\,dv = (n/(n+1))V_X^{n+1}$ and hence $\beta_n^*(V_Y) = (n/(n+1))V_Y^{n+1}$. The corresponding bid distribution is given by $F_n^*(x) = (((n+1)/n)x)^{n/(n+1)}$ for $x \in [0, n/(n+1)]$.

(ii) With asymmetric priors, i.e., $m \neq n$, we get from (1)

$$k' = \frac{k \cdot m \cdot v^{m-1}}{v \cdot n \cdot k^{n-1}} = \frac{m \cdot v^{m-2}}{n \cdot k^{n-2}},$$

[3] As noted in Marshall *et al*. (1994), these are the distributions of the maximum value of the members of a coalition of size m and n, when the members' valuations are distributed on the [0, 1] interval.

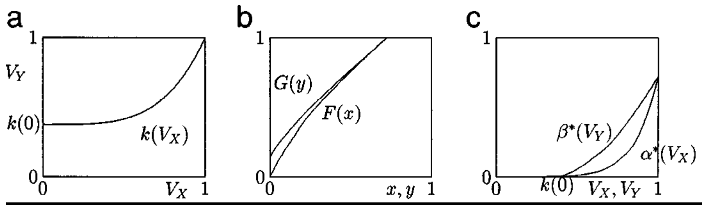

FIGURE 2

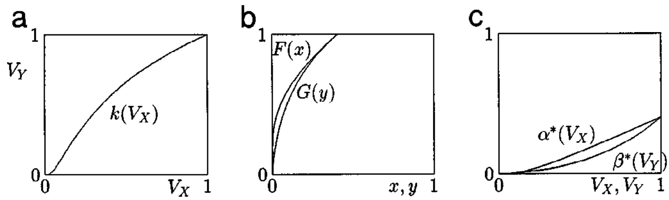

FIGURE 3

which has the unique solution

$$k(v) = \left(\frac{m}{n} \cdot \frac{(n-1)}{(m-1)} \cdot (v^{m-1} - 1) + 1\right)^{1/(n-1)} \qquad \text{satisfying } k(1) = 1.$$

Figure 2 gives the numerical computation of the solution for $m = 5$ and $n = 2$.

Note that the solution to (1), which must start in $(1, 1)$, does not run into the orgin (Fig. 2a). Accordingly, the bidding distribution of player Y, who on average has the *lower* valuation, has an atom at 0. One easily computes that with probability $G(0) = \frac{9}{64}$ she does not submit a (positive) bid (Fig. 2b). More precisely, all types of player Y with $V_Y \leq k(0) = \frac{3}{8}$ do not bid (β^* in Fig. 2c).

In contrast, the case $m = \frac{1}{2}$ and $n = 1$ depicted in Fig. 3 shows *both* equilibrium bid distributions being atomless ($k(0) = 0$).

The following lemma, which is easily proved, accounts for the above observed behavior of the solution $k(v)$ in terms of properties of the underlying priors.

LEMMA 6.

$$k(0) = (0) \Leftrightarrow \int_0^1 \frac{N'(w)}{w}\, dw = \int_0^1 \frac{M'(w)}{w}\, dw.$$

Note that the latter equality holds in particular when both integral values are infinite (as in our second example pictured above). Moreover, it holds trivially in the symmetric case. A similar condition appears in Nalebuff and Riley (1985), where the behavior of the infinitely many (!) solution functions of the "war of attrition" is classified as we will discuss further in the next section.

Finally, observe that in case (i) of our example the priors (weakly) converge to the mass *point* $\{1\}$ if $n \to \infty$; i.e., $M^{\infty}(V_X) = N^{\infty}(V_Y) = 0$, if $0 \leq V_X, V_Y < 1$, and $M(1) = N(1) = 1$. Thus, this limit represents the complete-information case, in which both valuations are known to be 1. The complete-information case has been studied extensively (independent proofs of uniqueness of equilibrium can be found in Hillman and Samet, 1987; Lippman and Mamer, 1988; Baye *et al.*, 1990a; Rosenthal, 1993). With equal valuations of the prize the unique (*mixed*) strategy equilibrium has both players bidding according to a uniform distribution on $[0, 1]$. We now observe from the representation of $F_n^*(x)$ given above that $\lim_{n\to\infty} F_n^*(x) = x$ for all $x \in [0, 1]$; i.e., with an ever diminishing degree of uncertainty over the opponent's valuation the equilibrium *bid distributions* of the incomplete information settings converge to the equilibrium bid distributions of the limiting complete information case! This observation generalizes to general priors when weak convergence onto the certainty case is maintained.

