Strategic Ironing in Pay-as-Bid Auctions: Equilibrium Existence with Private Information

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May, 2016

Abstract

I establish the existence of pure-strategy Bayesian-Nash equilibria in divisible-good pay-as-bid auctions with private information, and show that such equilibria can approximate equilibria in nearby multi-unit auctions. I show that equilibrium strategies exhibit *strategic ironing*, a reduction of bids below what might be expected. Strategic ironing has implications for the tractability of equilibrium strategies.

*kyle.woodward@unc.edu. I would like to thank Marek Pycia for valuable advice and insights. I would also like to thank Liran Einav, Teddy Kim, Jeffrey Mensch, Moritz Meyer-ter-Vehn, Joe Ostroy, Alessandro Pavan, Phil Reny, Valentin Verdier, and Bill Zame, as well as seminar audiences at UCLA, Northwestern, UNC, SWET, SAET, EC, and INFORMS for valuable discussions.

1 Introduction

Pay-as-bid auctions are frequently implemented when a single seller wants to allocate units of a perfectly homogeneous commodity. In practice, pay-as-bid auctions are used to sell sovereign debt, allocate electricity generation, distribute emissions permits, and were used in the Federal Reserve's quantitative easing program. Auctions for multiple units differ crucially from their single-unit counterparts, in that bids are frequently multidimensional objects in which offers for some units may affect the incentives to bid for others: bidders submit demand curves to the seller, who then computes the classical market-clearing price and allocations. I show that in the pay-as-bid context, in which the seller perfectly price-discriminates against reported demand, there is an equilibrium in pure strategies when bidders have private information. I also describe a bid-flattening effect I term *strategic ironing*, in which local downward bidding incentives imply reduced bids across the quantity domain.

In spite of their ubiquity, relatively little is concretely known about the behavior of bidders in pay-as-bid auctions. Beyond the apparent theoretical difficulty of computing fully general revenue and efficiency rankings, progress in the analysis of parameterized models has been hampered by the inability to efficiently compute equilibrium strategies in the case where goods, as in practice, are imperfectly divisible. Meaningful results have been obtained in certain settings—see, e.g., Engelbrecht-Wiggans and Kahn (2002), Ausubel et al. (2014), and Lotfi and Sarkar (2015)—but the general state of the art is best captured by Hortaçsu and Kastl (2012), who state, "Unfortunately, computing equilibrium strategies in (asymmetric) discriminatory multi-unit auctions is still an open question."[1]

Where discrete problems appear intractible, continuous approximations may offer sound and available economic insights (Geoffrion, 1976; Hall, 1986).[2] For example, the literature on single-unit auctions frequently relies upon the assumption that the set of available prices is dense. In the case of multi-unit auctions, bids may be approximated as objects determined on a dense domain of quantities, as well; there is no counterpart to this possibility in single-unit auctions, or even in combinatorial auctions. Wilson (1979) was the first to apply this approximation method in the context of multi-unit auctions, and this approximation has been used to establish

[1]There has been work in building approximate equilibria for multi-unit auctions; see, e.g., Armantier et al. (2008).

[2]In the opposite direction, the macroeconomic literature has long recognized issues regarding applying discrete estimation techniques to continuous problems (Sims, 1971; Phillips, 1974; Geweke, 1978). The key distinction here is that such models assume that the underlying economic fundamentals operate continuously, while in the pay-as-bid auction the true model of (ex post) quantity allocations is fundamentally discrete. More-closely related are the results of Boutilier et al. (1999), which posit an equilibrium for a discrete combinatorial auction. This equilibrium is intractable and is approximated by a continuous equilibrium, which is then discretized for computational approaches. Bontemps et al. (2000) develop a continuous search model which renders tractable discrete search problems.

results for parameterized models such as Ausubel et al. (2014), and **?**, but in the general case it has not even been known if an equilibrium exists. Without a sound basis for the existence of equilibrium strategies, it has been problematic to meaningfully apply the divisible-good model to policy debates. In this paper I establish the existence of a pure-strategy equilibrium, that this equilibrium has mathematically nice monotonicity properties, and that it may approximate equilibrium in large multi-unit auctions.

Although existence in this particular mechanism has remained an open problem, the question has been addressed in many related models. A thread of literature beginning with Athey (2001) examines equilibrium existence in models with private information and continuous payoffs. McAdams (2003) extends this result to include multidimensional private information, and Reny (2011) generalizes to the case of arbitrary lattices. These results cannot be directly applied because, as is common in auction models, payoff discontinuities cannot be ruled out ex ante; in fact, Section 4 proves that payoff discontinuities will certainly arise. Reny (1999) allows for discontinuous utility functions, but does not permit private information; his results have been extended by McLennan et al. (2011) and Barelli and Meneghel (2013). Merian-Weg and Häfner (2015) demonstrate the existence of an equilibrium in distributional strategies in a pay-as-bid auction with constrained bids, but do not obtain a pure-strategy existence result.

For the existence of a pure-strategy equilibrium all that is required is that marginal value functions are monotone and continuous in quantity and signal, and that the distribution of signals is convex and atomless. My proof approach builds directly on the existence results of McAdams (2003), and presents two novel features: first, there is a natural profile of "limiting strategies." I then show how these limiting strategies can be used to construct equilibrium strategies for all bidders.

I establish that in any equilibrium, bidders' actions are monotonic in their private information; additionally, the equilibrium I find must be close, in observed outcomes, to equilibrium in similarly-parameterized large multi-unit auctions for discrete goods.[3] This is of particular importance for the empirical value of the pay-as-bid share auction model, and is in stark contrast to the uniform-price auction; it has been observed that the uniform-price share auction model can admit a large set of "collusive-seeming" equilibria which cannot arise in the analogous discrete-good setting (Fabra et al., 2002; Kremer and Nyborg, 2004).[4]

One might hope that, knowing that an equilibrium exists and that it is in a neighborhood of equilibria of other mechanisms, a full characterization is close at hand. This is not the case. Section 4 explores a difficulty in the analytic charac-

[3] I do not prove the converse of this statement. That is, there might be equilibria of the divisible-good pay-as-bid auction which do not approximate any equilibrium of the discrete-good pay-as-bid auction. There may additionally be equilibria of the discrete-good auction which do not converge to an equilibrium of the divisible-good auction.

[4] This does not invalidate the share auction approach. Empirical work has been careful to address this feature of the uniform-price auction by inferring upper bounds for revenue comparison.

terization of equilibrium strategies while providing a two-bidder example in which behavior is (relatively) tractable. I demonstrate that the pay-as-bid format induces potentially dramatic bid reduction through a process which I term strategic ironing. This process distinguishes the pay-as-bid auction from the intuitive generalization of a first-price auction to many units. While it has been understood that the uniform-price auction is not simply a generalized second-price auction , the manner in which the pay-as-bid auction differs from a multi-unit generalization of a first-price auction has not previously been clarified.[5]

Explicit computation of equilibrium strategies in the pay-as-bid auction is complicated by two factors. First, there is an inherent asymmetry when the bidders' marginal values are strictly decreasing in quantity: one bidder bidding for a high-value quantity will be competing against other bidders for relatively low-value quantities. As presented in Maskin and Riley (2000), such asymmetric auctions can be solved as a system of differential equations, but generally lack a closed-form analytical solution. Second, I show that there are quantities for which the bidder's optimization problem is non-local; that is, the bid for a particular quantity is determined not by incentives for this quantity, but by incentives over an interval of quantities. This leads to strategic ironing: bidders subject their idealized bid functions to a monotonicity constraint, similar to the principal's process described in Myerson (1981). Crucially, this is where the first-price auction intuition breaks down: in single-object auctions there can be no non-local incentives, hence there is no analogy for this effect. Importantly, unlike in Myerson (1981), strategic ironing cannot be avoided by reparameterizing the model, but is a fundamental feature of the divisible-good model. Somewhat ironically, for such non-local incentives the best analogy may be the multi-unit uniform-price auction. I show that over ironed intervals, a small increase in bid must increase the payment made for all units on the interval, just as in the uniform-price auction; elsewhere on the bid curve, a small increase in bid will increase only the payment for that unit.[6]

Bid flattening has been observed in certain pay-as-bid auction models. Kastl (2012) constructs a divisible-good model in which bidders are constrained to submit step functions with a bounded number of steps. This is meant to capture behavior in certain real-world auctions, where bidders are often constrained to submit no more than a certain (small) number of bid points; Merian-Weg and Häfner (2015) look at a similar model.[7] By contrast, I show here that strategic ironing implies that bids

[5]In part, this is because the pointwise conditions for bid optimality are exactly the first-order conditions for a single-unit auction. Through strategic ironing, optimality conditions are expressed across *infra*-marginal units, which obviously has no analogue in a single-unit auction.

[6]It is important to maintain a distinction here: incentives within an ironed interval differ from those implied by the uniform-price format. In particular, slightly raising a bid within such an interval will affect the payment for larger quantities. This forward effect is not present in uniform-price auctions, where bids for lower quantities are independent of payments for higher quantities. Nonetheless, incentives on ironed intervals look like a combination of incentives from the non-ironed pay-as-bid and uniform-price formats.

[7]For example, the Federal Reserve's implementation of quantitative easing allowed submission

will be flat, to a certain extent, regardless of the constraints imposed by the pay-as-bid implementation. Engelbrecht-Wiggans and Kahn (1998) examine a model of a two-unit auction in which bidders demand up to two units, and find that bids for the two units may be flat with positive probability. Engelbrecht-Wiggans and Kahn (2002) extend these results to find bids which are flat—and indeed, ironed—for all bidders, with probability one. Their particular model relies crucially on demand only barely exceeding supply and hence is degenerate in the divisible-good case. I show that this flattening behavior persists with private information, and indeed is more general than almost-entire fulfillment might suggest. Without private information, Gresik (2001) obtains bids which are almost flat, in that bids are only submitted at the clearing price, plus or minus one bid increment.[8]

I continue now in Section 2 by explicitly defining the share auction model I employ. In Section 3 I prove the existence of a pure-strategy equilibrium, demonstrate that equilibrium strategies must be monotonic the agents' value functions, and establish that equilibrium in the divisible-good model can approximate equilibrium in the multi-unit model. In Section 4 I describe strategic ironing and prove that it must occur in equilibrium, and compute equilibrium strategies in a two-bidder example. Section 5 concludes.

2 Model

An auctioneer is selling Q units of a perfectly-divisible commodity to a set of $n \geq 2$ agents, $i \in \{1, \ldots, n\}$; Q is deterministic and inelastic.[9] Agent i has private signal $s_i \sim F$ and marginal value function $v^i : [0, Q] \times [0, 1] \to \mathbb{R}_+$, where $v^i(q; s)$ is her marginal value for the q^{th} unit of the good when her private information is s. v^i is strictly decreasing in q and strictly increasing and continuous in s. F has convex support $[0, 1]$ and admits density f; for any agent $j \neq i$, s_i and s_j are independent. I adopt the notational convention that superscripts represent functions, while subscripts represent vector indexes (or, when present with a superscript, partial derivatives).

Bidders compete for shares of the aggregate quantity in a pay-as-bid auction. Bidder i submits a weakly positive, weakly decreasing bid function b^i to the auctioneer, expressing the bidder's willingness to pay for the q^{th} unit. For simplicity, I take a mechanism design approach and consider b^i to be a function of both quantity and the agent's private information, so that $b^i(q; s)$ is agent i's submitted bid for quantity q when her signal is s. I will denote the bidder's implicit demand functions—the

of up to nine bid points. Korean Treasury auctions permit the submission of five bid points (Kang and Puller, 2008).

[8] In a sense, these results extend models in which bidders have unit demand and may submit only a single bid; for a seminal example, see Harris and Raviv (1981).

[9] Theorem 1, which proves the existence of an equilibrium in pure strategies, can permit elastic supply, provided that supply is monotonically increasing in the strength of submitted bids.

inverses of the bid function—by $\overline{\varphi}^i$ and $\underline{\varphi}^i$:

$$\overline{\varphi}^i\left(p;s\right)=\sup\left\{q:\ b^i\left(q;s\right)\geq p\right\},\quad \underline{\varphi}^i\left(p;s\right)=\inf\left\{q:\ b^i\left(q;s\right)\leq p\right\}.$$

If there is no q such that $b^i(q;s)\geq p$, then $\overline{\varphi}^i(p;s)=0$, and if there is no q such that $b^i(q;s)\leq p$, then $\underline{\varphi}^i(p;s)=Q$; because bids are defined only on the domain of available quantities, $\overline{\varphi}^i(0;s)=Q$.[10] The auctioneer compiles the submitted bid functions and computes the market-clearing price $p^\star$,

$$p^\star=\sup\left\{p:\sum_{i=1}^n\underline{\varphi}^i\left(p;s_i\right)\leq Q\leq\sum_{i=1}^n\overline{\varphi}^i\left(p;s_i\right)\right\}.^{11}$$

Given this price, the auctioneer allocates to each agent her demand at this price. If $\underline{\varphi}^i(p^\star;s_i)=\overline{\varphi}^i(p^\star;s_i)$ for all i (roughly, if $b^i(\cdot;s_i)$ is strictly decreasing for each agent), then $q^i(s_1,\ldots,s_n)=\underline{\varphi}^i(p^\star;s_i)$. Otherwise, the auctioneer employs a tiebreaking rule pro-rata on the margin,

$$q^i\left(s_1,\ldots,s_n\right)=\underline{\varphi}^i\left(p^\star;s_i\right)+\left(\frac{\overline{\varphi}^i\left(p^\star;s_i\right)-\underline{\varphi}^i\left(p^\star;s_i\right)}{\sum_{j=1}^n\overline{\varphi}^j\left(p^\star;s_j\right)-\underline{\varphi}^j\left(p^\star;s_j\right)}\right)\times\left(Q-\sum_{j=1}^n\underline{\varphi}^j\left(p^\star;s_j\right)\right).$$

Straightforward continuity arguments are sufficient to ensure that these allocation rules agree when rationing is unnecessary; as it turns out,[12] the tiebreaking rule is not essential to the existence of a pure-strategy equilibrium, but it is important in the later consideration of the effects of strategic ironing.

Once allocations are determined, the auctioneer price discriminates against bidders according to their reported bids, charging the full area under their bid functions. Bidder utility is quasilinear in payments, so the utility obtained from being allocated quantity q is

$$u^i\left(q;s_i\right)=\int_0^q v^i\left(x;s_i\right)-b^i\left(x;s_i\right)dx.$$

Accounting for the fact that other bidders possess private information, the bidder attempts to maximize expected utility,

$$\mathbb{E}_{s_{-i}}\left[u^i\left(q_i;s_i\right)\right]=\underbrace{\int_0^1\cdots\int_0^1}_{n-1}\int_0^{q^i\left(s_1,\ldots,s_n;b^1,\ldots,b^n\right)}v^i\left(x;s_i\right)-b^i\left(x;s_i\right)dx\,ds_{-i}.$$

[10]That $\overline{\varphi}^i(0;s)=Q$ for all i and all s ensures that all acceptable bid functions will generate well-defined market outcomes. In particular, there is no issue with determining the proper rationed quantities if all bidders “bid c everywhere,” for any constant c.

[11]Although the auctioneer does not know the agent’s private information, he observes $\overline{\varphi}^i(\cdot;s_i)$ and $\underline{\varphi}^i(\cdot;s_i)$ as they are recoverable from the submitted bid function $b^i(\cdot;s_i)$.

[12]And as also noted in, e.g., Merian-Weg and Häfner (2015).

Where useful, I will consider the equilibrium quantity distribution $G^i(\cdot; b)$ rather than the distribution over opponents' signals,

$$G^i(q; b) = \Pr\left(q_i \leq q \middle| b^i = b\right).$$

Then expected utility can be written as

$$\mathbb{E}_{q_i}\left[u^i(q_i; s_i)\middle| b^i\right] = \int_0^Q \int_0^q v^i(x; s_i) - b^i(x; s_i)\, dx dG^i\left(q; b^i\right).^{13}$$

Throughout, I restrict attention to Bayesian-Nash equilibria, in which each agent maximizes her expected utility conditional on her own private information. In particular, an equilibrium is a set of strategies $(b^1, \ldots, b^n)$ such that each agent i, conditional on her signal s_i, is maximizing her expected utility,

$$b^i(\cdot; s_i) \in \arg\max_b \mathbb{E}_{q_i}\left[u^i(q_i; s_i)\middle| b\right].$$

2.1 ε-discrete auctions

Central to the proof of equilibrium existence is the concept of an ε-discrete auction, a coarsening of the divisible-good model to a multi-unit auction. Given $\varepsilon > 0$, the auxiliary model $\mathcal{M}^\varepsilon$ is derived from the divisible-good model, but bids are constrained to be constant over intervals of the form $(t\varepsilon, t\varepsilon + \varepsilon]$, where $t \in \mathbb{N}$, and for each q there exists $k \in \mathbb{N}$ such that $b^i(q; s) = k\varepsilon$. Additionally, quantity allocations must be of the form $q_i = q\varepsilon$ for some $q \in \mathbb{N}$; rationing will occur pro-rata on the margin, up to ε-indivisibility. Issues of indivisibility in allocations are addressed by equiprobable lotteries for the units in question. **Define allocation rule.**

In the ε-discrete auction, bids are assumed to not exceed values. This assumption is without loss of generality with regard to the existence of an equilibrium in pure strategies in the discretized model, since the set of available strategies is still finite. Knowing that agents submit bids weakly below their values is crucial to the proof of existence in the divisible-good model, as it rules out the possibility of certain degenerate equilibria which converge to non-equilibrium strategies.[14] It is useful to

[13]Since quantity allocations are always positive, $q_j \geq 0$ and hence $q_i \leq Q$. This does not imply that Q is a tight bound for $\mathrm{Supp}(G^i(\cdot; b))$, only that it is a bound.

[14]To make this point concrete, suppose that there are two bidders with marginal values $v^1(q; s) = 1 + s - q$ and $v^2(q; s) = 4 + s - q$, and that quantity $Q = 1$ is available for allocation. In each discretized auction there is an equilibrium in which bidder 1 bids 2 for all units regardless of her type, and bidder 2 bids $2 + \varepsilon$ for all units regardless of her type. As $\varepsilon \searrow 0$ the strategies converge to both agents bidding 2 for all units, which is not an equilibrium (holding fixed the pro-rata on the margin tiebreaking rule). A related example in a different context is given in Jackson et al. (2002). Equilibria like this can be avoided by assuming that value functions are not too dispersed, but the existence question can be addressed without this additional assumption on the primitives of the model.

view this as a discrete-unit auction on a finite price grid, with Q/ε discrete units available for auction, hence $\mathcal{M}^\varepsilon$ is the *ε-discrete auction.*[15, 16]

In the ε-discrete auction, agent i's utility is no different from the divisible-good auction; the only distinction is that the set of available strategies is greatly reduced.

Because the primitives of the game I am analyzing, as well as its available strategies, are all limits of functions available in a refining sequence of ε-discrete auctions, I will occasionally refer to the base divisible-good model as the *divisible-good limit.*

3 Equilibrium existence

Since quantity allocations are weakly positive, there are necessarily some quantities which an agent can never win, regardless of the bid function she submits. As a particularly rough bound, the agent can never receive more than $q_i = Q$ units, hence there are no optimality conditions to determine bids for quantities $q_i > Q$. In general, similar logic will hold for some $\overline{q} < Q$. I therefore distinguish between quantities which an agent can win with some probability, which I term relevant quantities, and quantities which can never be obtained, which I term irrelevant quantities.

Definition 1 (Relevant quantity)**.** *Quantity q is a* relevant quantity *for agent i if $q \in Supp(G^i(\cdot; b^i))$.*

Recalling that $G^i(\cdot; b^i)$ is the equilibrium distribution of agent i's allocated quantity conditional on her submitted bid function, there is an important qualification to Definition 1: whether or not a quantity is relevant is not exogenous, but depends on both the bid b^i of agent i and on the bids $(b^j)_{j\neq i}$ of her opponents. Although this is a static model, there is a sense in which some irrelevant quantities may be considered off-path: while pointwise optimality conditions cannot determine behavior for these units, no-deviation constraints for agent i's opponents may restrict the values that her bids may take.[17]

To begin analysis of the existence problem, Lemma 1 first establishes that submitted bids are (generally) monotone in bidders' private information.

Lemma 1 (Best-response monotonicity)**.** *Let $(b^j)_{j\neq i}$ be the profile of strategies played by agents other than i, and suppose that b^i is a best-response. Then for almost all relevant quantities, b^i is weakly monotonic in agent i's private information s_i.*

[15]In the arguments employed, it is implicit that $Q/\varepsilon \in \mathbb{N}$. This assumption may be relaxed by considering equiprobable lotteries for the unit $q \in (\lfloor Q/\varepsilon \rfloor, \lceil Q/\varepsilon \rceil)$, but it is sufficient here to simply assume that $\varepsilon > 0$ divides Q.

[16]It is not essential that q and s are discretized onto grids with identical spacing, or even that the discretized spaces are regularly-spaced grids. All that is necessary is that the grids converge to dense sets on their respective spaces. The choice of equal spacing is made for simplicity.

[17]For more on this in a setting without private information, see **?**.

Proof. See Appendix A. □

Lemma 1's result is similar to the monotonicity results found throughout the auction literature: a bidder with a higher signal has a higher value, hence more to gain by winning a unit. It follows that if a lower-signal agent is indifferent between raising her bid and not, the higher-signal agent would prefer, on the margin, to obtain a higher winning probability in exchange for a slightly lower per-unit surplus.

That Lemma 1 holds only for almost all relevant quantities is a technical consideration. If left- or right-continuity was imposed on the submitted bid function this qualification could be widened to include all relevant quantities;[18] the qualification accounts for behavior at potential quantity-discontinuities in the bid function, which may be resolved in either direction without affecting the agent's interim utility.[19]

That this is true of all best responses depends crucially on the independence of agents' types, and on the fact that values are strictly increasing in signal. If the latter were not the case, agents might be indifferent across an interval of bids for a particular quantity, and there might be some inversion of the type order with regard to the bids submitted.[20] If signals were not independent, an agent could take, for example, a high signal as a sign that she should collude with her opponent, again leading to possible inversion of bids with respect to information; see, e.g., McAdams (2007a,b). As the existence result of Theorem 1 builds on a sequence of monotone equilibria, the results employed would imply the existence of a monotone equilibrium, but would not independently guarantee that any equilibrium must be monotone.

Lemma 1 allows me to focus on monotone strategies. This is particularly useful since Theorem 1 obtains "limiting" strategies at only a countable subset of possible signals; the remainder are found constructively, subject to the monotonicity constraint implied by Lemma 1: assurance of monotonicity allows the proof to focus attention on a relatively constrained set of possible actions for an agent receiving any particular signal.

Theorem 1 (Equilibrium existence)**.** *The divisible-good pay-as-bid auction admits a Bayesian-Nash equilibrium in pure strategies.*

Proof. See Appendix B. □

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[18]Due to the tiebreaking rule, enforcing left- or right-continuity is without loss of generality. Nevertheless it is simpler to remain agnostic on this front.

[19]As a concrete example of this minor issue, consider the functions $f, f' : [0, Q] \to \mathbb{R}_+$, $f(x) = \chi[x \leq 1]$ and $f'(x) = 2\chi[x < 1]$. In this case, f' is the "higher" function, with exception to the point $x = 1$. Since measure-zero issues of this form will not affect agents' utilities—obtaining or not obtaining a marginal unit has zero effect on gross utility—claiming that dominance holds almost-everywhere is technically correct but the qualification is not important to equilibrium outcomes.

[20]For example, if the agent is indifferent between a "high" and "low" bid over some quantity interval on which her value function is constant in signal, a low-signal agent could resolve indifference by selecting the high bid and the high-signal agent could resolve indifference by selecting the low bid.

The proof of Theorem 1 builds upon the results of Athey (2001), McAdams (2003), and Reny (2011). In particular, these results are sufficient to assert the existence of a pure-strategy equilibrium in any ε-discrete auction coarsening the divisible-good model.[21] Considering a sequence of such equilibria corresponding to ever-finer discrete approximations—that is, a decreasing sequence $\langle \varepsilon_k \rangle_{k=0}^{\infty} \searrow 0$—the monotonicity of equilibrium bid functions (by constraint with respect to quantity; proved in Lemma 1 with respect to private information) implies that strategies are converging pointwise on any countable grid of points $\{((q_i; s_i))_{i=1}^{n}\}$ (Widder, 1941).

Letting the countable set of quantity-signal pairs be all rational pairs of quantity and signal for each agent,[22] the limit is the outline of a bid function on the interval $[0, Q]$. Since bids are monotone and bounded,[23] taking any agent who receives a rational signal there can be at most a countable number of discontinuities arising in quantity space in the outlined bid function. These are measure-zero, and hence the bid function's value at these points may be determined arbitrarily subject to the monotonicity constraint. Following Lemma 1, we can assume that the bids submitted by irrational-signal bidders will lie between the bids submitted by their rational-signal counterparts.[24]

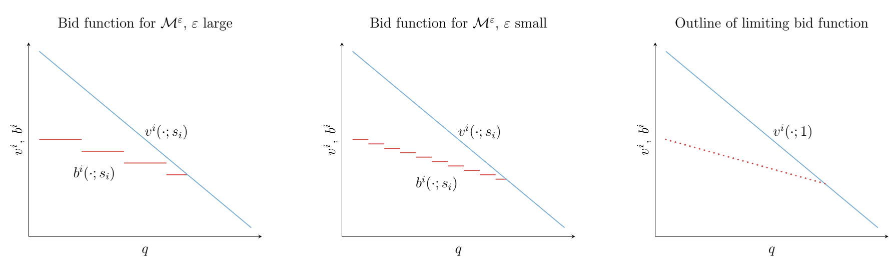

Figure 1: Illustration of refinement of equilibrium strategies to a countable set of quantity points, the first step in the proof of Theorem 1.[26] In the argument in the proof, this convergence is occurring simultaneously with respect to both arguments of the bid function, quantity and signal.

[21] Usefully, multi-unit auctions feature as examples in both McAdams (2003) and Reny (2011).

[22] The use of the term *rational* throughout this paper is meant to refer to the rational numbers, not to any form of agent rationality. An unstated assumption in all results is that agents are rational in the economic sense.

[23] Boundedness is a safe assumption, except potentially at the point $q = 0$. At this point, tiebreaking pro rata on the margin implies that boundedness is completely innocuous; a trivial counterpoint here would be to break ties on the basis of which agent placed the highest bid at $q = 0$.

[24] This assumption is safe when rational-signal bidders are best-responding. The proof of existence shows that rational-signal bidders are best-responding when irrational-signal bidders are bidding monotonically (in signal), and that irrational-signal bidders are best-responding when bidding between nearby rational-signal bidders.

[26] The right panel of Figure 1 illustrates convergence to a finite set of points due to obvious constraints.

So long as all bid functions are strictly monotone in one dimension or the other, this construction immediately yields best responses: small changes in an agent's strategy will yield at most small changes in gross utility, or small changes in allocation probabilities; this follows from the inverse bid functions being continuous.[27] Since the limit is built from equilibria in ε-discrete auctions, the limit will be a best response, otherwise a nearby ε-discrete auction will admit a profitable deviation.

representation of no issues when tiebreaking is unnecessary

Figure 2: figure.

When tiebreaking is involved the situation is less straightforward. Ties can only occur with positive probability if there is some chance that two agents receive quantities along a "mutual flat" of their submitted bid functions (see Figure 3). Helpfully, if an agent's limiting bid function is constant over an interval, convergence to points within this interval must be uniform. It can be shown—and this is the chief difficulty in the proof—that any discrete gains available from deviation at the limit must also be available near the limit; since bids are best responses in the ε-discrete auction, they must be best responses in the divisible-good auction.

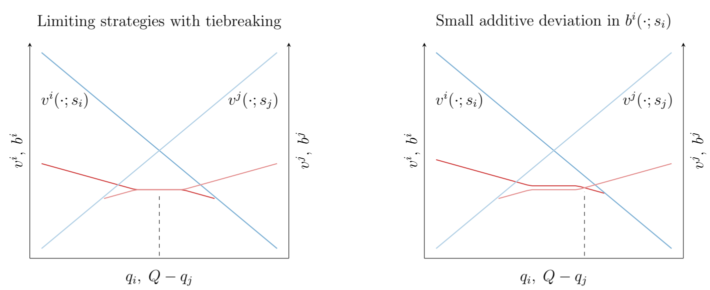

Figure 3: Limiting strategies necessitate use of the tiebreaking rule when there is a possibility that agents are allocated on a "mutual flat" of their respective bid functions; in the left panel, bidders i and j receive quantities strictly within a flat interval of their bid functions. When this arises, small deviations in an agent's bid function can yield discontinuous changes in the resulting allocation; in the right panel, bidder i increases her bid to $\hat{b}^i(\cdot; s) = b^i(\cdot; s) + \xi$, for small ξ. This deviation costs at most $Q\xi$ while yielding a discontinuous improvement in gross utility.

From this point, simple continuity arguments are sufficient to construct best responses for irrational-signal agents. Having then built mutual best responses for all agents regardless of their private information, the construction represents a Bayesian-Nash equilibrium.

[27] In Appendix D I consider the signal-inverse functions $\overline{\psi}^i(q; p)$ and $\underline{\psi}^i(q; p)$; discussion is deferred since they are not of immediate use to intuitive arguments.

Continuity and strict monotonicity of marginal value with respect to signal are crucial to the proof of Theorem 1. These properties together ensure that when I build actions for agents when they receive signals which are not within the "limiting" set, these actions are best responses. Explicitly, the existence result of McAdams (2003) is central to the proof of Theorem 1. Employing the more-general existence result of Reny (2011) would give the same result in the Theorem, and would also imply the following Remark.

Remark 1 (Symmetric equilibrium). *When all bidders have ex ante symmetric marginal values, $v^i \equiv v$, there is a symmetric pure-strategy Bayesian-Nash equilibrium of the divisible-good pay-as-bid auction with private information.*[28]

Remark 1 establishes that the symmetric models can imply symmetric behavior, although they may also admit asymmetric equilibria. Because Reny (2011) generalizes the results of McAdams (2003), its application in the course of the proof of Theorem 1 is equally valid. Reny's Theorem 4.5—establishing the existence of a symmetric pure-strategy equilibrium—is then sufficient to establish Remark 1.

The method of proof applied to demonstrate the existence of a pure-strategy equilibrium suggests the following useful result.

Theorem 2 (Equilibrium approximation). *Let $\langle \mathcal{M}^{\varepsilon_r} \rangle_{r=1}^{\infty}$ be a refining sequence of ε-discrete models such that $(b^{i,\varepsilon_r})_{i=1}^{n}$ is a pure-strategy equilibrium of $\mathcal{M}^{\varepsilon_r}$. Suppose that $C \subseteq [0, Q] \times [0, 1]$ is dense with respect to the componentwise order and contains $\{0, Q\} \times \{0, 1\}$; if each $(b^{i,\varepsilon_r})_{r=1}^{\infty}$ converges pointwise almost everywhere on C, there is an equilibrium of the divisible-good pay-as-bid auction such that:*

(i) The quantities allocated to agent i in the discretized models, q^{i,ε_r}, converge in probability to the quantities allocated to agent i in the divisible-good model, q^i; and,

(ii) The market-clearing price implied by $(b^{i,\varepsilon_r})_{i=1}^{n}$ is converging in probability to the market-clearing price of the divisible-good limit.

Proof. Consider the equilibrium $(\beta^i)_{i=1}^{n}$ of the divisible-good model which is suggested by the proof of Theorem 1.

I first establish that for any i, $q^{i,\varepsilon_r} \to q^i$ almost everywhere. Suppose to the contrary that there is a positive-measure set $S^i \subseteq [0, 1]^n$ such that, for $s \in S^i$, $q^{i,\varepsilon_r}(s) \not\to q^i(s)$. By market clearing, it is without loss to assume that for all $s \in S^i$, $\lim_{r \nearrow \infty} q^{i,\varepsilon_r}(s) < q^i(s)$. From almost-everywhere pointwise convergence of $(b^{i,\varepsilon_r})_{r=1}^{\infty}$ to β^i on C, this implies a discontinuous upward jump in ex post utility at the limit; for reasons similar to those given in the proof of Theorem 1, if this manifests as an *interim* discontinuity in utility, agent i has a utility-improving deviation for r sufficiently large, violating the assumption that b^{i,ε_r} is a best response in $\mathcal{M}^{\varepsilon_r}$. Since the distribution of private information is massless, this establishes point (i).

[28] I would like to thank Jeffrey Mensch for this Remark.

I now demonstrate that if β^i is continuous at $(q; s_i)$, then $\langle b^{i,\varepsilon_r}(q; s_i)\rangle_{r=1}^{\infty} \to \beta^i(q; s_i)$. Because β^i is monotonic in both dimensions, if β^i is continuous at $(q; s_i)$ it is continuous at all $(q'; s_i')$ in a neighborhood of $(q; s_i)$. Let $\xi > 0$ be small; by density of C, there are $(q_L, s_L), (q_R, s_R) \in C$ with

$$\begin{aligned} q - \xi < q_L \ &< q < \ q_R < q + \xi, \\ s_i + \xi > s_L \ &> s_i > \ s_R > s_i - \xi. \end{aligned}$$

The distance between any b^{i,ε_r} and β^i may be written as

$$\begin{aligned} \left|b^{i,\varepsilon_r}(q; s_i) - \beta^i(q; s_i)\right| &\leq \left|b^{i,\varepsilon_r}(q_L; s_L) - \beta^i(q_R; s_R)\right| \\ &\leq \left|b^{i,\varepsilon_r}(q_L; s_L) - \beta^i(q_L; s_L)\right| + \left|\beta^i(q_L; s_L) - \beta^i(q_R; s_R)\right|. \end{aligned}$$

Since $\langle b^{i,\varepsilon_r}(q_L; s_L)\rangle_{r=1}^{\infty} \to \beta^i(q_L, s_L)$ and β^i is continuous, letting $r \nearrow \infty$ and $\xi \searrow 0$ gives $\langle b^{i,\varepsilon_r}(q; s_i)\rangle_{r=1}^{\infty} \to \beta^i(q; s_i)$.

Now, consider the difference $||p^{\varepsilon_r} - p||$; appealing to the definition of market clearing and the triangle inequality,

$$\begin{aligned} ||p^{\varepsilon_r} - p|| &= \int_{[0,1]^n} \left|b^{i,\varepsilon_r}\left(q^{i,\varepsilon_r}(s); s_i\right) - \beta^i\left(q^i(s); s_i\right)\right| ds \\ &\leq \int_{[0,1]^n} \left|b^{i,\varepsilon_r}\left(q^{i,\varepsilon_r}(s); s_i\right) - \beta^i\left(q^{i,\varepsilon_r}(s); s_i\right)\right| ds \\ &\quad + \int_{[0,1]^n} \left|\beta^i\left(q^{i,\varepsilon_r}(s); s_i\right) - \beta^i\left(q^i(s); s_i\right)\right| ds. \end{aligned}$$

Point (i) establishes that the right-hand integral converges to zero. The left-hand integrand is zero whenever β^i is continuous at $(q^i(s); s_i)$, which is true almost everywhere (Lavrič, 1993). Hence both integrals are zero, and $p^{\varepsilon_r}(s) \to p(s)$ for almost all s. Point (ii) follows immediately. $\square$

The proof of Theorem 1 establishes the existence of such sequences as required by the antecedent of Theorem 2. The chief implication of Theorem 2 is that, given a sequence of convergent equilibria of the discretized pay-as-bid auction, the divisible-good pay-as-bid auction admits a pure-strategy equilibrium such that observed outcomes of the discretized auction converge in probability to outcomes in the divisible-good auction.

Remark 2. *There is a pure-strategy equilibrium of the divisible-good pay-as-bid auction which outcome-approximates a pure-strategy equilibrium of the discrete-good pay-as-bid auction with sufficiently-fine quantities and prices.*

Density of the convergence set and the assumption the a particular series of observed auction equilibria converges to the divisible-good auction equilibrium are relatively strong requirements, but these are difficult to weaken. Nonetheless, these particular issues have been discussed at length when applied to other asymptotic settings. The content of Theorem 2 is intuitive and informative: the divisible-good pay-as-bid auction admits an equilibrium which approximates outcomes in the multi-unit auction.

There is a second caveat that this approximation only holds with respect to a particular equilibrium of the divisible-good auction. While there are no known issues of equilibrium multiplicity in the pay-as-bid setting (**?**), there is also no guarantee of uniqueness. Moreover, a nearby discrete-good auction might not be present in the convergent sequence used to construct equilibrium in the divisible-good auction, hence the nature of convergence will play an important role in determining whether observed discrete-auction behavior is approximated by the divisible-good model.

4 Strategic ironing

In practice, as in this model, submitted bids are required to be weakly decreasing in quantity; strategic ironing occurs when this monotonicity constraint is binding over a nontrivial interval of quantities. In short, the agent's *ideal* bid function would be nonmonotonic, but the requirement that bids are weakly decreasing in quantity induces the agent to submit a bid which is flat over an interval. As compared to ironing in the classical principal-agent sense (Myerson, 1981), strategic ironing in pay-as-bid auctions is not a possible wrinkle under certain parameterizations but a necessity regardless of the fundamentals of the model.

This feature has been noted in multi-unit auctions in, e.g., Engelbrecht-Wiggans and Kahn (2002) and Kastl (2012).[29] In divisible-good pay-as-bid auctions, the intuition is as follows: consider a symmetric equilibrium in a symmetric model, the existence of which is guaranteed by Remark 1. Due to equilibrium symmetry, each agent, should she receive a high signal, is endogenously guaranteed a strictly positive quantity of the commodity—in particular, she will be allocated at least Q/n units. This reduces the incentive to submit strictly positive bids for units $q \in [0, Q/n)$, and ideally her bid would be zero on this interval. Since bids are constrained to be weakly decreasing, this is at odds with the fact that she will generally want to submit a strictly positive bid for at least some units $q > Q/n$. The agent must then iron her bid function, increasing her bids for lower quantities and decreasing her bids for larger quantities, vis a vis the ideal bid function not subject to the monotonicity constraint.[30]

[29] In Kastl's paper, ironing arises by constraint; in Engelbrecht-Wiggans and Kahn's, it is fundamental to the model.

[30] One could describe a model in which bids are potentially nonmonotonic, but the concept of a market-clearing price becomes difficult to pin down. Additionally, (i) in practice bids must be monotonically decreasing; (ii) a self-interested auctioneer will generally prefer to accept higher bids before lower bids (neglecting potential equilibrium effects); and (iii) depending on the structure of private information, it is possible that allowing null bids for low quantities would eliminate the majority of the auction's proceeds.

4.1 Optimality conditions

Each agent's optimization problem can be solved by applying the calculus of variations; helpful derivations can be found in Février et al. (2002) and **?**, among others. In particular, the agent's maximization problem can be restated as

$$\max_b \mathbb{E}_{q_i}\left[\int_0^{q_i} v^i(q;s) - b(q)\,dq\,\middle|\, b\right], \text{ s.t. } b \text{ is decreasing;}$$
$$\rightsquigarrow \max_b \int_0^Q \left(v^i(q;s) - b(q)\right)\left(1 - G^i(q;b)\right) dq, \text{ s.t. } b \text{ is decreasing.}$$

Where the monotonicity constraint is not binding, this suggests a simple pointwise optimality condition:

$$b(q) = v^i(q;s) + \frac{1 - G^i(q;b)}{G^i_b(q;b)}.$$

Note that since G^i is the probability that the quantity allocated is less than q, it is roughly the probability that the unit q is *lost*; in analogy to the single-unit case, $1-G^i$ is the probability of winning unit q. Since increasing the bid for unit q reduces the probability that it is lost, G^i_b is negative; then $(1-G^i(q;b))/G^i_b(q;b)$ is the wedge between the true value for the unit, $v^i(q;s)$, and the submitted bid $b(q)$.

Where the monotonicity constraint is binding the optimality conditions are more complicated. Suppose that b is constant on (q_ℓ, q_r); then optimality requires

$$\int_{q_\ell}^{q_r} \left(v^i(q;s) - b(q)\right)\left|G^i_b(q;b)\right| dq = \int_{q_\ell}^{q_r} \left(1 - G^i(q;s)\right) dq.$$

Intuitively, on these ranges the pointwise optimality condition holds "on average" (Kastl, 2012); that is, if the integrals are eliminated, the equation is identical to the pointwise optimality equation. This averaging reflects the apparent fact that for some quantities $q' \in (q_\ell, q_r)$ the agent would rather bid less than $b(q')$, and for other quantities she would rather bid more than $b(q')$.

4.2 Necessity and quantification

To quantify the presence of strategic ironing, it is useful to define two auxilliary quantities, $\underline{q}$ and $\check{q}$.

Definition 2 (Minimum-possible quantity)**.** *The* minimum-possible *quantity* $\underline{q}^i$ *when agent* i *receives signal* s *is* $\underline{q}^i(s) = \inf Supp(G^i(\cdot;b))$*. In equilibrium,* $\underline{q}^i(s) = q^i(s, 1_{-i})$*.*

As mentioned, $\underline{q}^i$ is endogenous to the strategies being played, and depends on the equilibrium quantity distribution $G^i(\cdot;b)$.

Definition 3 (End of initial flat)**.** *The* end of the initial flat *of agent* i*'s bid function when she receives signal* s *is* $\check{q}^i(s) = \sup\{q : b^i(q;s) = b^i(0;s)\}$*.*

It is immediate that $\check{q}^i(s) \geq \underline{q}^i(s)$; strategic ironing implies that this inequality is strict.

Definition 4 (Strategic ironing)**.** *A profile of equilibrium bidding strategies* $(b^i)_{i=1}^n$ *exhibits* strategic ironing *if there is a bidder* i *and a signal* s_i *such that* $\check{q}^i(s_i) > \underline{q}^i(s_i)$.