PROPOSITION 1. *Let $M_k(V_X)$ and $N_k(V_Y)$, $k \in \mathbb{N}$, be sequences of priors with positive densities on* $(0, 1)$ *such that $M_k^{-1}(z)$ and $N_k^{-1}(z)$ converge to $w(z) \equiv 1$ on the interval* $(0, 1]$. *Then the equilibrium bid distributions $F_k^*(x)$ and $G_k^*(y)$ of the first-price all-pay auction with priors M_k and N_k converge to $F^*(x) = x$ and $G^*(y) = y$ on* $[0, 1]$ *as $k \to \infty$.*

Proposition 1, which is proven in the Appendix, is a manifestation of the "approximation" theorem of Harsanyi (1973), which demonstrates how behavior under complete information governed by *mixed* strategies can be understood as (a limit of) behavior following from the use of *pure* strategies under (a slight degree of) incompleteness of information. Milgrom and Weber (1985) showed a similar convergence result for *symmetric* equilibria of the two-player second-price all-pay auction. With regard to the theory of rent seeking, Proposition 1 supplies the important insight that "full rent-dissipation" results obtained from complete-information models have to be understood as limits of only "partial rent-dissipation" in the *same* models with incomplete information.

4. A GENERAL CLASS OF ALL-PAY AUCTIONS

We now consider a family of all-pay auctions with incomplete information, which has as its extreme members the first-price all-pay auction examined above and the second-price all-pay auction (examined in Nalebuff and Riley, 1985).

Let $\lambda \in [0, 1]$ be an index which defines an all-pay λ-auction by the following price-rule: bids of x and y by Players X and Y result in payments for the players of

$$\begin{array}{rl} (1-\lambda)\cdot x + \lambda \cdot y & \text{if } x > y \\ x & \text{if } x \leq y \end{array} \qquad \text{for player } X$$

and, symmetrically,

$$\begin{array}{rl} (1-\lambda)\cdot y + \lambda \cdot x & \text{if } y > x \\ y & \text{if } y \le x \end{array} \qquad \text{for player } Y;$$

i.e., the winner pays an amount between the first and second price, while the loser always pays her bid. These λ-price rules were first considered by Güth and van Damme (1986) to assess distributional aspects of standard (winner-only-pays) auction design. Riley (1991) interprets them as *imperfectly* discriminatory rules with realistic appeal in "rent-seeking" contexts. Finally, Plum (1992) uses them explicitly to relate the solution theory of first-price auctions with incomplete information to second-price auctions with incomplete information in the standard case. Technically, these λ-price rules constitute a "homotopy map" between the first- and second-price all-pay auction; i.e., the former is *continuously* deformed into the latter via homotopy parameter $\lambda \in [0, 1]$. The case $\lambda = 0$ is characterized by a unique Bayesian equilibrium (Theorem 1) and the case $\lambda = 1$ by a continuum of Bayesian equilibria (Nalebuff and Riley, 1985). We now turn to the question of how the equilibrium sets are transformed into one another as the homotopy parameter increases.

Recall our simplifying assumption that $\text{supp}(M(V_X)) = \text{supp}(N(V_Y)) = [0, 1]$. Expected payoffs now read (recall our previous notation)

$$\Pi_X^\lambda(x, \beta \mid V_X) = V_X \cdot (\beta^{-1}(x)) - x - \lambda \cdot \int_0^x (y - x)\, dN(\beta^{-1}(y))$$

(as $(1-\lambda)\cdot x + \lambda \cdot y = x + \lambda(y - x)$) and

$$\Pi_Y^\lambda(y, \alpha \mid V_Y) = V_Y \cdot M(\alpha^{-1}(y)) - y - \lambda \cdot \int_0^y (x - y)\, dM(\alpha^{-1}(x)).$$

Accordingly, maximization of Π_X^λ and Π_Y^λ yields first-order equilibrium conditions as

$$\begin{aligned} V_X \cdot N'(\beta^{-1}(x)) \cdot (\beta^{-1})'(x) &= 1 - \lambda \cdot N(\beta^{-1}(x)) \\ V_Y \cdot M'(\alpha^{-1}(y)) \cdot (\alpha^{-1})'(y) &= 1 - \lambda \cdot M(\alpha^{-1}(y)). \end{aligned} \qquad (\text{FOC}_\lambda)$$

(FOC_λ) demands the expected gain and cost of a marginal increase in each player's bid to be the same. Note, in particular, that for $\lambda = 1$ the first-order conditions for the "war of attrition" with incomplete information as identified in Riley (1980, Eqs. (A19) and (A20)) result.