Strategic ironing requires that there is a bidder who submits a bid which is flat beyond the minimum quantity she can receive.[31] Under the constraint that marginal values are bi-Lipschitz continuous, strategic ironing must arise in any pure-strategy equilibrium.

Theorem 3 (Strategic ironing)**.** *Suppose that each* v^i *is bi-Lipschitz continuous: there is* M *such that* $|\partial v^i/\partial q| \leq M$ *and such that* $|\partial[v^i]^{-1}/\partial v| \leq M$. *If* $(b^i)_{i=1}^n$ *constitute a pure-strategy Bayesian-Nash equilibrium in the divisible-good pay-as-bid auction, then there is an agent* i *and a signal* s *such that* $\underline{q}^i(s) < \check{q}^i(s)$. *Agent* i*'s bid function exhibits strategic ironing.*

Proof. See Appendix D. □

Theorem 3 follows from the observation that for all $q < \underline{q}(s)$, $G^i(q;b) = 0$ and $G^i_b(q;b) = 0$. That is, there is no possibility of receiving less than $\underline{q}(s)$ units, and since the minimum quantity $\underline{q}(s)$ is obtained when all opponents receive signal $s_{-i} = 1$, a slight increase in bid for units $q < \underline{q}(s)$ results in no change to the allocation probability. Then if $\check{q}(s) = \underline{q}(s)$—that is, if there is no ironing—the integral $\int_0^{\check{q}(s)}(1 - G^i(x;b))dx = \check{q}(s)$, while the integral $\int_0^{\check{q}(s)}(v^i(x;s) - b(x))|G^i_b(x;b)|dx = 0$. The only way to resolve this equilibrium equality is by increasing $\check{q}(s)$ past $\underline{q}(s)$. That Theorem 3 is nontrivial arises from the fact that there is no guarantee that G^i_b is bounded.

The bi-Lipschitz constraint in Theorem 3 is fairly innocuous: the modulus of either $v^i(\cdot;s)$ or its inverse may be arbitrarily small or large. Additionally, when values are not bi-Lipschitz, the arguments employed in the proof of Theorem 3 suggest permissible structures in equilibrium bids.

Remark 3. *With arbitrary value functions, equilibrium either exhibits strategic ironing or the minimum-attainable quantities achieved by each agent lie at points of discontinuity of either* $v^i(\cdot;s)$ *or its inverse.*

The implications of Theorem 3 are more general than its quantifiers may suggest. For example, in a symmetric equilibrium all agents' bid functions will exhibit strategic ironing, at the same set of signals. The proof of Theorem 3 explicitly shows

[31] The bid function being flat is, in general, an indication that the bid-monotonicity constraint is binding. While it is plausible that, given equilibrium bid functions, one could find value functions which yield the same bid functions without paying attention to the bid-monotoncity constraint above $\underline{q}^i(s_i)$, the proof of Theorem 3 implies that as long as marginal values are bi-Lipschitz continuous we can ignore this possibility and focus only on the binding monotonicity constraint.

that agent i's bid function is ironed when she receives a high signal $s_i = 1$, but it is intuitive to see that even if ironing occurs only when agent i receives this signal, for nearby, relatively-high signals the bid function she submits will be *nearly* flat above $\underline{q}^i(s)$, if it is not perfectly so.

The form of strategic ironing implied by Theorem 3 implies equation (1), termed the *ironing equation*.

$$\int_0^{\check{q}(s)} \left(1 - G^i\left(q; b^i\left(\cdot; s\right)\right)\right) dq = \int_0^{\check{q}(s)} \left(v^i\left(q; s\right) - b^i\left(q; s\right)\right) \left|G_b^i\left(q; b^i\left(\cdot; s\right)\right)\right| dq. \quad (1)$$

The ironing equation is similar to the optimality condition obtained by Kastl (2012); the distinction here is that Kastl's optimality conditions rely upon exogenous constraints in the number of bids submitted while the ironing equation here is endogenous to the model. As Theorem 3 demonstrates, any pure-strategy equilibrium must exhibit strategic ironing.

4.3 Two-agent example

Undergoing revision.

The nature of strategic ironing is best illustrated by an example. Consider two agents with private information $s_i \sim \mathcal{U}(0,1)$ and marginal value functions $v^i(q; s) = [\alpha_0 + \alpha_s s - \alpha_q q]^+$. Assuming that the bid function is everywhere montone-decreasing—that is, the optimality conditions are simply the pointwise optimality equation—the solution to the differential equation implied by first-order optimality is

$$b^i\left(q; s\right) = \alpha_0 + \frac{1}{2}\alpha_s s - \frac{1}{3}\alpha_q\left(Q + q\right).$$

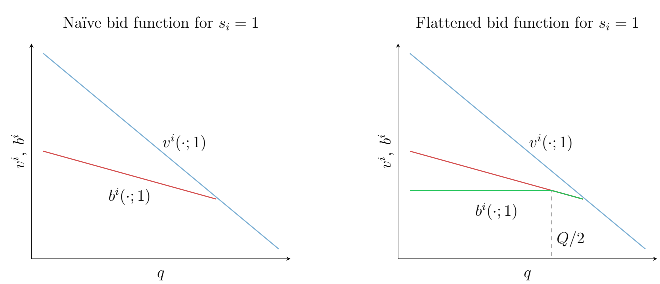

Figure 4: Naïve application of the agent's first-order conditions yields a bid function which is strictly decreasing across its domain (left panel). By noting that an agent with a high signal, $s_i = 1$, never receives less than $Q/2$, it is apparent that the bid function should be constant for all quantities $q < Q/2$ (right panel). Nonetheless, strategic ironing implies that bids should be flattened even further.

Lemma 1 claims that bids should be increasing in signal, and this bid function satisfies this condition. Since, ceteris paribus, a given bidder will win larger quantities against opponents who submit lower bid functions, the bidder will obtain her ex post minimum quantity when facing an opponent who receives her highest possible signal, $s_{-i} = 1$. Bid functions are strictly decreasing in q, so when her opponent receives $s_{-i} = 1$ the market-clearing price is uniquely determined as

$$\alpha_0 + \frac{1}{2}\alpha_s s_i - \frac{1}{3}\alpha_q \left(Q + q_i\right) = p = \alpha_0 + \frac{1}{2}\alpha_s - \frac{1}{3}\alpha_q \left(Q + q_{-i}\right).$$

Market clearing gives

$$q_i = \frac{3}{4}\frac{\alpha_s}{\alpha_q}\left(s_i - 1\right) + \frac{1}{2}Q.$$

Constraining attention to the case in which $s_i = 1$, when an agent receives her highest possible signal the least she can receive is $Q/2$.

This implies that quantities $q < Q/2$ are never marginal, and in turn that there is no direct incentive to bid competitively for these units. One might naïvely apply the monotonicity constraint ex post, and assert that $b^i(q;1) = b^i(Q/2;1)$ for all $q < Q/2$, but this is not mindful of the need to balance the desire to bid competitively for quantities $q > Q/2$ against the desire to bid nothing for quantities $q < Q/2$.

To properly solve the agent's ironing problem here,[32] note that when agent $-i$'s bid function is unchanged,

$$q_i = \frac{3}{4}\frac{\alpha_s}{\alpha_q}\left(s_i - s_{-i}\right) + \frac{1}{2}Q \quad \Longrightarrow \quad 1 - G^i\left(q;b\right) = \min\left\{s_i + \frac{4}{3}\frac{\alpha_q}{\alpha_s}\left(\frac{1}{2}Q - q\right), 1\right\}.$$

Similarly, for $q \geq \underline{q}^i(s_i)$, $|G^i_b(q;b)| = 2/\alpha_s$. The left-hand side of the ironing equation is

$$\begin{aligned}\int_0^{\check{q}} \left(1 - G^i\left(q;b\right)\right) dq &= \underline{q}\left(s_i\right) + \frac{1}{2}\left(\check{q} - \underline{q}\left(s_i\right)\right)\left(1 + \left[1 - G^i\left(\check{q};b\right)\right]\right) \\ &= \underline{q}\left(s_i\right) + \frac{1}{2}\left(\check{q} - \underline{q}\left(s_i\right)\right)\left(1 + s_i + \frac{4}{3}\frac{\alpha_q}{\alpha_s}\left(\frac{1}{2}Q - \check{q}\right)\right).\end{aligned}$$

The right-hand side of the ironing equation is

$$\begin{aligned}&\int_0^{\check{q}} \left(v^i\left(q;s_i\right) - b\left(\check{q}\right)\right)\left|G^i_b\left(q;b\right)\right| dq \\ &\quad = \frac{1}{2}\frac{2}{\alpha_s}\left(\check{q} - \underline{q}\left(s_i\right)\right)\left(v^i\left(\underline{q}\left(s_i\right);s_i\right) + v^i\left(\check{q};s_i\right) - 2b\left(\check{q}\right)\right) \\ &\quad = \left(\check{q} - \underline{q}\left(s_i\right)\right)\left(s_i - \frac{\alpha_q}{\alpha_s}\underline{q}\left(s_i\right) - \frac{1}{3}\frac{\alpha_q}{\alpha_s}\check{q} + \frac{2}{3}\frac{\alpha_q}{\alpha_s}Q\right).\end{aligned}$$

[32] In this case, I am solving a partial-equilibrium version: agent i strategically irons, but agent $-i$'s strategy is as suggested by the pointwise optimality conditions. This exercise is purely for exposition. Full equilibrium strategies are determined later.

This results in a quadratic equation, the solution for which is

$$\check{q}(s) = \underline{q}(s) + \sqrt{3\frac{\alpha_s}{\alpha_q}\underline{q}(s)}.$$

While this equation can be solved, equilibrium effects imply that this is not the end of the story: this ironing incentive also exists for agent $-i$, and her bid will also be lower than suggested by application of the pointwise first-order conditions. This fundamentally alters agent i's bidding incentives, hence this solution is not indepedently valuable.

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5 Conclusion

I have shown that the divisible-good pay-as-bid auction with private information admits a pure-strategy Bayesian-Nash equilibrium, and that when bidders are symmetric there must exist a symmetric pure-strategy Bayesian-Nash equilibrium. Helpfully, equilibrium strategies are monotone in each agent's private information; this holds with respect to best responses more generally. The proof strategy I use to establish existence suggests a natural approximation result which I am able to verify: equilibrium outcomes in the divisible-good auction may be near outcomes in discrete pay-as-bid auctions with large numbers of units.

These results provide novel evidence in favor of the pay-as-bid auction. In particular, the format has commonly been believed to be intractable with respect to equilibrium analysis; although the results here do not establish that equilibrium may in general be easily computed, I provide an equilibrium construction in a two-agent example in Section 4.3, and that there is an equilibrium at all is suggestive that the model might be tractably analyzed.[33] Monotonicity ensures that empirical investigations have access to ready intuition about higher bids indicating higher valuations.

Knowing that a pure-strategy equilibrium exists, I compute an explicit pure-strategy equilibrium in an example with two bidders and affine demand functions. To the best of my knowledge, this is the first equilibrium construction in the divisible-good pay-as-bid auction with private information. It is possible that the methods used here can be employed to construct equilibrium in more general contexts.

While the existence and monotonicity results most directly address concern with regard to the utility of the divisible-good model, the approximation result in Theorem 2 is arguably the most useful to researchers arguing from counterfactuals generated by this model. It is often taken as given that continuous models offer

[33] It is worth noting that although the divisible-good *uniform-price* auction may not suffer from the problem of equilibrium nonexistence with $n \geq 3$ bidders, it is not yet known how to generally compute equilibirum strategies in the format. In light of this fact and the results here, tractability arguments in favor of the uniform-price auction hold equally with respect to the pay-as-bid auction.

approximations of the discretized situations they attempt to simplify. By establishing probabilistic convergence, I have shown that this assumption is well-founded in the case of the pay-as-bid auction. This provides suggestive evidence that theoretical outcome analysis from the divisible-good model may in many cases successfully predict real-world discrete outcomes to within a reasonable tolerance.

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A Proof of Lemma 1: Best-response monotonicity

Prior to proving Lemma 1, I establish a set of auxiliary results which simplify the analysis.

Lemma 2 (Stochastic ordering)**.** *Suppose that b and b' are bid functions, and let $\overline{b} = b' \vee b$ and $\underline{b} = b' \wedge b$. The distribution of allocated quantity conditional on the bids submitted by other agents, $G^i(\cdot;\cdot)$, satisfies $G^i(\cdot;\overline{b}) \succeq_{FOSD} G^i(\cdot;b)$ and $G^i(\cdot;\overline{b}) \succeq_{FOSD} G^i(\cdot;b')$, and $G^i(\cdot;b) \succeq_{FOSD} G^i(\cdot;\underline{b})$ and $G^i(\cdot;b') \succeq_{FOSD} G^i(\cdot;\underline{b})$.*

Proof. I demonstrate only the first inequality, and the rest follow similarly. Let S_{-i} be defined as

$$S_{-i}\left(q;b''\right) = \left\{s_{-i} :\ q^i\left(s_{-i};b''\right) \leq q\right\}.$$

Since $\overline{b} \geq b$ and bids are strictly decreasing, for any $s_{-i} \in S_{-i}(q;\overline{b})$ it must be that $s_{-i} \in S_{-i}(q;b)$; relatively speaking, were bidder i to submit $\overline{b}$ rather than b the stop-out price would rise and q_{-i} would fall. It follows that $S_{-i}(q;\overline{b}) \subseteq S_{-i}(q;b)$ for all q. With $G^i(q;b'') = \Pr(s_{-i} \in S_{-i}(q;b''))$, it follows that $G^i(q;\overline{b}) \leq G^i(q;b)$ for all q, and hence $G^i(\cdot;\overline{b}) \succeq_{\text{FOSD}} G^i(\cdot;b)$. □

It is useful to introduce some notation in advance of further analysis. Suppose that b' is a putative best-response bid function for bidder i, and that b is some other bid function. Let $D_+^\varepsilon(x) = [x, x+\varepsilon)$ and $D_-^\varepsilon(x) = (x-\varepsilon, x]$. Let $X' = \{q : b'(q) = b(q)\} \cup \{0, Q\}$, and let $X = \mathrm{Cl}X' \setminus \mathrm{IntCl}X'$ be the set of boundaries of intersections. There are $x_L, x_R \in X$, $x_L < x_R$, such that there is no $x \in X$ with $x_L < x < x_R$. Taking any such x_L, x_R, define $\overline{b}$ and $\underline{b}$ as

$$\overline{b}(q) = \begin{cases} \max\{b(q), b'(q)\} & \text{if } q \in [x_L, x_R], \\ b'(q) & \text{otherwise;} \end{cases}$$

$$\underline{b}(q) = \begin{cases} \min\{b(q), b'(q)\} & \text{if } q \in [x_L, x_R], \\ b(q) & \text{otherwise.} \end{cases}$$

Lemma 3 (Incentive inequalities)**.** *Suppose that b^i is a best-response bidding strategy, and let $b = b^i(\cdot; s)$ and $b' = b^i(\cdot; s')$, $s' \neq s$. The following inequalities must be valid:*

$$\begin{aligned} &\mathbb{E}_{q_i}\left[\int_0^{q_i} v\left(x; s'\right) - b(x)\,dx \middle| b\right] \\ \leq\ &\mathbb{E}_{q_i}\left[\int_0^{q_i} v\left(x; s'\right) - \underline{b}(x)\,dx \middle| \underline{b}\right] \\ \leq\ &\mathbb{E}_{q_i}\left[\int_0^{q_i} v\left(x; s'\right) - b'(x)\,dx \middle| b'\right] \end{aligned}$$

Proof. To economize on notation, let $v'(\cdot) = v^i(\cdot; s')$, $G(\cdot) = G^i(\cdot; b)$, $\underline{G}(\cdot) = G^i(\cdot; \underline{b})$, $G'(\cdot) = G^i(\cdot; b')$, and $\overline{G}(\cdot) = G^i(\cdot; \overline{b})$; further, I will drop the arguments from functions where no confusion is likely to occur. The desired result is therefore

$$\mathbb{E}_{q_i}\left[\int v' - b dx \middle| b\right] \leq \mathbb{E}_{q_i}\left[\int v' - \underline{b} dx \middle| \underline{b}\right] \leq \mathbb{E}_{q_i}\left[\int v' - b' dx \middle| b'\right].$$

By incentive compatibility, it must be that $\mathbb{E}_{q_i}[\int v' - b dx | b] \leq \mathbb{E}_{q_i}[\int v' - b' dx | b']$ and $\mathbb{E}_{q_i}[\int v' - \underline{b} dx | \underline{b}] \leq \mathbb{E}_{q_i}[\int v' - b' dx | b']$. Thus it is sufficient to show

$$\mathbb{E}_{q_i}\left[\int v' - b dx \middle| b\right] \leq \mathbb{E}_{q_i}\left[\int v' - \underline{b} dx \middle| \underline{b}\right]. \tag{2}$$

If $b(q) \leq b'(q)$ for all $q \in [x_L, x_R]$, then $\underline{b} = b$ and the proof is trivial. Let x_L, x_R be as above and assume that $b(q) \geq b'(q)$ for all $q \in [x_L, x_R]$. I first perform some helpful manipulation of inequality (2),

$$\begin{aligned}
\mathbb{E}_{q_i}\left[\int v' - \underline{b}dx \middle| \underline{b}\right] &= \int\int v' - \underline{b}dxd\underline{G} \\
&= \int\int v' - bdxd\underline{G} + \int\int b - \underline{b}dxd\underline{G} \\
&= \int_{b=\underline{b}}\int v' - bdxdG + \int_{b>\underline{b}}\int v' - bdxdG' + \int\int b - \underline{b}dxd\underline{G} \\
&= \mathbb{E}_{q_i}\left[\int v' - bdx \middle| b\right] - \int_{b>\underline{b}}\int v' - bdxdG \\
&\quad + \int_{b>\underline{b}}\int v' - bdxdG' + \int\int b - \underline{b}dxd\underline{G}.
\end{aligned}$$

Then the desired result is equivalent to

$$\int_{b>\underline{b}}\int v' - bdxdG - \int_{b>\underline{b}}\int v' - bdxdG' \leq \int\int b - \underline{b}dxd\underline{G}.$$

In turn, this is

$$\int_{b>\underline{b}}\int v' - bdxdG - \int_{b>\underline{b}}\int v' - \underline{b}dxdG' \leq \int_{b=\underline{b}}\int b - \underline{b}dxdG. \tag{3}$$

Looking at $\bar{b}$, incentive compatibility gives

$$\mathbb{E}_{q_i}\left[\int v' - \bar{b}dx \middle| \bar{b}\right] \leq \mathbb{E}_{q_i}\left[\int v' - b'dx \middle| b'\right].$$

Following the same manipulations as with respect to $b, \underline{b}$ gives

$$\int_{b>\underline{b}}\int v' - \bar{b}dxdG - \int_{b>\underline{b}}\int v' - b'dxdG' \leq \int_{b=\underline{b}}\int b - \underline{b}dxdG. \tag{4}$$

To obtain the final result, compare the left-hand sides of (3) and (4). This comparison is

$$\int_{b>\underline{b}}\int v' - bdxdG - \int_{b>\underline{b}}\int v' - \underline{b}dxdG' \gtrless \int_{b>\underline{b}}\int v' - \bar{b}dxdG - \int_{b>\underline{b}}\int v' - b'dxdG'.$$

This comparison is equivalent to

$$\int_{b>\underline{b}}\int \bar{b} - bdxdG \gtrless \int_{b>\underline{b}}\int b' - \underline{b}dxdG'.$$

By construction, these relate as

$$\left[\int_0^{x_L} b' - bdx\right]\left(G^i\left(x_R; b\right) - G^i\left(x_L; b\right)\right) \gtrless \left[\int_0^{x_L} b' - bdx\right]\left(G^i\left(x_R; b'\right) - G^i\left(x_L; b'\right)\right).$$

Since these two sides are equivalent, it must be that inequality (4) holds with its left-hand side replaced by the left-hand side of inequality (3), which is exactly the desired result. □

Lemma 1 (Best-response monotonicity). *Let $\langle b^j \rangle_{j \neq i}$ be the profile of strategies played by agents other than i, and suppose that b^i is a best-response. Then for almost all relevant quantities, b^i is weakly monotonic in agent i's private information s_i.*

Proof. Let $s_i < s_i'$, and define $b = b^i(\cdot; s)$ and $b' = b^i(\cdot; s')$. By monotonicity of value functions in signal, incentive compatibility, and Lemma 3, it follows that

$$\mathbb{E}_{q_i}\left[\int v - \underline{b}dx \middle| \underline{b}\right] \leq \mathbb{E}_{q_i}\left[\int v - bdx \middle| b\right]$$
$$\leq \mathbb{E}_{q_i}\left[\int v' - bdx \middle| b\right] \leq \mathbb{E}_{q_i}\left[\int v' - \underline{b}dx \middle| \underline{b}\right].$$

Subtraction gives

$$\mathbb{E}_{q_i}\left[\int v' - vdx \middle| b\right] \leq \mathbb{E}_{q_i}\left[\int v' - vdx \middle| \underline{b}\right].$$

If there exists q such that $b(q) > \underline{b}(q)$ and $G^i(q; b') < 1$, then Lemma 2 gives $G^i(\cdot; b) \succ_{\text{FOSD}} G^i(\cdot; \underline{b})$. With this stochastic ordering, the above inequality cannot hold. Thus it must be that for all q with $G^i(q; b') < 1$, $b(q) = \underline{b}(q)$, and hence $b \leq b'$. □

Lemma 4 (Relevant bids strictly below values). *If b is a best response and q is such that $G^i(q; b) < 1$, then $b(q) < v^i(q; s_i)$.*

Proof. To begin, I show that if $b(q) = v^i(q; s_i)$, then $\lim_{q' \searrow q} b(q') < b(q)$. For $\varepsilon > 0$, consider a deviation $\hat{b}^\varepsilon$ given by

$$\hat{b}^\varepsilon(q') = \begin{cases} b(q') & \text{if } q' \leq q, \\ b(q + \varepsilon) & \text{if } q' \in (q, q + \varepsilon), \\ b(q') & \text{otherwise.} \end{cases}$$

Let $p_\in(\varepsilon) = \Pr(q_i \in (q, q + \varepsilon) | b)$ and $p_\geq(\varepsilon) = \Pr(q_i \geq q + \varepsilon | b) = 1 - G^i(q + \varepsilon; b)$. The costs associated with the deviation $\hat{b}$ are bounded above by $\int_q^{q+\varepsilon} v^i(x; s_i) - b(x)dxp_\in(\varepsilon)$, and the benefits are bounded below by $\int_q^{q+\varepsilon} b(x) - b(q + \varepsilon)dxp_\geq(\varepsilon)$. Note that, by assumption, $\lim_{\varepsilon \searrow 0} p_\in(\varepsilon) = 0$ and $\lim_{\varepsilon \searrow 0} p_\geq(\varepsilon) > 0$. Additionally,

$$\int_q^{q+\varepsilon} v^i(x; s_i) - b(x)\,dxp_\in(\varepsilon) \leq \left[v^i(q; s_i) - b(q + \varepsilon)\right] p_\in(\varepsilon)\,\varepsilon,$$
$$\int_q^{q+\varepsilon} b(x) - b(q + \varepsilon)\,dxp_\geq(\varepsilon) \geq \left[b\left(q + \frac{1}{2}\varepsilon\right) - b(q + \varepsilon)\right] p_\geq(\varepsilon)\,\frac{1}{2}\varepsilon.$$

Comparing costs to benefits gives

$$\begin{aligned}
\left[v^i(q;s_i) - b(q+\varepsilon)\right] p_{\in}(\varepsilon)\varepsilon &\gtrless \frac{1}{2}\left[b\left(q+\frac{1}{2}\varepsilon\right) - b(q+\varepsilon)\right] p_{\geq}(\varepsilon)\varepsilon \\
\rightsquigarrow [b(q) - b(q+\varepsilon)]\left(p_{\in}(\varepsilon) - \frac{1}{2}p_{\geq}(\varepsilon)\right) &\gtrless \frac{1}{2}\left[b\left(q+\frac{1}{2}\varepsilon\right) - b(q)\right] p_{\geq}(\varepsilon) \\
\rightsquigarrow \left[\frac{b(q+\varepsilon) - b(q)}{\varepsilon}\right]\left(1 - \frac{2p_{\in}(\varepsilon)}{p_{\geq}(\varepsilon)}\right) &\gtrless \frac{b\left(q+\frac{1}{2}\varepsilon\right) - b(q)}{\varepsilon}.
\end{aligned}$$

Because $p_{\in}(\varepsilon) \searrow 0$, it is safe to assume that the right-hand term in the left-hand side is at least 2/3; then taking the limit at $\varepsilon \searrow 0$ leads to

$$\begin{aligned}
\frac{2}{3}\limsup_{\varepsilon\searrow 0}\frac{b(q+\varepsilon) - b(q)}{\varepsilon} &\gtrless \limsup_{\varepsilon\searrow 0}\frac{b\left(q+\frac{1}{2}\varepsilon\right) - b(q)}{\varepsilon} \\
\rightsquigarrow \frac{2}{3}\limsup_{\varepsilon\searrow 0}\frac{b(q+\varepsilon) - b(q)}{\varepsilon} &\gtrless \frac{1}{2}\limsup_{\varepsilon\searrow 0}\frac{b(q+\varepsilon) - b(q)}{\varepsilon}
\end{aligned}$$

As long as $\limsup_{\varepsilon\searrow 0}|(b(q+\varepsilon) - b(q))/\varepsilon|$ is finite, the left-hand side is less than the right-hand side,[34] hence benefits outweigh costs and deviation is profitable.[35]

Now suppose that $\lim_{\varepsilon\searrow 0}(b(q+\varepsilon) - b(q))/\varepsilon \to -\infty$, and consider a deviation $\hat{b}^{\delta,\lambda}$ given by

$$\hat{b}^{\delta,\lambda}(q') = \begin{cases} b(q') & \text{if } q' \leq q-\delta, \\ \min\{b(q)-\lambda, b(q')\} & \text{otherwise.} \end{cases}$$

Let $\overline{q}(\lambda) = \overline{\varphi}(b(q)-\lambda)$, $p_{\in}(\delta,\lambda) = \Pr(q_i \in (q-\delta, \overline{q}(\lambda)))$, and $p_{\geq}(\delta,\lambda) = \Pr(q_i \geq \overline{\varphi}(b(q)-\lambda))$. As before, $\lim_{\lambda\searrow 0} p_{\geq}(\delta,\lambda) > 0$, however it is now possible that $\lim_{\delta\searrow 0} p_{\in}(\delta,\lambda) > 0$. The costs associated with the deviation are bounded above by $\int_{q-\delta}^{\overline{q}(\lambda)} v^i(x;s_i) - b(x)dx p_{\in}(\delta,\lambda)$ and the benefits are bounded below by $\int_{q-\delta}^{\overline{q}(\lambda)} b(x) - (b(q)-\lambda)dx p_{\geq}(\delta,\lambda)$. Note that, for some $\gamma > 0$,

$$\begin{aligned}
\int_{q-\delta}^{\overline{q}(\lambda)} v^i(x;s_i) - b(x)\,dx p_{\in}(\delta,\lambda) &\leq \left(\left[v^i(q-\delta;s_i) - v(q)\right]\delta + [\overline{q}(\lambda) - q]\lambda\right) p_{\in}(\delta,\lambda) \\
&\leq \left[v^i(q-\delta;s_i) - v(q)\right]\delta + [\overline{q}(\lambda) - q]\lambda, \\
\int_{q-\delta}^{\overline{q}(\lambda)} b(x) - (b(q)-\lambda)\,dx p_{\geq}(\delta,\lambda) &\geq \delta\lambda p_{\geq}(\delta,\lambda) \\
&\geq \gamma\delta\lambda.
\end{aligned}$$

Comparing costs to benefits gives

$$\left[v^i(q-\delta;s_i) - v(q)\right]\delta + [\overline{q}(\lambda) - q]\lambda \gtrless \gamma\delta\lambda.$$

[34] Recall that b is weakly decreasing, hence the limit is weakly negative.

[35] If the limit goes to zero, the same analysis can be applied to successively higher derivatives to obtain the same inequality for sufficiently small ε.

First, note that for any $\lambda > 0$, continuity of $v^i(\cdot; s_i)$ implies that there is $\overline{\delta} > 0$ such that, for all $\delta < \overline{\delta}$,

$$v^i(q - \delta; s_i) - v(q) < \frac{1}{2}\gamma\lambda.$$

Second, strict monotonicity of $v^i(\cdot; s_i)$, $b(q) = v^i(q; s_i)$ and $(b(q+\varepsilon) - b(q))/\varepsilon \to -\infty$ imply that $\overline{q}(\lambda) \to q$ as $\lambda \searrow 0$. Then given a $\delta > 0$ there is $\overline{\lambda} > 0$ such that for all $\lambda < \overline{\lambda}$,

$$\overline{q}(\lambda) - q < \frac{1}{2}\gamma\delta$$

Summing the two inequalities, it follows that for δ and λ sufficiently small,

$$\left[v^i(q - \delta; s_i) - v(q)\right]\delta + \left[\overline{q}(\lambda) - q\right]\lambda < \gamma\delta\lambda.$$

Then deviation is profitable, establishing the desired result. **co-selection of $\overline{\delta}$ and $\overline{\lambda}$?** $\square$

Lemma 5 (Semi-strict monotonicity)**.** *In any equilibrium $(b^i)_{i=1}^n$, each bidding strategy b^i is* semi-strictly monotone*: it does not admit any relevant co-ironed intervals.*

Proof. **clean up!**

Suppose otherwise: there is some agent i with co-ironed intervals $\mathcal{I}_s^i$ and $\mathcal{I}_q^i$ such that $\Pr(q_i \in \mathrm{Int}\mathcal{I}_q^i | s_i) > 0$ for all $s_i \in \mathcal{I}_s^i$. It is straightforward to see that if there is another agent j with relevant co-ironed intervals $\mathcal{I}_s^j$ and $\mathcal{I}_q^j$ such that $b^j(q_j; s_j) = b^i(q_i; s_i)$ for all $(q_j, s_j) \in \mathcal{I}_q^j \times \mathcal{I}_s^j$ and $(q_i, s_i) \in \mathcal{I}_q^i \times \mathcal{I}_s^i$, and $\Pr(q_i \in \mathcal{I}_q^i, q_j \in \mathcal{I}_q^j) > 0$, a small (interim) deviation by any $s_i \in \mathcal{I}_s^i$ or $s_j \in \mathcal{I}_s^j$ will yield a discrete improvement in utility, hence it is sufficient to analyze the case in which, for all $s_i \in \mathcal{I}_s^i$ and almost all s_{-i} such that $q^i(s_i, s_{-i}) \in \mathcal{I}_q^i$, $b^j(\cdot; s_j)$ is strictly decreasing at $q^j(s_j, s_i, s_{-ij})$.

Note that, because bids are monotone in signal, for any s_i and almost all s_{-i} such that $q^i(s_i, s_{-i}) \in \mathcal{I}_q^i$, any $s_j \in s_{-i}$ is such that $\Pr(q_j = q^j(s_j, s_i, s_{-ij}) | s_j) = p^j(s_j) > 0$. Suppose that $b^j(\cdot; s_j)$ is right-continuous at $q^j(s_j, s_i, s_{-ij})$, and that $b^j(q^j(s_i, s_j, s_{-ij}); s_j) = b^i(q^i(s_i, s_{-i}); s_i) < v^j(q^j(s_j, s_i, s_{-ij}); s_j)$; consider a deviation $\hat{b}^\varepsilon$,

$$\hat{b}^\varepsilon(q) = \begin{cases} \max\left\{b^j(q; s_j), b^j\left(q^j(s_j, s_i, s_{-ij}); s_j\right) + \varepsilon^2\right\} & \text{if } q \le q^j(s_j, s_i, s_{-ij}), \\ b^j(q; s_j) & \text{otherwise.} \end{cases}$$

This deviation incurs additional costs of at most $Q\varepsilon^2 + [b^j(q^j; s_j) - b^j(q^j + \varepsilon; s_j) + \varepsilon^2]\varepsilon$. Since $q^i(s_i, s_{-i}) \in \mathrm{Int}\mathcal{I}_q^i$ **not assumed, but argument is okay**, for ε sufficiently small the deviation $\hat{b}^\varepsilon$ will obtain an additional quantity of ε with positive probability. Then the utility gain of this deviation is bounded below by $(v^j(q^j + \varepsilon; s_j) - b^j(q^j; s_j) - \varepsilon^2)\varepsilon$. Comparing gains to losses yields

$$Q\varepsilon^2 + \left[b^j\left(q^j; s_j\right) - b^j\left(q^j + \varepsilon; s_j\right) + \varepsilon^2\right]\varepsilon \gtrless \left[v^j\left(q^j + \varepsilon; s_j\right) - b^j\left(q^j; s_j\right) - \varepsilon^2\right]\varepsilon$$

$$\rightsquigarrow \quad Q\varepsilon + \left[b^j\left(q^j; s_j\right) - b^j\left(q^j + \varepsilon; s_j\right) + \varepsilon^2\right] \gtrless v^j\left(q^j + \varepsilon; s_j\right) - b^j\left(q^j; s_j\right) - \varepsilon^2.$$

Letting $\varepsilon \searrow 0$, the assumption that b^j is continuous and $b^j < v^j$ leads to $0 \gtreqless v^j(q^j; s_j) - b^j(q^j; s_j)$, hence the benefits outweigh the costs and the deviation is profitable. It follows that for almost all relevant s_j, either $b^j(q^j; s_j) = v^j(q^j; s_j)$ or $\lim_{q' \searrow q^j} b^j(q'; s_j) < b^i(q^i)$. In the former case, Lemma 4 establishes that either $b^j_{q+}(q^j; s_j) = -\infty$, or $G^j(q^j; b^j(\cdot; s_j)) = 1$.

Let $\underline{s}_i = \min \mathcal{I}^i_s$ (it may be assumed that $\mathcal{I}^i_s$ is closed without loss of generality) and suppose that $\mathcal{I}^i_q$ extends maximally rightward, in the sense that there is no q' such that for all $q \in \mathcal{I}^i_q$, $q' > q$, and $b^i(q'; s_i) = b^i(q; s_i)$. □

B Auxiliary results for Theorem 1: Equilibrium existence

Lemma 6 (Upward deviations not discretely harmful)**.** *Suppose that b and $\hat{b}$ are bid functions, $\hat{b} \geq b$, and that $||b - \hat{b}||_{\ell_1} = \varepsilon$. Then against any profile of opponents' bids b^{-i},*

$$U^i\left(\hat{b}, b^{-i}; s_i\right) \geq U^i\left(b, b^{-i}; s_i\right) - \varepsilon.$$

Proof. Recall the expected utility representation,

$$U^i\left(\hat{b}, b^{-i}; s_i\right) = \int_0^Q \left(v^i(x; s_i) - \hat{b}(x)\right)\left(1 - G^i\left(x; \hat{b}\right)\right) dx.$$

By Lemma 2, $G^i(\cdot; \hat{b}) \succeq_{\text{FOSD}} G^i(\cdot; b)$, hence

$$\int_0^Q \left(v^i(x; s_i) - \hat{b}(x)\right)\left(1 - G^i\left(x; \hat{b}\right)\right) dx \geq \int_0^Q \left(v^i(x; s_i) - \hat{b}(x)\right)\left(1 - G^i(x; b)\right) dx.$$

Substitution into the right-hand equation reveals

$$\begin{aligned}\int_0^Q \left(v^i(x; s_i) - b(x)\right)\left(1 - G^i(x; b)\right) dx = &\int_0^Q \left(v^i(x; s_i) - \hat{b}(x)\right)\left(1 - G^i(x; b)\right) dx \\ &+ \int_0^Q \left(b(x) - \hat{b}(x)\right)\left(1 - G^i(x; b)\right) dx.\end{aligned}$$

Since G^i is a CDF, $0 \leq 1 - G^i(x; b) \leq 1$ for all x, hence

$$\int_0^Q \left(b(x) - \hat{b}(x)\right)\left(1 - G^i(x; b)\right) dx \geq \int_0^Q b(x) - \hat{b}(x)\, dx = -\left|\left|b - \hat{b}\right|\right|_{\ell_1} = -\varepsilon.$$

Then we have

$$\begin{aligned}U^i\left(\hat{b}, b^{-i}; s_i\right) &\geq \int_0^Q \left(v^i(x; s_i) - b(x)\right)\left(1 - G^i(x; b)\right) dx - \varepsilon \\ &= U^i\left(b, b^{-i}; s_i\right) - \varepsilon.\end{aligned}$$

□

Corollary 1. *Suppose that b and $\hat{b}$ are bid functions and that there is $\varepsilon > 0$ such that, for all q, $b(q) \leq \hat{b}(q) \leq b(q)+\varepsilon/Q$. Then against any profile of opponents' bids b^{-i},*

$$U^i\left(\hat{b}, b^{-i}; s_i\right) \geq U^i\left(b, b^{-i}; s_i\right) - \varepsilon.$$

Lemma 7 (Satisfaction of assumptions)**.** *For $\varepsilon > 0$, $\mathcal{M}^\varepsilon$ satisfies the assumptions of McAdams (2003).*

Proof. I verify each of the numbered assumptions made in McAdams (2003).

Assumption 1. In any model $\mathcal{M}^\varepsilon$, agent i's bid for unit q when her signal is s_i is bounded above by $v^i(q; s_i) + \varepsilon$, hence it is safe to consider the action space $\times_{k=0}^{Q/\varepsilon}\{0, \varepsilon, \ldots, v^i(k\varepsilon; s_i)+\varepsilon\}$.[36] As a finite subset of $\mathbb{R}^{Q/\varepsilon}$ constrained only by monotoncity, the action space is a lattice.

Assumption 2. All players have signal $s_i \sim \mathcal{U}(0,1)$, with common support $[0,1]$ and density bounded and bounded away from zero.

Assumption 3. Ex post payoffs are bounded since values are bounded and bids are constrained to be weakly positive.

Assumption 4. Let b and b' be bid functions in model $\mathcal{M}^\varepsilon$, and let $\underline{b} \equiv b \wedge b'$ and $\overline{b} \equiv b \vee b'$. Quasisupermodularity requires that

$$\begin{aligned} & U^i\left(b', b^{-i}; s\right) \geq U^i\left(\underline{b}, b^{-i}; s\right) \\ \implies & U^i\left(\overline{b}, b^{-i}; s\right) \geq U^i\left(b', b^{-i}; s\right). \end{aligned} \tag{4}$$

It is also necessary to verify that strict inequality in the former implies strict inequality in the latter. This will follow immediately from the weak-inequality derivation results.

Let $DG^i(q; \hat{b}) = G^i(q; \hat{b}) - \lim_{q' \nearrow q} G^i(q'; \hat{b})$ be the probability that agent i receives quantity q in equilibrium. Note that this is no longer a density, since quantities are discrete. Let $\underline{b} \equiv b' \wedge b$. Letting $Q/\varepsilon = T$, the antecedent utility expression in the quasisupermodularity requirement (4) is

$$\sum_{t=0}^{T} \int_0^{t\varepsilon} v^i(x; s) - b'(x)\, dx DG^i\left(t\varepsilon; b'\right) - \sum_{t=0}^{T} \int_0^{t\varepsilon} v^i(x; s) - \underline{b}(x)\, dx DG^i\left(t\varepsilon; \underline{b}\right) \geq 0.$$

Rewrite this as

$$\begin{aligned} & \sum_{t=0}^{T} \int_0^{t\varepsilon} v^i(x; s)\, dx \left[DG^i\left(t\varepsilon; b'\right) - DG^i\left(t\varepsilon; \underline{b}\right)\right] \\ & \quad \geq \sum_{t=0}^{T} \int_0^{t\varepsilon} b'(x)\, dx DG^i\left(t\varepsilon; b'\right) - \int_0^{t\varepsilon} \underline{b}(x)\, dx DG^i\left(t\varepsilon; \underline{b}\right). \end{aligned}$$

[36] Recall that Section 3 constrains attention to ε which evenly divide Q. If ε does not evenly divide $\overline{v}$, the preceding claim holds with regard to the Cartesian product taken up to $\lceil Q/\varepsilon \rceil$.