THEOREM 2. *Let V_X, V_Y be independently distributed on* $[0, 1]$ *according to M and N whose densities are assumed to be differentiable and positive on* $(0, 1)$. *Then there exists a unique Bayesian equilibrium for all* $\lambda \in [0, 1)$.

We give only a sketch of the proof since it completely parallels the proof of Theorem 1. The mapping

$$\begin{aligned} k_\lambda\colon [0,1] &\to [0,1] \\ V_X &\to \beta^{-1}(\alpha(V_X)) \end{aligned}$$

is, again, well-defined and yields the differential equation

$$k'_\lambda(v) = (\beta^{-1})'(\alpha(v)) \cdot \alpha'(v). \tag{$*$}$$

Rewriting and substituting (FOC_λ) into ($*$) now yields

$$k'_\lambda(v) = \frac{1-\lambda \cdot N(k_\lambda(v))}{1-\lambda \cdot M(v)} \cdot \frac{k_\lambda(v) \cdot M'(v)}{v \cdot N'(k_\lambda(v))} \qquad (1)_\lambda$$

$$\alpha'(v) = \frac{k_\lambda(v) \cdot M'(v)}{1-\lambda \cdot M(v)}. \qquad (2)_\lambda$$

The first observation is that setting $\lambda = 0$ reproduces Eqs. (1) and (2) obtained in the proof of Theorem 1. In general, $(1)_\lambda$ is an ordinary differential equation of first-order, which admits a *unique* solution $k(v)$ satisfying $k(1) = 1$ *provided* $\lambda \neq 1$. For $\lambda = 1$, $(1)_\lambda$ is not defined for $v = 1$, as then $M(v) = 1$; this singularity singles out the "war of attrition" from our family and accounts for its drastically different solution behavior. For $0 \leq \lambda < 1$, $(1)_\lambda$ implies that $k(1) = 1$ must hold at a *finite* maximal bid of type 1 of either player, which uniquely determines the solution trajectory. In contrast, for $\lambda = 1$ the singularity in $(1)_\lambda$ means that $k(1) = 1$ holds at an indeterminate *infinite* maximal bid (see Riley, 1980), making the boundary less effectual on the form of the solution trajectories and accounting for the multiplicity of equilibria in the "war of attrition."

Theorem 2 suggests an equilibrium *selection* procedure for the "war of attrition": select the equilibrium which is obtained as the limit of the sequence of unique equilibria as $\lambda \to 1$. This technically suggested procedure is also behaviorally well justifiable. It is an idealization that the winner of a "war of attrition" *exactly* bears the cost (effort, etc.) expended by the loser; more realistically, it has to be a little more: the very moment when the "loser" decides to "give up," i.e., stops making further efforts, her opponent is still fully engaged in the contest. The loser's expenditure hence can serve only as a lower bound on the winner's effort, and by letting λ go to 1 we approach this lower bound arbitrarily closely.

Let us illustrate this point with the help of an example analyzed by Nalebuff and Riley (1985):

Suppose M and N are uniform distributions over [0, 1], i.e., $M(V) = N(V) = V$. In this case $(2)_\lambda$ reads

$$\alpha'(V) = \frac{k_\lambda(V) \cdot M'(V)}{1-\lambda \cdot M(V)} = \frac{V}{1-\lambda \cdot V},$$

and thus

$$\alpha^*(V_X) = \int_0^{V_X} \frac{V}{1-\lambda V}\, dV$$

$$= \begin{cases} \frac{1}{2}V_X^2 & \text{if } \lambda = 0 \\ -\frac{V_X}{\lambda} - \frac{1}{\lambda^2}\ln|1-\lambda\cdot V_X| & \text{if } \lambda \in (0,1). \end{cases}$$

Note that the solution depends *continuously* on λ as $\ln(1+x) = \sum_{t=1}^{\infty}(-1)^{t-1}\cdot(x^t/t)$ and hence

$$-\frac{V_X}{\lambda} - \frac{1}{\lambda^2}\ln|1-\lambda\cdot V_X| = \frac{V_X^2}{2} + \frac{\lambda\cdot V_X^3}{3} + \cdots + \frac{\lambda^{t-2}\cdot V_X^t}{t} + \cdots$$
$$= \frac{V_X^2}{2} + \sum_{t=3}^{\infty}\frac{\lambda^{t-2}}{t}\cdot V_X^t.$$