Noting that the "order of integration" (in this case, summing then integrating) may be changed, this is

$$\sum_{t=1}^{T} \int_{(t-1)\varepsilon}^{t\varepsilon} v^i(x;s)\,dx \left[G^i(t\varepsilon;\underline{b}) - G^i\left(t\varepsilon;b'\right)\right]$$
$$\geq \sum_{t=1}^{T} \int_{(t-1)\varepsilon}^{t\varepsilon} b'(x)\,dx \left[1 - G^i\left(t\varepsilon;b'\right)\right] - \int_{(t-1)\varepsilon}^{t\varepsilon} \underline{b}(x)\,dx \left[1 - G^i(t\varepsilon;\underline{b})\right].$$

Working in parallel, the same series of steps rewrites the implication of quasisupermodularity (4) as

$$\sum_{t=1}^{T} \int_{(t-1)\varepsilon}^{t\varepsilon} v^i(x;s)\,dx \left[G^i(t\varepsilon;b) - G^i\left(t\varepsilon;\overline{b}\right)\right]$$
$$\geq \sum_{t=1}^{T} \int_{(t-1)\varepsilon}^{t\varepsilon} \overline{b}(x)\,dx \left[1 - G^i\left(t\varepsilon;\overline{b}\right)\right] - \int_{(t-1)\varepsilon}^{t\varepsilon} b(x)\,dx \left[1 - G^i(t\varepsilon;b)\right].$$

Since $G^i(x;b) - G^i(x;\overline{b}) = G^i(x;\underline{b}) - G^i(x;b')$ for all x, the left-hand terms of these inequalities are equal. Thus it is sufficient to show

$$\sum_{t=1}^{T} \int_{(t-1)\varepsilon}^{t\varepsilon} b'(x)\,dx \left[1 - G^i\left(t\varepsilon;b'\right)\right] - \int_{(t-1)\varepsilon}^{t\varepsilon} \underline{b}(x)\,dx \left[1 - G^i(t\varepsilon;\underline{b})\right]$$
$$\geq \sum_{t=1}^{T} \int_{(t-1)\varepsilon}^{t\varepsilon} \overline{b}(x)\,dx \left[1 - G^i\left(t\varepsilon;\overline{b}\right)\right] - \int_{(t-1)\varepsilon}^{t\varepsilon} b(x)\,dx \left[1 - G^i(t\varepsilon;b)\right].$$

Since $\overline{b}(x) + \underline{b}(x) = b'(x) + b(x)$ for all x, this simplifies to

$$\sum_{t=1}^{T} \int_{(t-1)\varepsilon}^{t\varepsilon} \overline{b}(x)\,dxG^i\left(t\varepsilon;\overline{b}\right) + \int_{(t-1)\varepsilon}^{t\varepsilon} \underline{b}(x)\,dxG^i(t\varepsilon;\underline{b})$$
$$\geq \sum_{t=1}^{T} \int_{(t-1)\varepsilon}^{t\varepsilon} b'(x)\,dxG^i\left(t\varepsilon;b'\right) + \int_{(t-1)\varepsilon}^{t\varepsilon} b(x)\,dxG^i(t\varepsilon;b).$$

Bids are for discrete units and are constrained to be constant on ε-intervals, hence for any valid bid function f it is the case that $\int_{(t-1)\varepsilon}^{t\varepsilon} f(x)dx = \varepsilon f(t\varepsilon)$. Then the above inequality can be written

$$\sum_{t=1}^{T} \overline{b}(t\varepsilon)\,G^i\left(t\varepsilon;\overline{b}\right) + \underline{b}(t\varepsilon)\,G^i(t\varepsilon;\underline{b}) \geq \sum_{t=1}^{T} b'(t\varepsilon)\,G^i\left(t\varepsilon;b'\right) + b(t\varepsilon)\,G^i(t\varepsilon;b).$$

Since at any $t\varepsilon$, either $b = \overline{b}$ and $b' = \underline{b}$ or $b = \underline{b}$ and $b' = \overline{b}$, this weak inequality holds with equality and weak quasisupermodularity is satisfied. Strict quasisupermodularity can be established in the same manner.

Assumption 5. Let b' and b be actions available to agent i in model $\mathcal{M}^{\varepsilon}$, and suppose that $b' > b$ and

$$U^i\left(b', b^{-i}; s\right) \geq U^i\left(b; b^{-i}; s\right).$$

Let $DG^i(q;\hat{b}) = G^i(q;\hat{b}) - \lim_{q' \nearrow q} G^i(q';\hat{b})$ be the probability that agent i receives quantity q in equilibrium. Letting $Q/\varepsilon = T$, the utility expression above is

$$\sum_{t=0}^{T} \int_0^{t\varepsilon} v^i\left(x; s\right) - b'\left(x\right) dx DG^i\left(t\varepsilon; b'\right) \geq \sum_{t=0}^{T} \int_0^{t\varepsilon} v^i\left(x; s\right) - b\left(x\right) dx DG^i\left(t\varepsilon; b\right).$$

This expression is rearranged as

$$\begin{aligned} &\sum_{t=0}^{T} \int_0^{t\varepsilon} v^i\left(x; s\right) dx \left(DG^i\left(t\varepsilon; b'\right) - DG^i\left(t\varepsilon; b\right)\right) \\ \geq &\sum_{t=0}^{T} \int_0^{t\varepsilon} b'\left(x\right) dx DG^i\left(t\varepsilon; b'\right) - \sum_{t=0}^{T} \int_0^{t\varepsilon} b\left(x\right) dx DG^i\left(t\varepsilon; b\right). \end{aligned}$$

Fixing b' and b, the right-hand side is constant. Therefore restrict attention to the left-hand side, and (as in the analysis of Assumption 4 above) rewrite it as

$$\begin{aligned} &\sum_{t=0}^{T} \int_0^{t\varepsilon} v^i\left(x; s\right) dx \left(DG^i\left(t\varepsilon; b'\right) - DG^i\left(t\varepsilon; b\right)\right) \\ = &\sum_{t=1}^{T} \int_{(t-1)\varepsilon}^{t\varepsilon} v^i\left(x; s\right) dx \left[\left(1 - G^i\left(t\varepsilon; b'\right)\right) - \left(1 - G^i\left(t\varepsilon; b\right)\right)\right] \\ = &\sum_{t=1}^{T} \int_{(t-1)\varepsilon}^{t\varepsilon} v^i\left(x; s\right) dx \left[G^i\left(t\varepsilon; b\right) - G^i\left(t\varepsilon; b'\right)\right]. \end{aligned}$$

When $b' > b$, Lemma 2 gives us that $G^i(q;b) \geq G^i(q;b')$ for all q. Hence the left-hand side is increasing in v^i; it follows that the left-hand side is greater under $s' > s$, establishing single-crossing. Strict single crossing can be shown in the same manner. □

The proof of equilibrium existence begins by defining "limiting strategies," derived from equilibria of sequential refinements of the ε-discrete model $\mathcal{M}^{\varepsilon}$.[37] Because

[37] This proof does not require that the discretized model has equal spacing of quantities and bids, or that the limiting set of quantity-bid pairs is a subset of the rational plane. These are convenient assumptions for analysis and notation, and are sufficient. The latter assumption can be replaced by analyzing the Cartesian product of any countable, dense quantity- and bid-sets which contain the endpoints of their respective non-discretized underlying sets.

the proofs below frequently consider sets constrained to the rational numbers, the following shorthands will be used:

$$\mathcal{S} = [0,1] \cap \mathbb{Q}, \qquad \mathcal{S}^C = [0,1] \setminus \mathcal{S};$$
$$\mathcal{Q} = [0,Q] \cap \mathbb{Q}, \qquad \mathcal{Q}^C = [0,Q] \setminus \mathcal{Q};$$
$$\mathcal{B} = \left[0, \max_j v^j(0;1)\right] \cap \mathbb{Q}, \qquad \mathcal{B}^C = \left[0, \max_j v^j(0;1)\right] \setminus \mathcal{B}.$$

Definition 5 (Limiting strategies)**.** *Bids* $(\beta^{i,\square})_{i=1}^n$ *and distributions* $(G^{i,\square})$ *are* limiting strategies *and* limiting beliefs*, respectively, if there exists a monotone decreasing sequence* $\langle \varepsilon_t \rangle_{t=1}^\infty$*,* $\varepsilon_t \searrow 0$*, and a sequence of equilibria of the* ε_t*-discrete model* $\mathcal{M}^{\varepsilon_t}$*,* $(\beta^{i,\varepsilon_t})_{i=1}^n$ *and associated quantity distributions* $(G^{i,\varepsilon_t})_{i=1}^n$ *such that:*

1. $\beta^{i,\square}$ *is monotonically decreasing in its first argument and monotonically increasing in its second;*
2. *For all* $(q,s) \in \mathcal{Q} \times \mathcal{S}$*,* $\beta^{i,\varepsilon_t}(q;s) \to \beta^{i,\square}(q;s)$ *pointwise;*
3. $G^{i,\square}$ *is monotonically increasing in its first argument and monotonically decreasing in its second;*
4. *For all* $(q,b) \in \mathcal{Q} \times \mathcal{B}$*,* $G^{i,\varepsilon_t}(q;b) \to G^{i,\square}(q;b)$ *pointwise.*

When coordinates are rational, limiting strategies are the natural limits of equilibria of the ε_t-discrete auction. When signals are irrational, bids at any rational quantity are the supremum over bids for this same quantity for lower-signal versions of this agent. When quantities—or, in the case of $G^{i,\square}$, bids—are irrational, any values which satisfy the monotonicity constraints are permissible.

Lemma 8 establishes that limiting strategies exist.

Lemma 8 (Existence of limiting strategies and beliefs)**.** *Given any monotone decreasing sequence* $\langle \varepsilon_t \rangle_{t=1}^\infty$*, there is a subsequence* $\langle \varepsilon_{t_k} \rangle_{k=1}^\infty$*, strategies* $(\beta^{i,\square})_{i=1}^n$*, and distributions* $(G^{i,\square})_{i=1}^n$ *such that* $(\beta^{i,\square})$ *are limiting strategies and* $G^{i,\square}$ *are limiting beliefs.*

Proof. Lemma 7 establishes that for all t, there is a pure-strategy equilibrium $(\beta^{i,\varepsilon_t})$ of the ε_t-discrete model $\mathcal{M}^{\varepsilon_t}$. Selection results (Widder, 1941) imply that for any countable $\tilde{\mathcal{Q}} \times \tilde{\mathcal{S}}$ and any countable $\tilde{\mathcal{Q}}' \times \tilde{\mathcal{B}}$ there is a subsequence $\langle \varepsilon_{t_k} \rangle_{k=1}^\infty$ such that $\beta^{i,\varepsilon_{t_k}}(q;s) \to \beta^{i,\square}(q;s)$ pointwise for all i and all $(q,s) \in \tilde{\mathcal{Q}} \times \tilde{\mathcal{S}}$, and $G^{i,\varepsilon_{t_k}}(q;b) \to G^{i,\square}(q;b)$ pointwise for all i and all $(q,b) \in \tilde{\mathcal{Q}}' \times \tilde{\mathcal{B}}$. For any such sets monotonicity of $\beta^{i,\square}$ and $G^{i,\square}$ are guaranteed by the fact that β^{i,ε_t} and G^{i,ε_t} are monotone for all i and all ε_t. The desired result follows from letting $\tilde{\mathcal{Q}} = \tilde{\mathcal{Q}}' = \mathcal{Q}$, $\tilde{\mathcal{S}} = \mathcal{S}$, and $\tilde{\mathcal{B}} = \mathcal{B}$. □

Lemma 9 (ℓ_1 convergence on $\mathcal{S}$)**.** *Let* $(\beta^{i,\square})_{i=1}^n$ *be limiting strategies associated with some sequence* $\langle (\beta^{i,\varepsilon_t})_{i=1}^n \rangle_{t=1}^\infty$ *of* $\mathcal{M}_{\varepsilon_t}$*-equilibria. Then for all* i *and all* $s \in \mathcal{S}$*,* $||\beta^{i,\varepsilon_t}(\cdot;s) - \beta^{i,\square}(\cdot;s)||_{\ell_1} \to 0$.

Proof. This is a localized version of the Helly selection theorem.

Suppose that $\beta^{i,\Box}$ is continuous at some $q \in [0, Q]$. Then for all $\lambda > 0$ there is a $\delta > 0$ such that $|\beta^{i,\Box}(q; s) - \beta^{i,\Box}(q + \delta'; s)| < \lambda$ for all $\delta' \in (-\delta, \delta)$. Since $\mathcal{Q}$ is dense, there are $q_\ell, q_r \in \mathcal{Q} \times (-\delta, \delta)$ such that $q_\ell < q < q_r$; by pointwise convergence of β^{i,ε_t} to $\beta^{i,\Box}$ on $\mathcal{Q} \times \mathcal{S}$, there is a T such that $|\beta^{i,\varepsilon_t}(q'; s) - \beta^{i,\Box}(q'; s)| < \lambda$ for $q' \in \{q_\ell, q_r\}$ and all $t > T$.

The difference between $\beta^{i,\Box}(q_\ell; s)$ and $\beta^{i,\Box}(q_r; s)$ is bounded,

$$\left|\beta^{i,\Box}(q_\ell; s) - \beta^{i,\Box}(q_r; s)\right| = \left|\beta^{i,\Box}(q_\ell; s) - \beta^{i,\Box}(q; s)\right| + \left|\beta^{i,\Box}(q; s) - \beta^{i,\Box}(q_r; s)\right| < 2\lambda.$$

This implies

$$\begin{aligned}\left|\beta^{i,\varepsilon_t}(q_\ell; s) - \beta^{i,\varepsilon_t}(q_r; s)\right| \leq \Big[& \left|\beta^{i,\varepsilon_t}(q_\ell; s) - \beta^{i,\Box}(q_\ell; s)\right| \\ & + \left|\beta^{i,\Box}(q_\ell; s) - \beta^{i,\Box}(q_r; s)\right| \\ & + \left|\beta^{i,\Box}(q_r; s) - \beta^{i,\varepsilon_t}(q_r; s)\right| \Big] < 4\lambda.\end{aligned}$$

Since $\beta^{i,\varepsilon_t}(\cdot; s)$ is monotone, this further implies that $|\beta^{i,\varepsilon_t}(q'; s) - \beta^{i,\varepsilon_t}(q; s)| < 4\lambda$ for $q' \in \{q_\ell, q_r\}$. Then

$$\begin{aligned}\left|\beta^{i,\varepsilon_t}(q; s) - \beta^{i,\Box}(q; s)\right| \leq \Big[& \left|\beta^{i,\varepsilon_t}(q; s) - \beta^{i,\varepsilon_t}(q_\ell; s)\right| \\ & + \left|\beta^{i,\varepsilon_t}(q_\ell; s) - \beta^{i,\Box}(q_\ell; s)\right| \\ & + \left|\beta^{i,\Box}(q_\ell; s) - \beta^{i,\Box}(q; s)\right| \Big] < 6\lambda.\end{aligned}$$

Since $\lambda > 0$ may be arbitrarily small, it follows that there is T' such that $|\beta^{i,\varepsilon_t}(q; s) - \beta^{i,\Box}(q; s)| < \lambda$ for all $t > T'$. Then $\beta^{i,\varepsilon_t}(q; s) \to \beta^{i,\Box}(q; s)$ whenever $\beta^{i,\Box}$ is continuous at q.

Since $\beta^{i,\Box}(\cdot; s)$ is a monotone bounded function, it has at most a measure-zero set of discontinuities; hence $\beta^{i,\varepsilon_t}(q; s) \to \beta^{i,\Box}(q; s)$ for almost all q, and it is immediate that $||\beta^{i,\varepsilon_t}(\cdot; s) - \beta^{i,\Box}(\cdot; s)||_{\ell_1} \to 0$. □

Limiting strategies and beliefs are defined almost nowhere. However, since bid functions are monotonic in all dimensions and map into a compact space, limiting strategies can be used to naturally define functions on all of $[0, Q] \times [0, 1]$.

Definition 6 (Completed limiting strategies). $\beta^{i,\Box}$ *is a* completed limiting strategy *if it is a limiting strategy and*

1. *For all $s \in \mathcal{S}$ and all $q \in \mathcal{Q}^C$, $\beta^{i,\Box}(q; s) = \inf_{q' \in [0,q) \cap \mathbb{Q}} \beta^{i,\Box}(q'; s)$;*
2. *For all $s \in \mathcal{S}^C$ and all $q \in [0, Q]$, $\beta^{i,\Box}(q; s) = \sup_{s' \in [0,s) \cap \mathbb{Q}} \beta^{i,\Box}(q; s)$.*

Definition 7 (Supremum-limit strategies[38]). $(\overline{\beta}^i)_{i=1}^n$ *is a profile of* supremum-limit *strategies if there is a profile of limiting strategies* $(\beta^{i,\square})_{i=1}^n$ *such that, for any* i,

1. *For all* $q \in \mathcal{Q}$ *and* $s > 0$, $\overline{\beta}^i(q; s) = \sup_{s' \in [0,s) \cap \mathcal{S}} \beta^{i,\square}(q; s')$;
2. *For all* $q \in \mathcal{Q}^C$ *and* $s > 0$, $\overline{\beta}^i(q; s) = \inf_{q' \in [0,q) \cap \mathcal{Q}} \beta^{i,\square}(q'; s)$.

That is, a strategy profile $(\overline{\beta}^i)_{i=1}^n$ is a profile of supremum-limit strategies if each bid function is left-continuous in quantity and right-continuous in signal, and it is (roughly) derived from a limiting strategy profile $(\beta^{i,\square})_{i=1}^n$. The choice of left-continuity in quantity is not essential, but is made for completeness' sake.

In what follows, I will fix a particular convergent sequence of discretized equilibria $\langle(\beta^{i,\varepsilon_t})_{i=1}^n\rangle_{t=1}^\infty$, an associated limiting strategy profile $(\beta^{i,\square})_{i=1}^n$, and an associated supremum-limit strategy profile $\overline{\beta}^i$.

Lemma 10 (Almost-everywhere convergence to supremum-limit strategies). *For all* i *and almost all* s_i, $||\beta^{i,\varepsilon_t}(\cdot; s_i) - \overline{\beta}^i(\cdot; s_i)||_{\ell_1} \to 0$.

Proof. Note that any limiting strategy $\beta^{i,\square}$ has at most a measure-zero set of discontinuities (Lavrič, 1993). Let $\tilde{\beta}^i$ be a completion of $\beta^{i,\square}$ such that $\tilde{\beta}^i(q; s) = \beta^{i,\square}(q; s)$ whenever $\beta^{i,\square}$ is continuous at $(q; s)$; then $||\beta^{i,\square} - \tilde{\beta}^i||_{\ell_1} = 0$; adapting arguments from Lemma 9 implies that $||\beta^{i,\varepsilon_t} - \tilde{\beta}^i||_{\ell_1} \to 0$.

Let S_δ be the set of δ-nonconverging signals s,

$$S_\delta = \left\{ s : \lim_{t \nearrow \infty} \left|\left|\beta^{i,\varepsilon_t}(\cdot; s) - \tilde{\beta}^i(\cdot; s)\right|\right|_{\ell_1} > \delta \right\},$$
$$\rightsquigarrow S_0 = \left\{ s : \lim_{t \nearrow \infty} \left|\left|\beta^{i,\varepsilon_t}(\cdot; s) - \tilde{\beta}^i(\cdot; s)\right|\right|_{\ell_1} > 0 \right\} = \bigcup_{y \in \mathbb{N}} S_{1/2^y}.$$

Letting μ^k be the Lebesgue measure on $\mathbb{R}^k$, $\mu^1(S_0) \leq \sum_{y=0}^\infty \mu^1(S_{1/2^y})$; hence if $\mu^1(S_0) > 0$ there is y such that $\mu^1(S_{1/2^y}) > 0$. Consider the measure of all points of nonconvergence,

$$\begin{aligned}
&\mu^2\left(\left\{(q,s) : \left|\beta^{i,\varepsilon_t}(q;s) - \tilde{\beta}^i(q;s)\right| \not\to 0\right\}\right) \\
&= \int_{[0,1]} \mu^1\left(\left\{q : \lim_{t \nearrow \infty} \left|\beta^{i,\varepsilon_t}(q;s) - \tilde{\beta}^i(q;s)\right| > 0\right\}\right) d\mu^1(s) \\
&\geq \int_{s \in S_{1/2^y}} \mu^1\left(\left\{q : \lim_{t \nearrow \infty} \left|\beta^{i,\varepsilon_t}(q;s) - \tilde{\beta}^i(q;s)\right| > 0\right\}\right) d\mu^1(s).
\end{aligned}$$

[38] The term "limit-supremum strategy" is more grammatically apt, but is not used to distinguish these strategies from the mathematical notion of limit supremum.

Note that for any $s \in S_{1/2^y}$, the boundedness of $\tilde{\beta}^i$ is sufficient to imply that

$$\mu^1\left(\left\{q : \lim_{t \nearrow \infty}\left|\beta^{i,\varepsilon_t}(q;s) - \tilde{\beta}^i(q;s)\right| > 0\right\}\right) \geq \frac{1}{2^y v^i(0;1)}.$$

Then we have

$$\begin{aligned}
&\mu^2\left(\left\{(q,s) : \left|\beta^{i,\varepsilon_t}(q;s) - \tilde{\beta}^i(q;s)\right| \not\to 0\right\}\right) \\
&\geq \int_{s \in S_{1/2^y}} \frac{1}{2^y v^i(0;1)} d\mu^1(s) = \frac{\mu^1\left(S_{1/2^y}\right)}{2^y v^i(0;1)} > 0.
\end{aligned}$$

Then $\mu^2(\{(q,s) : |\beta^{i,\varepsilon_t}(q;s) - \tilde{\beta}^i(q;s)| > 0\}) > 0$, contradicting the fact that $||\beta^{i,\varepsilon_t} - \tilde{\beta}^i||_{\ell_1} \to 0$. Since $\overline{\beta}^i$ is a completion of $\beta^{i,\square}$, it follows that $||\beta^{i,\varepsilon_t}(\cdot;s_i) - \overline{\beta}^i(\cdot;s_i)||_{\ell_1} \to 0$ for almost all s_i. □

Definition 8 (Co-ironed intervals)**.** *Given a bidding strategy b, non-degenerate intervals $\mathcal{I}_s \subseteq [0,1]$ and $\mathcal{I}_q \subseteq [0,Q]$ represent a* co-ironed interval *if for all $s, s' \in \mathcal{I}_s$ and $q, q' \in \mathcal{I}_q$, $b(q;s) = b(q';s')$.* ***add quantity-relevance***

Lemma 11 (Discontinuity implies co-ironing)**.** *For any agent i and any signal s_i, interim expected utility is discontinuous at bid function b, with respect to deviations in the ℓ_1 norm, given opponents' strategy profile $(b^j)_{j \neq i}$ only if there is some agent j with co-ironed intervals $\mathcal{I}_s^j$ and $\mathcal{I}_q^j$ in her bidding strategy b^j.*

Proof. Recall the formula for interim expected utility,

$$U^i\left(b, b^{-i}; s_i\right) = \int_0^Q \left(v^i(x;s_i) - b(x)\right)\left(1 - G^i(x;b)\right) dx.$$

If G^i is continuous in b, the integrand is continuous in b and hence interim expected utility is continuous in b. For interim expected utility to be discontinuous in b, it must be that G^i is discontinuous in b, in the sense that $\int_0^Q G^i(x;b)dx$ is discontinuous in b;[39] that is, G^i is discontinuous in b with respect to the ℓ_1 norm.