Obviously, for $\lambda \to 1$ we get as limit strategy

$$\alpha_1^*(V_X) = -V_X - \ln(1-V_X)$$

and a maximal bid of

$$\lim_{V_X\to 1}\alpha_1^*(V_X) = \infty.$$

This strategy then yields $\beta_1^*(V_Y) = -V_Y - \ln(1-V_Y)$; i.e., what is obtained in the limit is the only *symmetric* equilibrium of the solution family of the "war of attrition" identified by Nalebuff and Riley (1985). The reason for the selection of the symmetric equilibrium is, of course, the continual validity of Corollary 1 for all $\lambda \in [0, 1)$.

APPENDIX

Recall that F and G resp. α and β are assumed to refer to a Bayesian equilibrium.

Proof of Lemma 1. Suppose $x = \alpha(V_X) \notin \text{supp}(G)$. Then there is an open neighborhood of x, $U(x)$, such that for all $x' \in U(x)$, $G(x') = G(x)$. Suppose $G(x) > 0$, then if X lowers her bid from x to $x' < x$, but $x' \in U(x)$, her probability of winning (i.e., her expected gain) does not change, while her cost will decline by $(x - x')$. This improves her expected payoff and, thus, x cannot be optimal in contradiction to $x = \alpha(V_X)$. Consequently, $G(x) = 0$. A symmetric argument holds for player Y. ■

Proof of Lemma 2. Suppose G is not continuous at z; i.e., let $z \in (0, \alpha(1)]$ and $\delta > 0$ such that $G(z) > G(z-\varepsilon) + \delta$ for all $\varepsilon < \varepsilon(\delta)$. If we define $\underline{G}(z) := \lim_{\varepsilon\to 0} G(z-\varepsilon)$ then the following inequalities hold:

$$\begin{aligned}
\Pi_X(z-\varepsilon, \beta \mid V_X) &= G(z-\varepsilon)\cdot V_X - (z-\varepsilon) \\
&< \underline{G}(z)\cdot V_X - (z-\varepsilon) \\
&< \left(\frac{1}{2}\underline{G}(z) + \frac{1}{2}G(z) - \frac{\delta}{2}\right)\cdot V_X - (z-\varepsilon) \\
&= \Pi_X(z, \beta \mid V_X) + \varepsilon - \frac{\delta}{2}\cdot V_X \\
&< (G(z)-\delta)\cdot V_X - (z-\varepsilon) \\
&< \Pi_X(z+\varepsilon, \beta \mid V_X) + 2\varepsilon - \delta\cdot V_X .
\end{aligned}$$

As $V_X \geq \varrho = z/G(z)$ we have that for all $\varepsilon < \min\{\varepsilon(\delta), \delta/4\cdot\varrho\} =: \bar{\varepsilon}$

$$\Pi_X(z-\varepsilon, \beta \mid V_X) < \Pi_X(z, \beta \mid V_X) - \frac{\delta}{4}\cdot V_X < \Pi_X(z+\varepsilon, \beta \mid V_X) - \frac{\delta}{2}\cdot V_X,$$

which means that X will not bid in $[z-\bar{\varepsilon}, z]$. But then $F(y)$ is constant for $y \in [z-\bar{\varepsilon}, z]$ and, by the same argument as in Lemma 1, $y = z$ cannot be a best response for player Y in contradiction to $z = \alpha(V_X)$ for some $V_X \in [0,1]$. Thus G must be continuous at z. A symmetric argument holds for F. ■

Proof of Lemma 3. By definition of Bayesian equilibrium we must have that

$$\Pi_X(v, \beta \mid V_X) \geq \Pi_X(v', \beta \mid V_X) \quad \text{and} \quad \Pi_X(v', \beta \mid V_X') \geq \Pi_X(v, \beta \mid V_X'),$$

which implies

$$(\Pi_X(v, \beta \mid V_X) - \Pi_X(v', \beta \mid V_X)) + (\Pi_X(v', \beta \mid V_X') - \Pi_X(v, \beta \mid V_X')) \geq 0.$$

Upon rearranging terms we get

$$(\Pi_X(v, \beta \mid V_X) - \Pi_X(v, \beta \mid V_X')) + (\Pi_X(v', \beta \mid V_X') - \Pi_X(v', \beta \mid V_X)) \geq 0,$$

which is equivalent to

$$(V_X - V_X')\cdot G(v) + (V_X' - V_X)\cdot G(v') \geq 0.$$

Rearranging terms again yields

$$(V_X - V_X')\cdot(G(v) - G(v')) \geq 0.$$

Since the first term is positive by assumption, the second must be nonnegative, which proves the claim. ■