Suppose that for no opponent j does there exist a co-ironed interval given by $\mathcal{I}_s^j$ and $\mathcal{I}_q^j$. In particular, suppose that for all j, whenever a non-degenerate interval $\mathcal{I}_q^j$ is such that there is an $s \in [0,1]$ and for all $q, q' \in \mathcal{I}_q^j$, $b^j(q;s) = b^j(q';s)$, there is no $s' \neq s$ such that $b^j(q;s') = b^j(q';s')$ unless $q = q'$.

[39] This follows from the fact that the Lemma is looking for necessary conditions for discontinuity, not sufficient conditions. In this case, it is without loss to ignore the $v^i(x;s_i) - b(x)$ term in the integrand.

Consider the effect of increasing b to $b^\varepsilon = b + \varepsilon$.[40] By definition,

$$W(q;p) = \left\{ s_{-i} : \sum_{j\neq i} \underline{\varphi}^j(p;s_j) \leq Q - q \leq \sum_{j\neq i} \overline{\varphi}^j(p;s_j) \right\},$$
$$G^i(q;b^\varepsilon) = \Pr(s_{-i} \in W(q;b^\varepsilon(q))).$$

Since allocation probabilities are monotone in bid, for any q, $W(q;b(q)) \subseteq W(q;b^\varepsilon(q))$. Suppose then that $s_{-i} \in \cap_{\varepsilon>0} W(q;b^\varepsilon(q)) \setminus W(q;b(q))$; then

$$\lim_{\varepsilon\searrow 0} \sum_{j\neq i} \underline{\varphi}^j(b^\varepsilon(q);s_j) \leq Q - q \leq \lim_{\varepsilon\searrow 0} \sum_{j\neq i} \overline{\varphi}^j(b^\varepsilon(q);s_j) < \sum_{j\neq i} \underline{\varphi}^j(b(q);s_j).$$

what is missing here? this should be a simple argument.

[41] □

Lemma 12 (Utility approximation)**.** *Let $(b^j)_{j\neq i}$ be a profile of bidding strategies in $\mathcal{M}^\varepsilon$ for agents other than i. For any bid function $b \leq v^i(\cdot;s_i)$ for agent i, let $\hat{b}$ be a bid function in $\mathcal{M}^\varepsilon$ given by*

$$\hat{b}(q) = \left\lceil \frac{b\left(\left\lfloor \frac{q}{\varepsilon} \right\rfloor \varepsilon\right)}{\varepsilon} \right\rceil \varepsilon.$$

Then for $\varepsilon > 0$ sufficiently small, there is $C \in \mathbb{R}_+$, independent of ε, such that $U^i(\hat{b}, b^{-i}; s_i) \geq U^i(b, b^{-i}; s_i) - C\varepsilon$.

Proof. The utility difference between the two bid functions can be expressed as

$$\begin{aligned} U^i\left(b, b^{-i}; s_i\right) - U^i\left(\hat{b}, b^{-i}; s_i\right) &= \int_0^Q \left(v^i(x;s_i) - b(x)\right)\left(1 - G^i(x;b)\right) dx \\ &\quad - \int_0^Q \left(v^i(x;s_i) - \hat{b}(x)\right)\left(1 - G^i\left(x;\hat{b}\right)\right) dx \\ &= \int_0^Q \left(\hat{b}(x) - b(x)\right)\left(1 - G^i\left(x;\hat{b}\right)\right) \\ &\quad + \left(v^i(x;s_i) - b(x)\right)\left(G^i\left(x;\hat{b}\right) - G^i(x;b)\right) dx. \end{aligned}$$

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[40] Because only the smoothness of the change in G^i is of consequence, the order of the change in G^i is irrelevant and it is sufficient to consider additive deviations.

[41] Note that $W(q;p)$ has the property that for all $x \in W(q;p)$, $x' \in W(q;p)$ whenever $x' < x$. Let $D = \{\{x\}^n : x \in [0,1]\}$ be the diagonal in $[0,1]^n$ and let $\bar{W}(q;p) = \partial W(q;p) \setminus D$; then for all $x \in \bar{W}(q;p)$, $x - \{\delta\}^n \notin \bar{W}(q;p)$. Then

$$\mu\left(\left[\bigcup_{\delta\in[0,1]} \bar{W}(q;p) - \delta\right] \cup D\right) \leq \mu([-1,1])^n = 2^n.$$

Since the sets $\bar{W}(q;p) - \delta$ are disjoint, it follows that $\sum_{\delta\in[0,1]} \mu(\bar{W}(q;p) - \delta) \leq 2^n$. It is immediate that $\mu(\bar{W}(q;p)) = 0$.

Because $\hat{b} \geq b$, Lemma XXX implies that $G^i(\cdot;\hat{b}) \succeq_{\text{FOSD}} G^i(\cdot;b)$, hence the second term in the final right-hand side is weakly negative. Then

$$U^i\left(b, b^{-i}; s_i\right) - U^i\left(\hat{b}, b^{-i}; s_i\right) \leq \int_0^Q \left(\hat{b}(x) - b(x)\right)\left(1 - G^i\left(x;\hat{b}\right)\right) dx \\ \leq \int_0^Q \hat{b}(x) - b(x)\, dx.$$

Note that since bids are decreasing in quantity, for any $q > \varepsilon$, $\hat{b}(q) < b(q-\varepsilon) + \varepsilon$. Then

$$U^i\left(b, b^{-i}; s_i\right) - U^i\left(\hat{b}, b^{-i}; s_i\right) < \int_0^Q \left[b\left(\max\left\{x-\varepsilon, 0\right\}\right) - b(x)\right] + \varepsilon dx \\ = \int_0^{Q-\varepsilon} b(x)\, dx - \int_0^Q b(x)\, dx + \varepsilon b(0) + \varepsilon Q \\ = -\int_{Q-\varepsilon}^Q b(x)\, dx + (b(0) + Q)\,\varepsilon \leq (b(0) + Q)\,\varepsilon.$$

Since $b(0) \leq v^i(0; s_i) + \varepsilon$, it follows that

$$U^i\left(\hat{b}, b^{-i}; s_i\right) > U^i\left(b, b^{-i}; s_i\right) - \left(v^i(0; s_i) + Q\right)\varepsilon + \varepsilon^2.$$

Then for ε sufficiently small—namely, for any $\gamma \in \mathbb{R}_{++}$ and $\varepsilon < \gamma$—take $C = v^i(0; s_i) + Q + \gamma$, establishing the desired result. □

Corollary 2 (Existence of utility approximation). *Given any unconstrained best response b^i to opponents' bidding strategies $(b^j)_{j\neq i}$ in $\mathcal{M}^\varepsilon$, there is a feasible bid function $\hat{b}$ in $\mathcal{M}^\varepsilon$ that generates utility that is worse by no more than $O(\varepsilon)$.*

Lemma 13 (No discrete utility gain for ℓ_1-convergent bids). *If $s_i > 0$ is such that $||\beta^{i,\varepsilon_t}(\cdot; s_i) - \overline{\beta}^i(\cdot; s_i)||_{\ell_1} \to 0$, then agent i's interim utility is continuous in her bid at the supremum-limit strategy profile $(\overline{\beta}^j)_{j=1}^n$.*

Proof. **(this is a proof)** □

Lemma 14 (Best responses for ℓ_1-convergent bids). *If $s_i > 0$ is such that $||\beta^{i,\varepsilon_t}(\cdot; s_i) - \overline{\beta}^i(\cdot; s_i)||_{\ell_1} \to 0$, then $\overline{\beta}^i(\cdot; s_i)$ is a best response for agent i when her opponents play the supremum-limit strategy profile $(\overline{\beta}^j)_{j\neq i}$.*

Proof. NEEDS LEMMA 19! Move it up? **(this is a proof)** □

Lemma 15 (Best responses for ℓ_1-nonconvergent bids). *If $s_i > 0$ is such that $||\beta^{i,\varepsilon_t}(\cdot; s_i) - \overline{\beta}^i(\cdot; s_i)||_{\ell_1} \not\to 0$, then $\overline{\beta}^i(\cdot; s_i)$ is a best response for agent i when her opponents play the supremum-limit strategy profile $(\overline{\beta}^j)_{j\neq i}$.*

Proof. If $\overline{\beta}^i(\cdot;s_i)$ is not a best response when agent i's signal is s_i, there is some $\hat{\beta}$ and $\varepsilon > 0$ such that $U^i(\hat{\beta};s_i) \geq U^i(\overline{\beta}^i(\cdot;s_i);s_i) + 3\varepsilon$. By continuity of $v^i(q;\cdot)$, the definition of $\overline{\beta}^i$, there is $\delta > 0$ such that for all $s' \in (s_i - \delta, s_i)$,

$$\begin{aligned}\left|\left|\overline{\beta}^i\left(\cdot;s'\right) - \overline{\beta}^i\left(\cdot;s_i\right)\right|\right|_{\ell_1} &< \varepsilon,\\ \left|\left|v^i\left(\cdot;s'\right) - v^i\left(\cdot;s_i\right)\right|\right|_{\ell_1} &< \varepsilon.\end{aligned}$$

By Lemma 10, it is without loss of generality to assume that $||\beta^{i,\varepsilon_t}(\cdot;s') - \overline{\beta}^i(\cdot;s')||_{\ell_1} \to 0$, hence by Lemma 14 it is without loss of generality to assume that $\overline{\beta}^i(\cdot;s')$ is a best response when agent i's signal is s'.

Taking as given any bid function $\tilde{\beta}$,

$$\begin{aligned}U^i\left(\tilde{\beta};s'\right) &= \int_0^Q \left(v^i\left(x;s'\right) - \tilde{\beta}\left(x\right)\right)\left(1 - G^i\left(x;\tilde{\beta}\right)\right)dx\\ &= \int_0^Q \left(v^i\left(x;s_i\right) - \tilde{\beta}\left(x\right)\right)\left(1 - G^i\left(x;\tilde{\beta}\right)\right)dx\\ &\quad \int_0^Q \left(v^i\left(x;s'\right) - v^i\left(x;s_i\right)\right)\left(1 - G^i\left(x;\tilde{\beta}\right)\right)dx\\ &> \int_0^Q \left(v^i\left(x;s_i\right) - \tilde{\beta}\left(x\right)\right)\left(1 - G^i\left(x;\tilde{\beta}\right)\right)dx - \varepsilon\\ &= U^i\left(\tilde{\beta};s_i\right) - \varepsilon.\end{aligned}$$

That is, the expected utility generated by any strategy is continuous in s. Then utility can be compared:

$$\begin{aligned}U^i\left(\overline{\beta}^i\left(\cdot;s'\right);s'\right) &= \int_0^Q \left(v^i\left(x;s'\right) - \overline{\beta}^i\left(x;s'\right)\right)\left(1 - G^i\left(x;\overline{\beta}^i\left(\cdot;s'\right)\right)\right)dx\\ &= \int_0^Q \left(v^i\left(x;s_i\right) - \overline{\beta}^i\left(x;s_i\right)\right)\left(1 - G^i\left(x;\overline{\beta}^i\left(\cdot;s'\right)\right)\right)dx\\ &\quad + \int_0^Q \left(v^i\left(x;s'\right) - v^i\left(x;s_i\right)\right)\left(1 - G^i\left(x;\overline{\beta}^i\left(\cdot;s'\right)\right)\right)dx\\ &\quad + \int_0^Q \left(\overline{\beta}^i\left(x;s_i\right) - \overline{\beta}^i\left(x;s'\right)\right)\left(1 - G^i\left(x;\overline{\beta}^i\left(\cdot;s'\right)\right)\right)dx\\ &< \int_0^Q \left(v^i\left(x;s_i\right) - \overline{\beta}^i\left(x;s_i\right)\right)\left(1 - G^i\left(x;\overline{\beta}^i\left(\cdot;s'\right)\right)\right)dx + \varepsilon\\ &\leq \int_0^Q \left(v^i\left(x;s_i\right) - \overline{\beta}^i\left(x;s_i\right)\right)\left(1 - G^i\left(x;\overline{\beta}^i\left(\cdot;s_i\right)\right)\right)dx + \varepsilon\\ &= U^i\left(\overline{\beta}^i\left(\cdot;s_i\right);s_i\right) + \varepsilon \leq U^i\left(\hat{\beta};s_i\right) - 2\varepsilon < U^i\left(\hat{\beta};s'\right) - \varepsilon.\end{aligned}$$

Then $U^i(\hat{\beta}; s') > U^i(\overline{\beta}^i(\cdot; s'); s')$, contradicting the fact that $\overline{\beta}^i(\cdot; s')$ is a best response when agent i's signal is s'. □

Definition 9 (Almost-equilibrium)**.** *A strategy profile* $(\beta^i)_{i=1}^n$ *is an* almost-equilibrium *if there is a probability-zero set of types for each agent,* $X = \times_{i=1}^n X_i$*, such that* $(\beta^i)_{i=1}^n$ *is an equilibrium in a game where agent* i*'s type space is* $S_i' = S_i \setminus X_i$*.* ***rephrase!***

Corollary 3 (Supremum-limit almost-equilibrium)**.** *The supremum-limit strategy profile* $(\overline{\beta}^i)_{i=1}^n$ *is an almost-equilibrium of the divisible-good pay-as-bid auction, excluding only* $s_i \in X_i = \{0\}$*.*

Proof. Lemma 14 establishes that $\overline{\beta}^i(\cdot; s)$ is a best response when $s_i = s > 0$ if $||\beta^{i,\varepsilon_t}(\cdot; s) - \overline{\beta}^i(\cdot; s)||_{\ell_1} \to 0$; Lemma 15 establishes the same for the case when $||\beta^{i,\varepsilon_t}(\cdot; s) - \overline{\beta}^i(\cdot; s)||_{\ell_1} \not\to 0$. Then for all agents j and all $s_j > 0$, $\overline{\beta}^i(\cdot; s_j)$ is a best response to the opponents' strategy profile $(\overline{\beta}^i)_{i \neq j}$, establishing the desired result. (immediate consequence) □

Lemma 16. *[Strict monotonicity in almost-equilibrium] For almost all relevant* q*, agent* i*'s supremum-limit bid function* $\overline{\beta}^i(q; \cdot)$ *is strictly increasing.*

Proof. **(this is a proof)** □

Lemma 17. *[Best responses for low-signal agents] When* $s_i = 0$, $\underline{\beta}^i(q) = \inf_{s'>0} \overline{\beta}^i(q; s')$ *is a best response for agent* i*.*

Proof. By Lemma 16, for all $j \neq i$ the bid function $\overline{\beta}^j(q; \cdot)$ is strictly increasing for almost all q and all $s_j > 0$.[42] The arguments from Lemma 13 then imply that $U^i(\cdot; 0)$ is continuous at $\underline{\beta}^i$; the arguments from Lemma 15 are then sufficient to show that $\underline{\beta}^i$ is a best response. □

Theorem 4 (Equilibrium construction)**.** *For each agent* i *let* β^i *be a strategy given by*

$$\beta^i(q; s) = \begin{cases} \overline{\beta}^i(q; s) & \text{if } s > 0, \\ \inf_{s'>0} \overline{\beta}^i(q; s') & \text{if } s = 0. \end{cases}$$

Then $(\beta^i)_{i=1}^n$ *is an equilibrium in the divisible-good pay-as-bid auction.*

[42] Since $\Pr(s_j = 0) = 0$, the selection of a strategy for agent j when her signal is $s_j = 0$ is irrelevant to agent i's best-response behavior; it is then without loss of generality to say that $\overline{\beta}^j(q; \cdot)$ is strictly increasing for almost all q, since whether or not $\overline{\beta}^j(q; 0) < \overline{\beta}^j(q; s)$ is of no importance to the result at hand.

Proof. Corollary 3 establishes that $(\beta^i)_{i=1}^n$ is an almost-equilibrium when types $s_i = 0$ are excluded. Including types $s_i = 0$, each $\beta^j(\cdot; s_j)$ is still a best response for $s_j > 0$ since the probability that she is facing a type $s_i = 0$ (for any number of opponents) is exactly zero. Lemma 17 establishes that $\beta^j(\cdot; 0)$ is a best response when $s_j = 0$. Then $(\beta^i)_{i=1}^n$ is a profile of mutual best responses, and hence is a Bayesian-Nash equilibrium. □

Corollary 4 (Equilibrium existence)**.** *The divisible-good pay-as-bid auction with private information admits a pure-strategy Bayesian-Nash equilibrium.*

C Auxilliary results for Theorem 2: Equilibrium approximation

this is now by constraint! (right?)

Lemma 18 (Upper bound on bids)**.** *For any $\varepsilon > 0$, in any pure-strategy equilibrium $\langle b^{i,\varepsilon}\rangle_{i=1}^n$ of the ε-discrete coarsening $\mathcal{M}^\varepsilon$,*

$$b^{i,\varepsilon}(0; s) \leq \max_j v^j(0; 1) + \varepsilon.$$

Proof. Suppose otherwise. Without loss, assume that $b^{i,\varepsilon}(0; s) \geq b^{j,\varepsilon}(0; s')$ for all j, s'. Let $\bar{v} = \lceil \max_j v^j(0; 1)/\varepsilon \rceil \varepsilon$, and let $\hat{q} = \max\{q : b^{i,\varepsilon}(q; s) \geq \bar{v}\}$. Suppose that the bidder deviates to

$$\hat{b}(q) = \begin{cases} \bar{v} & \text{if } q \leq \hat{q}, \\ b^{i,\varepsilon}(q; s) & \text{otherwise.} \end{cases}$$

The bidder evidently saves payment whenever her allocation is above $\hat{q}$. However, she also sacrifices quantity allocations whenever her allocation is weakly below $\check{q}$ under $b^{i,\varepsilon}$. Because bids have only been reduced where they strictly exceed the maximum marginal value for the initial unit, hence where they strictly exceed the marginal value of any unit, the bidder only sacrifices quantities which she was previously obtaining at negative margin, hence the deviation is profitable. □

Argument has changed; be sure old proof still works.

I have shown that equilibrium strategy profiles in the ε-discrete model $\mathcal{M}^\varepsilon$ have a pointwise convergent subsequence for all rational-valued quantities q and all rational-valued signals s; for agents receiving rational-valued signals, all extensions to the reals of their limiting schedule on the rationals generate the same payoff. Transforming the left-continuous extension so that it is left-continuous in signal, the interim payoffs of agents receiving rational signals are continuous in their own actions and in the strategies of other agents; hence the suggested transformation is valid. Because of strategic monotonicity, the interim payoffs of agents receiving rational-valued signals are independent of the particular strategies implemented by irrational-signal agents. As bid functions must be converging in the L^1 norm for all rational-signal agents, this implies that the extension of the agent i's limiting strategy is in fact a best response at the limit. Agents receiving irrational-valued signals are shown to have best-responses at the limit, and these best responses must satisfy strategic monotonicity. This completes a construction of an equilibrium in the divisible-good case with private information.

Lemma 19 (Utility convergence)**.** *In any equilibrium $\langle b^i \rangle_{i=1}^n$ of $\mathcal{M}^\varepsilon$, or of the divisible-good model,*

$$\lim_{s' \nearrow s_i} U^i \left(b^i \left(\cdot; s' \right), b^{-i}; s_i \right) = \lim_{s' \searrow s_i} U^i \left(b^i \left(\cdot; s' \right), b^{-i}; s_i \right).$$

That is is, expected utility converges from both sides.[43]

Proof. Suppose that there is $\varepsilon > 0$ such that

$$\lim_{s' \nearrow s_i} U^i \left(b^i \left(\cdot; s' \right), b^{-i}; s_i \right) + \varepsilon = \lim_{s' \searrow s_i} U^i \left(b^i \left(\cdot; s' \right), b^{-i}; s_i \right).$$

Because $v^i(q; \cdot)$ is continuous and strictly increasing, for any $\delta > 0$ there is $\underline{s} < s_i$ such that

$$U^i \left(b^i \left(\cdot; s' \right), b^{-i}; s' \right) > U^i \left(b^i \left(\cdot; s_i \right), b^{-i}; s_i \right) - \delta, \ \ \forall s' > \underline{s}.$$

Since $b^i(\cdot; s_i)$ is a best response when agent i receives signal s_i, it follows that

$$U^i \left(b^i \left(\cdot; s_i \right), b^{-i}; s' \right) > \lim_{s' \nearrow s_i} U^i \left(b^i \left(\cdot; s' \right), b^{-i}; s_i \right) + \varepsilon - \delta.$$

For δ sufficiently small, this is a contradiction.