Proof of Lemma 4. Suppose there is a "hole" (a, b), $0 < a < b < \max_{V_Y \in [0,1]} \beta(V_Y)$, over which (according to the previous lemmas) F is constant, while a and b belong to the support of F. Then—by Lemma 1—G must be constant over (a, b), and since $G(a) = G(b)$ it can never be optimal for player X to bid $x = b$ by the same argument as in Lemma 1. Hence, such a hole (a, b) in the interior of $[0, \max_{V_X \in [0,1]} \alpha(V_X)]$ cannot exist; neither can it exist in the interior of $[0, \max_{V_Y \in [0,1]} \beta(V_Y)]$. ■

Proof of Lemma 5. Suppose $G(0) = s > 0$. Then

$$\Pi_X(\varepsilon, \beta \mid V_X) = G(\varepsilon) \cdot V_X - \varepsilon \geq s \cdot V_X - \varepsilon > \frac{s}{2} \cdot V_X = \Pi_X(0, \beta \mid V_X)$$

for any $\varepsilon \leq (s/2) \cdot V_X$.

But this means that—for every type of player X—making a strictly positive bid dominates making a bid of 0. This proves our claim. ■

Proof of Proposition 1. From (E′) we know that

$$F_k(x) = M_k\left(\frac{1}{G_k'(x)}\right) \qquad \text{and} \qquad G_k(x) = N_k\left(\frac{1}{F_k'(x)}\right)$$

with $F_k(\bar{x}_k) = G_k(\bar{x}_k) = 1$, where $\bar{x}_k = \max_{V_X \in [0,1]}\{\alpha_k(V_X)\} \leq 1$. This can be rewritten as

$$G_k'(x) = 1 + \frac{1 - M_k^{-1}(F_k(x))}{M_k^{-1}(F_k(x))} \tag{1}$$

and

$$F_k'(x) = 1 + \frac{1 - N_k^{-1}(G_k(x))}{N_k^{-1}(G_k(x))}. \tag{2}$$

By hypothesis, M_k^{-1} and N_k^{-1} are monotonically increasing. Hence on *any* interval $[\varepsilon, 1]$, $\varepsilon > 0$, M_k^{-1} and N_k^{-1} converge *uniformly* to $w(z) \equiv 1$. Let $\varepsilon \in (0, 1)$ and $k(\varepsilon)$ be such that $N_k^{-1}(z) > 1/(1 + \varepsilon)$ for all $z \in [\varepsilon, 1]$ and for all $k \geq k(\varepsilon)$. By continuity of F_k (Lemma 2), $F_k(z) \geq \varepsilon$ for all z in some positive interval $[z_k, \bar{x}_k]$. Now choose $\underline{z}(\varepsilon) \geq 0$ as $\inf\{z \in [0, 1] \mid F_k(z) \geq \varepsilon \text{ for all } k \geq k(\varepsilon)\}$. By construction and Eq. (2) above, $F_k'(z) < 1 + \varepsilon$ for all $z \in [\underline{z}(\varepsilon), \bar{x}_k]$ and for all $k \geq k(\varepsilon)$. This implies that $\underline{z}(\varepsilon) \leq 2\varepsilon/(1 + \varepsilon)$. Let ε go to 0. Then $\underline{z}(\varepsilon)$ goes to 0 and consequently $\lim_{k\to\infty} F_k'(x) = 1$ for *all* $x \in (0, 1]$.

By a symmetric argument, $\lim_{k\to\infty} G_k'(x) = 1$ for all $x \in (0, 1]$ must hold.

Now let $\{\bar{x}_{k_l}\}$ be a convergent subsequence of $\{\bar{x}_k\}$. The above argument—together with $F_k(\bar{x}_k) = G_k(\bar{x}_k) = 1$ (Lemma 1)—then implies that G_{k_l} is uniformly convergent on $(0, 1]$ and that F ($F(x) = \lim_{l\to\infty} F_{k_l}(x)$) and G

($G(x) = \lim_{l\to\infty} G_{k_l}(x)$) are of the form $F(x) = G(x) = x + c$ on $[0, 1]$ with $c =$ const. Since at most one of the sequences of equilibrium bid distributions F_{k_l} and G_{k_l} can have an atom at 0 for k_l large enough (Lemma 5), we conclude that $c = 0$. This proves the proposition. ■

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