The case where $\lim_{s' \nearrow s_i} U^i(b^i(\cdot; s'), b^{-i}; s_i) > \lim_{s' \searrow s_i} U^i(b^i(\cdot; s'), b^{-i}; s_i)$ is analogous. Hence the stated result holds. ☐

Lemma 20 (Utility approximation)**.** *Suppose that $\langle b^j \rangle_{j \neq i}$ are strategies played by agents other than agent i in $\mathcal{M}^\varepsilon$, and that b^i is a best response for agent i in $\mathcal{M}^\varepsilon$. Let $\hat{b} : [0, Q] \to \mathbb{R}^+$ be an unconstrained monotonic bid function with $U^i(\hat{b}, b^{-i}; s) > U^i(b^i, b^{-i}; s)$, and let $f : [0, Q] \to \mathbb{R}^+$ be any monotonic function that agrees with b^i at available units, $f(t\varepsilon) = b^i(t\varepsilon; s)$. Then*

$$U^i \left(\hat{b}, b^{-i}; s \right) - U^i \left(f, b^{-i}; s \right) = O \left(\varepsilon \right).$$

[43] This does not imply expected utility equivalence of the limiting strategies themselves.

Proof. I first show that, given signal s, the ex post utility generated by $\hat{b}$ can be approximated by a constrained bid function f^ε from $\mathcal{M}^\varepsilon$ so that interim utility U^i is such that $U^i(\hat{b}, b^{-i}; s) - U^i(f^\varepsilon, b^{-i}; s) \leq C\varepsilon$ for some constant C.[44] I then show that the worst approximation $\hat{f}$ of b^i yields utility which is lower by at worst $O(\varepsilon)$. The result then follows.

Construct f^ε as

$$f^\varepsilon(q) = \left\lceil \frac{\hat{b}\left(\left\lfloor \frac{q}{\varepsilon} \right\rfloor \varepsilon\right)}{\varepsilon} \right\rceil \varepsilon.$$

That is, f^ε is the maximum of $\hat{b}$ over any discrete unit, rounded up to the nearest available point on the price grid. By construction, $f^\varepsilon(q) \geq \hat{b}(q)$ for all q; hence the gross utility generated by f^ε weakly exceeds that generated by $\hat{b}^i$ (see the proof of Lemma 2). It will then suffice to bound the extra payment required by f^ε versus $\hat{b}$. For this, a weak bound is the extra payment required if the agent obtains all possible units, $q_i = Q$. This comparison is

$$\begin{aligned}\int_0^Q f^\varepsilon(x) - \hat{b}(x)\, dx &\leq \sum_{t=1}^{Q/\varepsilon} \left[f^\varepsilon((t-1)\varepsilon) - \hat{b}(t\varepsilon)\right]\varepsilon \\ &\leq \sum_{t=1}^{Q/\varepsilon} \left[\hat{b}((t-1)\varepsilon) + \varepsilon - \hat{b}(t\varepsilon)\right]\varepsilon.\end{aligned}$$

Because bids are monotonic in quantity, and will not exceed values for the initial unit, this becomes

$$\int_0^Q f^\varepsilon(x) - \hat{b}(x)\, dx \leq \left[v^i(0; s) + Q\right]\varepsilon.$$

Letting $C = v^i(0; s) + Q$, the inequality is

$$U^i\left(\hat{b}, b^{-i}; s\right) - U^i\left(f^\varepsilon, b^{-i}; s\right) \leq C\varepsilon.$$

I now show that any unconstrained monotonic bid function which equals b^i at the multi-unit grid points must approximate the payoffs to within an order-ε difference. Let $\overline{b}$ and $\underline{b}$ be defined as ε-unit offsets of b^i,

$$\overline{b}(q) = b^i(q - \varepsilon; s), \quad \underline{b}(q) = b^i(q + \varepsilon; s).$$

Let f be any monotonic function satisfying $f(t\varepsilon) = b^i(t\varepsilon; s)$ for all $t \in \mathbb{N}$. Appealing to Lemma 2, the gross utility obtained under f is weakly dominated by that obtained under $\overline{b}$, and weakly dominates the gross utility obtained under $\underline{b}$. Moreover, the

[44] This inequality will trivially hold in absolute value: $\hat{b}$ is preferred to b^i, but is not feasible. Since f^ε is feasible and b^i is a best response within $\mathcal{M}^\varepsilon$, $\hat{b}$ generates strictly greater utility than f^ε.

payment made under f is weakly less than that made under $\overline{b}$ and weakly greater than that made under $\underline{b}$. This implies useful bounds with respect to these two functions.

The same logic applied in the first argument of this Lemma gives that the additional payment incurred by $\overline{b}$ versus b^i will not exceed $v^i(0;s)\varepsilon$; similarly, the savings of $\underline{b}$ versus b^i will not exceed $v^i(0;s)\varepsilon$. The gross utility obtained by $\overline{b}$ can be bounded by noticing that in against any profile of opponents' signals this bid function gives the agent no more than one ε-unit above what she would have obtained under b^i. Since the per-unit margin is bounded by $v^i(0;s)$, the additional gross utility does not exceed $v^i(0;s)\varepsilon$. A similar argument establishes that the gross utility sacrificed by $\underline{b}$ does not exceed $v^i(0;s)\varepsilon$.

Since for all q, $\underline{b}(q) \leq f(q) \leq \overline{b}(q)$, the utility generated by f may be bounded below by assuming that quantities arise from $\underline{b}$ while payments arise from $\overline{b}$, and above by assuming that quantities arise from $\overline{b}$ while payments arise from $\underline{b}$. Then the utility obtained under f must be such that

$$\left|U^i\left(b^i, b^{-i}; s\right) - U^i\left(f, b^{-i}; s\right)\right| \leq 2v^i\left(0; s\right)\varepsilon. \tag{5}$$

Now, since b^i is a best-response to b^{-i} in $\mathcal{M}^\varepsilon$, for any other available bid function $\tilde{b}$ in $\mathcal{M}^\varepsilon$ it must be the case that

$$U^i\left(\tilde{b}, b^{-i}; s\right) - U^i\left(b^i, b^{-i}; s\right) \leq 0.$$

Hence any unconstrained monotonic bid function $\hat{b}$ is such that

$$\left|U^i\left(\hat{b}, b^{-i}; s\right) - U^i\left(b^i, b^{-i}; s\right)\right| \leq \left[v^i\left(0; s\right) + Q\right]\varepsilon. \tag{6}$$

When f is an unconstrained monotonic bid function which meets b^i at the ε-units, and the unconstrained monotonic bid function $\hat{b}$ generates strictly greater utility than b^i, equations (5) and (6) can be summed to obtain

$$\left|U^i\left(f, b^{-i}; s\right) - U^i\left(\hat{b}, b^{-i}; s\right)\right| \leq \left[3v^i\left(0; s\right) + Q\right]\varepsilon.$$

□

D Proof of Theorem 3: Strategic ironing (general case)

Demonstrating the necessity of strategic ironing relies on a set of auxiliary Lemmas which establish that equilibrium strategies must satisfy certain technical properties. In this Section I show the necessity of strategic ironing in the case with $n \geq 3$ bidders; for technical reasons the case of $n = 2$ bidders is proved in Section E.

Lemma 21 (Equal upper bids)**.** *There must be at least two bidders i, j, $i \neq j$ such that* $\lim_{q \searrow 0} b^i(q;1) = \lim_{q \searrow 0} b^j(q;1) \geq \lim_{q \searrow 0} b^k(q;1)$ *for all* k.

Proof. This result follows from standard auction logic. Suppose that this is not the case; then there is an agent k and a $\varepsilon > 0$ such that

$$\lim_{q \searrow 0} b^k(q;1) = \max_\ell \left\{ \lim_{q \searrow 0} b^\ell(q;1) \right\} + \varepsilon.$$

Then there is a $\delta > 0$ such that for all $q \in [0, \delta)$ it is the case that $b^k(q;1) > b^k(0;1) - \varepsilon$. By reducing the bid for all such units by no more than ε, the agent reduces her payment in all outcomes without affecting the quantity received; hence she should do so.

□

Bidders of the sort described by Lemma 21 are the cornerstone of subsequent analysis. In particular, I focus on bidders who submit bids which will tie the highest-submitted bid, and who obtain a strictly positive quantity no matter the signals of their opponents.

Definition 10. *Agent i is a* maximal bidder *if* $\underline{q}^i(1) > 0$.

An immediate implication of Definition 10 is that if i is maximal, then $b^i(0;1) = \max_j b^j(0;1)$.

Henceforth I will assume that all bidders are maximal; for the purposes of the argument here, this assumption is without loss. Since only the behavior of maximal agents in a neighborhood of the quantity $\check{q}^i(1)$ is used in analysis, any bidder who is non-maximal is of no consequence. Should this assumption not be made, in what follows the population of n bidders would need to be replaced with the m maximal bidders; otherwise the results remain unchanged.

Lemma 22 (Zero probability of maximal ties)**.** *For all i, it must be that* $\Pr(q_i \in [\underline{q}^i(1), \check{q}^i(1))) = 0$.

Proof. Suppose that $\Pr(q_i \in [\underline{q}^i(1), \check{q}^i(1))) > 0$. Then $\mathbb{E}_{s_{-i}}[q_i | q_i < \check{q}^i(1), b^i(\cdot;1)] < \check{q}^i(1)$. Given $\varepsilon > 0$, define deviation $\hat{b}^\varepsilon$ by

$$\hat{b}^\varepsilon(q) = \begin{cases} \min\left\{b^i(q;1) + \varepsilon, v^i(q;1)\right\} & \text{if } q \leq \check{q}^i(1), \\ b^i(q;1) & \text{otherwise.} \end{cases}$$

This deviation requires extra payment for all obtained units, no greater than $\varepsilon q^i(1)$. It also yields greater expected utility by shifting expected quantity in this range from $\mathbb{E}_{s_{-i}}[q_i | q_i < \check{q}^i(1), b^i(\cdot;1)]$ to $\check{q}^i(1)$. This difference is strictly positive and independent of ε, and by Lipschitz bicontinuity of value functions the margin per unit is also strictly positive. The gain from this deviation is therefore bounded away from zero while the cost goes to zero, hence the deviation is profitable. □

Lemma 23 (Nondegenerate maximal bids)**.** *For any maximal bidder i, it must be that* $b^i(0;1) > 0$.

Proof. Suppose otherwise. Since bids are monotonic in value, all bidders must submit $b^j(\cdot;\cdot) = 0$. Then for any bidder j and for almost all s_j, the bidder receives quantities $q_j \in [\underline{q}^j(s_j), \check{q}^j(s_j)]$ with positive probability. For any such agent there is some $\varepsilon > 0$ such that bidding $\hat{b}^j(q) = \min\{v^j(q;s_j), \varepsilon\}$ yields strictly higher expected utility. □

With Lemma 22 I verify the intuitive result that bidders submitting the highest (ex ante) bid functions cannot be subject to ties for low quantities. This might lead appear to imply that these agents are never rationed; strategic ironing claims only that these agents are *almost never* rationed. Rationing will still arise when, for example, quantities are discontinuous in opponents' signals. I now address this point by contradiction: henceforth unless specified otherwise, assume that $\underline{q}^i(1) = \check{q}^i(1)$.

Lemma 24. *At least one maximal agent's bid function must be continuous at $\check{q}^i(1)$.*

Proof. Note that if $b^i(\cdot;1)$ is discontinuous at $q = \check{q}^i(1)$, then $G^i(\check{q}^i(1); b^i(\cdot;1)) > 0$—otherwise the agent should shade her bid slightly on the initial flat. Two cases arise.

Suppose first that $b^i(\check{q}^i(1);1) = v^i(\check{q}^i(1);1)$. Let $b^r = \lim_{q \searrow \check{q}^i(1)} b^i(q;1) < b^i(\check{q}^i(1);1)$, and let $\varepsilon > 0$. Define a deviation $\hat{b}^\varepsilon$ by

$$\hat{b}^\varepsilon(q) = \begin{cases} b^r & \text{if } q \in \left(\check{q}^i(1) - \varepsilon, \check{q}^i(1)\right], \\ b^i(q;1) & \text{otherwise.} \end{cases}$$

Note that this deviation saves the bidder payment of at least $(b^i(\check{q}^i(1);1) - b^r)(1 - G^i(\check{q}^i(1); b^i(\cdot;1)))\varepsilon = O(\varepsilon)$. The utility sacrificed is at most $M_v\varepsilon^2 G^i(\check{q}^i(1); b^i(\cdot;1)) = O(\varepsilon^2)$.[45] It follows that for ε sufficiently close to zero, deviation is profitable.

Now suppose that $b^i(\check{q}^i(1);1) < v^i(\check{q}^i(1);1)$, and that all agents $j \neq i$ have discontinuities in $b^j(\cdot;1)$ at $q = \check{q}^j(1)$. By deviating downward by small $\varepsilon > 0$ the agent will sacrifice quantity with zero probability—otherwise she could profit by increasing her bid to the right of $\check{q}^i(1)$—while saving $\varepsilon\check{q}^i(1)$. Deviation is therefore profitable. □

I now constrain attention to maximal bidders i such that $b^i(\cdot;1)$ is continuous at $q = \check{q}^i(1)$. By Lemma 24 there is at least one such bidder. When referring to agents $j \neq i$ I will continue to include agents whose bid functions are discontinuous at the ends of their respective initial flats. In all cases, undecorated indexed bid functions b^i represent putative best responses played in equilibrium.

Let Δ^b be the gap between the maximum bid $b(\check{q}^i(1);1)$ and a quantity slightly to the right of $\check{q}^i(1)$,

$$\Delta^b(\delta) = b\left(\check{q}^i(1);1\right) - b\left(\check{q}^i(1) + \delta;1\right).$$

[45] There is no concern here that deviating downward will discontinuously affect agent i's quantity: $\underline{q}^i(1) = \check{q}^i(1)$, and $\underline{q}^i(1) > 0$; hence if ties are broken pro-rata on the margin, $\underline{q}^j(1) = \check{q}^j(1)$ for all other maximal bidders j with $\underline{q}^j(1) > 0$.

Lemma 25. *For any maximal bidder i, it must be that*

$$\lim_{\delta \searrow 0} \frac{\Delta^{b^i}(\delta)}{G^i(\check{q}^i(1)+\delta; b^i)} = 0.$$

Proof. Consider a deviation b^δ defined by

$$b^\delta(q) = \begin{cases} b^i\left(\check{q}^i(1)+\delta;1\right) & \text{if } q \leq \check{q}^i(1)+\delta, \\ b^i(q;1) & \text{otherwise.} \end{cases}$$

By reducing her submitted bid, the agent saves costs on all units won, but sacrifices utility by also reducing the number of units she wins. A lower bound on cost savings from this deviation is $\check{q}^i(1)\Delta^{b^i}(\delta)(1-G^i(\check{q}^i(1);b^i))$: she saves $\check{q}^i(1)\Delta^{b^i}(\delta)$ whenever she wins quantity $q \geq \check{q}^i(1)+\delta$.

She will lose utility with probability $G^i(\check{q}^i(1)+\delta;b^i)$, and the per-unit margin will be bounded above by $\gamma^i(\delta) = v^i(0;1) - b^i(0;1) + \Delta^{b^i}(\delta)$. The number of units sacrificed will be bounded by the number of units lost when all other agents are maximal, plus δ; denoting the quantity lost to maximal opponents by $\Delta^{q^i}(\delta)$, the absolute bound on the quantity lost when facing any profile of opponents' signals is $\Delta^{q^i}(\delta)+\delta$. Since $\underline{q}^i(1) = \check{q}^i(1)$ by assumption, it is the case that $\Delta^{q^i}(\delta)+\delta \to 0$ when $\delta \searrow 0$. Putting these terms together, the utility loss is bounded above by $\gamma^i(\delta)(\Delta^{q^i}(\delta)+\delta)G^i(\check{q}^i(1)+\delta;b^i)$.

Incentive compatibility implies

$$\check{q}^i(1)\,\Delta^{b^i}(\delta)\left(1-G^i\left(\check{q}^i(1)+\delta;b^i\right)\right) \leq \gamma^i(\delta)\left(\Delta^{q^i}(\delta)+\delta\right)G^i\left(\check{q}^i(1)+\delta;b^i\right).$$

This may be rearranged as

$$\frac{\Delta^{b^i}(\delta)}{G^i(\check{q}^i(1)+\delta;b^i)} \leq \frac{\gamma^i(\delta)\left(\Delta^{q^i}(\delta)+\delta\right)}{\check{q}^i(1)\left(1-G^i(\check{q}^i(1)+\delta;b^i)\right)}.$$

As $\delta \searrow 0$, the right-hand side approaches 0. Since the left-hand side is weakly positive, it must then be that

$$\lim_{\delta \searrow 0} \frac{\Delta^{b^i}(\delta)}{G^i(\check{q}^i(1)+\delta; b^i)} = 0.$$

□

Lemma 26. *If i is a maximal bidder, then*

(i) $v^i(\check{q}^i(1);1) > b^i(\check{q}^i(1);1)$ *implies*

$$\limsup_{\delta \searrow 0} \frac{1}{\delta} G^i\left(\check{q}^i(1)+\delta;b^i\right) = 0, \text{ and}$$

(ii) $v^i(\check{q}^i(1);1) = b^i(\check{q}^i(1);1)$ *implies*

$$\liminf_{\delta \searrow 0} \frac{1}{\delta} G^i\left(\check{q}^i(1) + \delta; b^i\right) = +\infty.$$

Proof. To demonstrate point (i), let $\delta > 0$ and consider a deviation b^δ defined by

$$b^\delta(q) = \begin{cases} b^i(0;1) & \text{if } q \le \check{q}^i(1) + \delta, \\ b^i(q;1) & \text{otherwise.} \end{cases}$$

This deviation introduces extra costs, bounded by $\delta \Delta^{b^i}(\delta)$, with probability 1. With probability $G^i(\check{q}^i(1)+\delta;b^i)$ some quantity is gained, at maximum margin $v^i(\check{q}^i(1)+\delta;1) - b^i(\check{q}^i(1);1)$. Letting $\gamma = v^i(\check{q}^i(1);1) - b^i(\check{q}^i(1);1) > 0$, for sufficiently small δ the per-unit margin is at least $\gamma/2$. Lastly, the number of units gained will be written as

$$\Delta^{q^i}(\delta) = \delta - \mathbb{E}_{s_{-i}}\left[q^i - \check{q}^i(1) \,|\, q^i \le \check{q}^i(1) + \delta\right].$$

Incentive compatibility requires

$$\delta \Delta^{b^i}(\delta) \ge \left(\frac{\gamma}{2}\right) \Delta^{q^i}(\delta)\, G^i\left(\check{q}^i(1) + \delta; b^i\right).$$

This may be rearranged as

$$\frac{\Delta^{b^i}(\delta)}{G^i\left(\check{q}^i(1) + \delta; b^i\right)} \ge \left(\frac{\gamma}{2}\right) \frac{\Delta^{q^i}(\delta)}{\delta}.$$

Lemma 25 implies that the right-hand side goes to zero as δ becomes small; since the left-hand side is positive, it follows that

$$\lim_{\delta \searrow 0} \frac{1}{\delta} \Delta^{q^i}(\delta) = 0 \quad \Longrightarrow \quad \lim_{\delta \searrow 0} \frac{1}{\delta} \mathbb{E}_{s_{-i}}\left[q^i - \check{q}^i(1) \,|\, q^i \le \check{q}^i(1) + \delta\right] = 1.$$

This integral can be bounded in a natural way using a two-rectangle approximation, hence the subsequent of the above expression requires

$$\lim_{\delta \searrow 0} \frac{G^i\left(\check{q}^i(1) + \delta/2; b^i\right)}{G^i\left(\check{q}^i(1) + \delta; b^i\right)} = 0.$$

That is, for all $\kappa > 0$ there is $\varepsilon_\kappa > 0$ such that for all $0 < \delta < \varepsilon_\kappa$,

$$G^i\left(\check{q}^i(1) + \delta; b^i\right) \ge \kappa G^i\left(\check{q}^i(1) + \delta/2; b^i\right).$$

Now suppose that $\limsup_{\delta \searrow 0} G^i(\check{q}^i(1) + \delta; b^i)/\delta = \xi > 0$. Then for all $\lambda > 0$, there is ε_λ such that there are an infinite number of points $0 < \delta_k < \varepsilon_\lambda$ with $G^i(\check{q}^i(1) + \delta_k; b^i)/\delta_k > \xi - \lambda$.

Let $\lambda = \gamma/2$, $\kappa = 4$, and $\varepsilon = \min\{\varepsilon_\lambda, \varepsilon_\kappa\}$. Then there are an infinite number of δ_k with $0 < \delta_k < \varepsilon$ and $G^i(\check{q}^i(1) + \delta_k; b^i)/\delta_k > \xi/2$; but at any such δ_k it is also the case that $G^i(\check{q}^i(1) + 2\delta_k; b^i/(2\delta_k) > 2\xi$. For any $\varepsilon' < 2\varepsilon$, there must be an infinite number of such $2\delta^k$, hence $\limsup_{\delta \searrow 0} G^i(\check{q}^i(1)+\delta; b^i)/\delta \geq 2\xi$, a contradiction. Thus it must be that either this limit is zero or infinite. The latter can be ruled out by the fact that $G^i(\cdot; b^i) \in [0,1]$, hence the result is shown.

Point (ii) follows from the fact that bids are constrained above by values, and values are Lipschitz continuous. Hence $\Delta^{b^i}(\delta) \geq \delta/M_v$. Then from Lemma 25 it follows that

$$\lim_{\delta \searrow 0} \frac{\delta/M_v}{G^i\left(\check{q}^i\left(1\right) + \delta; b^i\right)} = 0.$$

This directly implies that

$$\lim_{\delta \searrow 0} \frac{1}{\delta} G^i\left(\check{q}^i\left(1\right) + \delta; b^i\right) = +\infty.$$

If G^i is continuous at $\check{q}^i(1)$, the derivative of G^i with respect to q is well-defined at $\check{q}^i(1)$, and $G^i_q(\check{q}^i(1); b^i) = +\infty$. Otherwise, there must be a mass point. □

Corollary 5. *If i is a maximal bidder, then $\lim_{\delta \searrow 0} G^i(\check{q}^i(1)+\delta; b^i)/\delta = G^i_q(\check{q}^i(1); b^i)$ is well-defined.*

Lemma 27. *Suppose that agent i is such that $b^i(\check{q}^i; 1) < v^i(\check{q}^i; 1)$. Then for all other maximal agents $k \neq i$,*

$$\lim_{q \searrow \check{q}^k(1)} G^k_b\left(q; b^i\right) = -\infty.$$

Proof. Consider agent i's distribution of quantity. Choosing some $k \neq i$, this function can be written as

$$G^i\left(q; b^i\right) = \underbrace{\int_0^1 \cdots \int_0^1}_{n-2} \int_{\psi^k\left(Q - q - \sum_{j \neq i,k} \varphi^j\left(b^i(q;1); s_j\right); b^i(q;1)\right)}^1 dF\left(s_k\right) \prod_{j \neq i,k} dF\left(s_j\right).$$

Since $s_j \sim \mathcal{U}(0,1)$ for all j, this can in turn be written as

$$G^i\left(q; b^i\right) = 1 - \underbrace{\int_0^1 \cdots \int_0^1}_{n-2} \psi^k\left(Q - q - \sum_{j \neq i,k} \varphi^j\left(b^i\left(q;1\right); s_j\right); b^i\left(q;1\right)\right) ds_{-i}.$$

Having constrained attention to bounded monotonic functions, all functions of interest are differentiable almost everywhere. Therefore examine the derivative of

G^i with respect to b where applicable, and take limits as needed. Where it exists, this derivative is

$$G_b^i\left(q;b^i\right) = \underbrace{\int_0^1 \cdots \int_0^1}_{n-2} \psi_q^k \left(Q - q - \sum_{j\neq i,k} \varphi^j\left(b^i\left(q;1\right);s_j\right);b^i\left(q;1\right)\right) \sum_{j\neq i,k} \varphi_b^j\left(b^i\left(q;1\right);s_j\right) - \psi_b^k\left(Q - q - \sum_{j\neq i,k} \varphi^j\left(b^i\left(q;1\right);s_j\right);b^i\left(q;1\right)\right) ds_{-ik}.$$

Substituting in for G_q^i and dividing by b_q^i gives

$$\frac{G_q^i\left(q;b^i\right)}{b_q^i\left(q;1\right)} = G_b^i\left(q;b^i\right) + \underbrace{\int_0^1 \cdots \int_0^1}_{n-2} \psi_q^k\left(Q - \sum_{j\neq k} \varphi^j\left(b^i\left(q;1\right);s_j\right);b^i\left(q;1\right)\right) \varphi_b^i\left(b^i\left(q;1\right);1\right) ds_{-ik}.$$

From Lemma 25, it must be that

$$\lim_{\delta\searrow 0} G_b^i\left(\check{q}^i\left(1\right)+\delta;b^i\right) + \underbrace{\int_0^1 \cdots \int_0^1}_{n-2} \psi_q^k\left(Q - \sum_{j\neq k} \varphi^j\left(b^i\left(\check{q}^i\left(1\right)+\delta;1\right);s_j\right);b^i\left(\check{q}^i\left(1\right)+\delta;1\right)\right) \times \varphi_q^i\left(b^i\left(\check{q}^i\left(1\right)+\delta;1\right);1\right) ds_{-ik} = -\infty.$$

Suppose that $\lim_{\delta\searrow 0} G_b^i(\check{q}^i(1)+\delta;b^i) \neq -\infty$. Then it must be that for all maximal $k \neq i$,[46]

$$\lim_{\delta\searrow 0} \underbrace{\int_0^1 \cdots \int_0^1}_{n-2} \psi_q^k \varphi_p^i ds_{-i,k} = -\infty.$$

[46]The two-maximal-agent case must be handled separately, but can be dealt with much more simply than this. See Section E.

Now, for all other maximal agents $k, j \neq i$, $k \neq j$, it is the case that

$$\begin{aligned} G_b^k\left(q; b^k\right) &= \underbrace{\int_0^1 \cdots \int_0^1}_{n-2} \psi_q^j \sum_{\ell \neq j,k} \varphi^\ell \left(b^k\left(q;1\right);1\right) - \psi_p^j \left(b^k\left(q;1\right);\cdot\right) ds_{-k,j} \\ &< \underbrace{\int_0^1 \cdots \int_0^1}_{n-2} \psi_q^j \sum_{\ell \neq j,k} \varphi^\ell \left(b^k\left(q;1\right);1\right) ds_{-k,j} \\ &< \underbrace{\int_0^1 \cdots \int_0^1}_{n-2} \psi_q^j \varphi^i \left(b^k\left(q;1\right);1\right) ds_{-k,j}. \end{aligned}$$

The last term has been shown to go to $-\infty$ as $\delta \searrow 0$, hence $\lim_{\delta \searrow 0} G_b^k(\check{q}^k(1)+\delta; b^k) = -\infty$ for all maximal agents $k \neq i$. □

Lemma 28. *If i is a maximal agent with $G^i(\check{q}^i(1); b^i) = 0$, then $v^i(\check{q}^i(1);1) > b^i(\check{q}^i(1);1)$.*

Proof. Suppose that $b^i(\check{q}^i(1);1) = v^i(\check{q}^i(1);1)$. Let $\varepsilon > 0$ and consider a deviation b^ε defined by

$$b^\varepsilon\left(q\right) = \begin{cases} b^i\left(q;1\right) & \text{if } q \leq \check{q}^i(1) - \varepsilon, \\ b^i\left(\check{q}^i\left(1\right);1\right) - \varepsilon & \text{if } q > \check{q}^i\left(1\right) - \varepsilon \text{ and } b^i\left(q;1\right) > b^i\left(\check{q}^i\left(1\right);1\right) - \varepsilon, \\ b^i\left(q;1\right) & \text{otherwise.} \end{cases}$$

This deviation saves the agent payment of at least ε^2, with probability 1. The costs of the deviation are associated with sacrificed quantities, which are easily bounded: under the deviation, the new minimum quantity $\check{q}^\varepsilon$ is such that $\check{q}^\varepsilon \geq \check{q}^i(1) - \varepsilon$, while the maximum quantity affected is bounded above by $\varphi^i(b^i(\check{q}^i(1);1) - \varepsilon;1) \leq \check{q}^i(1) + M_v\varepsilon$, since bids fall weakly below values, and $b^i(\check{q}^i(1);1) = v^i(\check{q}^i(1);1)$ by assumption. The total quantity lost is therefore bounded by $(1 + M_v)\varepsilon$. The per-unit margin lost to the left of $\check{q}^i(1)$ is bounded again by $M_v\varepsilon$, while on the right it is bounded by $M_v\varepsilon / b_q^i(\check{q}^i(1);1) \leq \varepsilon$. The utility loss is therefore bounded by $[(1 + M_v)\varepsilon]^2$, and this loss occurs with at most probability $G^i(\check{q}^i(1) + M_v\varepsilon; b^i)$.

Incentive compatibility requires that the expected benefits from this deviation be outweighed by the expected losses; that is,

$$\varepsilon^2 \leq \left[\left(1 + M_v\right)\varepsilon\right]^2 G^i\left(\check{q}^i\left(1\right) + M_v\varepsilon; b^i\right).$$

After cancelation, this is $G^i(\check{q}^i(1) + M_v\varepsilon; b^i) \geq 1/(1 + M_v)^2$ for all $\varepsilon > 0$. Since G^i is right-continuous and $G^i(\check{q}^i(1); b^i) = 0$, this inequality cannot hold for ε sufficiently small, and the deviation will be profitable. □

Lemma 29. *It cannot be that for all maximal agents i, $v^i(\check{q}^i(1);1) = b^i(\check{q}^i(1);1)$.*

Proof. Suppose otherwise. Consider the argument from Lemma 25, which required

$$\check{q}^i(1)\,\Delta^{b^i}(\delta)\left(1-G^i\left(\check{q}^i(1)+\delta;b^i\right)\right)\leq\gamma^i(\delta)\left(\Delta^{q^i}(\delta)+\delta\right)G^i\left(\check{q}^i(1)+\delta;b^i\right).$$

In the proof of Lemma 26, point (ii) it was demonstrated that $\Delta^{b^i}(\delta)\geq\delta/M_v$. Moreover, since all other maximal agents j have $v^j(\check{q}^j(1);1)=b^j(\check{q}^j(1);1)$, for small δ, $\Delta^{q^i}(\delta)\leq(n-1)M_v\Delta^{b^i}(\delta)$. These inequalities give

$$\begin{aligned}\check{q}^i(1)\left(1-G^i\left(\check{q}^i(1)+\delta;b^i\right)\right)&\leq\gamma^i(\delta)\left((n-1)\,M_v+\frac{\delta}{\Delta^{b^i}(\delta)}\right)G^i\left(\check{q}^i(1)+\delta;b^i\right)\\&\leq nM_v\gamma^i(\delta)\,G^i\left(\check{q}^i(1)+\delta;b^i\right).\end{aligned}$$

As $\delta\searrow 0$, the right-hand side goes to zero while the left-hand side goes to $\check{q}^i(1)>0$, contradicting the possibility that this equilibrium satisfies incentive compatibility. □

Lemma 30. *If there are m maximal agents, it cannot be that for $m-1$ maximal agents k, $v^k(\check{q}^k(1);1)=b^k(\check{q}^k(1);1)$.*

Proof. Lemma 29 establishes that it cannot be the case that all maximal agents k have $v^k(\check{q}^k(1);1)=b^k(\check{q}^k(1);1)$. Moreover, Lemma 27 argued that if there is a single maximal agent i with $v^i(\check{q}^i(1);1)>b^i(\check{q}^i(1);1)$, then the remaining $m-1$ maximal agents $k\neq i$ have $v^k(\check{q}^k(1);1)=b^k(\check{q}^k(1);1)$. From Lemma 28, this is only possible when all such agents have $G^k(\check{q}^k(1);1)=\pi_k>0$.

Because signals are independent, this is only possible if, given a particular j, all other maximal agents $k\neq j$ are such that $b^k(\check{q}^k(1);s_k)$ is constant for $s_k\in(1-\varepsilon_k,1]$ for some $\varepsilon_k>0$. Since this must be true for all agents $j\neq i$, it follows that all maximal agents k satisfy this requirement for some $\varepsilon_k>0$. By the same token, it must be that $G^i(\check{q}^i(1);b^i)\geq\prod_{j\neq i}\varepsilon_j>0$, contradicting Lemma 25, point (i). Thus the result is shown. □

Theorem 5. *If $\underline{q}^i(1)\in(0,Q)$, $b^i(\cdot;1)$ exhibits strategic ironing.*

Proof. Assuming that there is no ironing, $\underline{q}^i(1)=\check{q}^i(1)$, I have shown that: (Lemma 29) it cannot be that all maximal agents j have $b^j(\check{q}^j(1);1)=v^j(\check{q}^j(1);1)$; (Lemma 27) there can be at most one agent j with $b^j(\check{q}^j(1);1)<v^j(\check{q}^j(1);1)$; (Lemma 30) it cannot be that $n-1$ maximal agents have $b^j(\check{q}^j(1);1)=v^j(\check{q}^j(1);1)$. Since these results together contradict the existence of an equilibrium without ironing so long as the market is not cornered, any equilibrium must exhibit nontrivial ironing: $\underline{q}^i(1)>0$ implies $\check{q}^i(1)>\underline{q}^i(1)$. □

E Proof of Theorem 3: Strategic ironing (two-agent case)

As mentioned in the main text and the preceding Section, the two-agent case requires special care. Because the fundamental results are the same—even if the proofs differ—it is valid to lump this case in with the general n-agent case for exposition's sake. In particular, with two agents the probability expression $G^i(q; b^i)$ is not written as an integral, but is rather $G^i(q; b^i) = 1 - \psi^{-i}(Q - q; b^i(q))$. Because this is the only meaningful change, all results that do not involve the integral form of G^i continue to go through without modification. It is therefore only necessary to reestablish Lemma 27 in the two-agent case.

Lemma 31 (Unbounded marginal probability improvement (two agents))**.** *At least one agent has* $\lim_{q \searrow \check{q}^i(1)} G^i_b(q; b^i) = +\infty$.

Proof. With $G^i(q; b^i) = 1 - \psi^{-i}(Q - q; b^i(q))$, it is the case that

$$G^i_b\left(q; b^i\right) = -\psi^{-i}_p\left(Q - q; b^i\left(q\right)\right),$$
$$G^i_q\left(q; b^i\right) = \psi^{-i}_q\left(Q - q; b^i\left(q\right)\right) + G^i_b\left(q; b^i\right) b^i_q\left(q\right).$$

By implicit differentiation,

$$\psi^{-i}_p\left(Q - q; b^i\left(q\right)\right) = 1/b^{-i}_s\left(Q - q; \psi^{-i}\left(Q - q; b^i\left(q\right)\right)\right),$$
$$\psi^{-i}_q\left(Q - q; b^i\left(q\right)\right) = -b^{-i}_q\left(Q - q; \psi^{-i}\left(Q - q; b^i\left(q\right)\right)\right)\psi^{-i}_p\left(Q - q; b^i\left(q\right)\right).$$

It follows that at all q,

$$G^i_q\left(q; b^i\right) = \left(b^{-i}_q\left(Q - q; \psi^{-i}\left(Q - q; b^i\left(q\right)\right)\right) + b^i_q\left(q\right)\right) G^i_b\left(q; b^i\right).$$

Thus at all q,

$$\frac{G^i_q\left(q; b^i\right)}{b^i_q\left(q\right)} = \left(\frac{b^{-i}_q\left(Q - q; \psi^{-i}\left(Q - q; b^i\left(q\right)\right)\right)}{b^i_q\left(q\right)} + 1\right) G^i_b\left(q; b^i\right).$$

As $q \searrow \check{q}^i(s)$ the left-hand side goes to $+\infty$, and this holds for both agents. Thus either $b^{-i}_q/b^i_q \to +\infty$, or $G^i_b \to +\infty$. Note that if $b^{-i}_q/b^i_q \to +\infty$, then $b^{-j}_q/b^j_q \to 0$ for $j \neq i$, and thus $G^j_b \to +\infty$. It follows that at least one of $i \in \{1, 2\}$ has $\lim_{q \searrow \check{q}^i(s)} G^i_b(q; b^i) = +\infty$. □

The remainder of the necessity proof is identical to the generic n-agent case: it cannot be that all-but-one agent bids her true value at the end of the initial flat, and since in this case all-but-one is one, Lemma 31 implies that this must be the case when one agent's bid does not equal her value at the end of the initial flat. Since it also cannot be the case that both agents bid their true values at the end of the initial flat, any pure-strategy equilibrium must involve ironing, $\underline{q}^i(1) < \check{q}^i(1)$.

F Calculations for simulated models

Derivatives. When bids are strictly monotonic and $\check{q}^i(s) \geq Q/2$ the market price $p(s_1, s_2)$ will be determined by the bid placed along the flat of the agent with the lower of the two signals, $i \in \arg\min_j \{s_j\}$. Thus the market price equation is written as $p(s_1, s_2) = b^i(\check{q}^i(\underline{s}); \underline{s})$, where $\underline{s} = s_i = \min\{s_1, s_2\}$. I will therefore consider p as a function of only one variable, the lower of the two signals.

Now, let $s_1' > s_1 > s_2 = \underline{s}$. Since the market price is determined by s_2 alone, it must be that

$$b^1\left(q^1\left(s_1', s_2\right); s_1'\right) = b^1\left(q^1\left(s_1, s_2\right); s_1\right)$$

$$\begin{aligned} \Longrightarrow v^1\left(q^1\left(s_1', s_2\right); s_1'\right) + \frac{s_2}{G_b^1\left(q^1\left(s_1', s_2\right); b^1\left(\cdot; s_1'\right)\right)} \\ = v^1\left(q^1\left(s_1, s_2\right); s_1\right) + \frac{s_2}{G_b^1\left(q^1\left(s_1, s_2\right); b^1\left(\cdot; s_1\right)\right)}. \end{aligned}$$

In a strictly monotone equilibrium, $q^1(s, s_2) > Q/2$ when $s \in \{s_1, s_1'\}$. Then slightly increasing the bid for unit $q^1(s, s_2)$ will render this unit pivotal against some higher $s_2' > s_2$, but this alternative opponent will also be bidding along her initial flat. Since the opponent's bid is, by construction, still flat along this region, the change in win probability is the same for either of $s \in \{s_1, s_1'\}$. It follows that $G_b^1(q(s, s_2); b^1(\cdot; s))$ is constant for $s > s_2$. The above equation then becomes

$$s_1'\alpha_s - q^1\left(s_1', s_2\right)\alpha_q = s_1\alpha_s - q^1\left(s_1, s_2\right)\alpha_q.$$

From this, it follows that $q_{s_1}^1(s, s_2) = \alpha_s/\alpha_q$ whenever $s > s_2$.

By virtue of both paying the same price, it must also be that

$$\frac{\partial}{\partial s}\left[b^1\left(q^1\left(s, s_2\right); s\right)\right] = 0 = b_q^1 q_{s_1}^1 + b_s^1.$$

It follows that $b_s^1(q; s) = -(\alpha_s/\alpha_q) b_q^1(q; s)$ when $s > s_2$ and $q = q^1(s, s_2)$.

With respect to the lower of the two signals, it must also be that

$$\frac{\partial}{\partial s}\left[b^1\left(q^1\left(s_1, s\right); s_1\right)\right] = p_s\left(s\right) = b_q^1\left(q^1\left(s_1, s_2\right); s_1\right) q_{s_2}^1\left(s_1, s_2\right).$$

Intervals. Let $i \in \arg\min_j s_j$ as before. It is helpful to define two quantities, $\underline{q}(s)$ and $q^\ell(s)$ as the minimum possible quantity and the maximum rationed quantity along the flat, respectively. In particular,

$$\underline{q}\left(s\right) = q^i\left(s, 1_{-i}\right) \quad \text{and} \quad q^\ell\left(s\right) = \lim_{s \searrow s_i} q^i\left(s_i, s\right).$$

Since $q_{s_{-i}}^{-i}(s_i, s_{-i}) = \alpha_s/\alpha_q$ when $s_{-i} > s_i$, it follows that

$$\underline{q}\left(s_i\right) = q^\ell\left(s_i\right) - \frac{\alpha_s}{\alpha_q}\left(1 - s_i\right).$$

When bids are strictly monotonic and bids are continuous in signal, and so is $\check{q}(\cdot)$, it will be the case that

$$q^{\ell}(s_i) = Q - \check{q}(s_i).$$

That is, when s_i is only slightly smaller than s_{-i}, bidder i loses the difference between $\check{q}(s_i)$ and $Q/2$—as she would when opposing $s_{-i} = s_i$—and then this difference once more.

Ironing. The first-order condition with respect to the right endpoint of the initial flat is

$$b_q^i(\check{q}; s) \int_0^{\check{q}} \left(1 - G^i\left(q; b^i\right)\right) dq = -b_q^i(\check{q}; s) \int_0^{\check{q}} \left(v^i(q; s) - b^i(\check{q}; s)\right) G_b^i\left(q; b^i\right) dq.$$

The b_q^i terms cancel. This leaves two sides of the equation,

$$\begin{aligned} \text{LHS}(\check{q}; s) &\equiv \int_0^{\check{q}} \left(1 - G^i\left(q; b^i\right)\right) dq, \\ \text{RHS}(\check{q}; s) &\equiv \int_0^{\check{q}} \left(v^i(q; s) - b^i(\check{q}; s)\right) G_b^i\left(q; b^i\right) dq. \end{aligned}$$

In what follows I will suppress arguments from functions which are determined by a single signal; this is done for space efficiency.

Left-hand side. I analyze $\text{LHS}(\check{q}; s)$ by splitting the integral into parts,

$$\text{LHS}(\check{q}; s) = \check{q} - \left[\int_0^{\underline{q}} G^i\left(q; b^i\right) dq + \int_{\underline{q}}^{q^{\ell}} G^i\left(q; b^i\right) dq + \int_{q^{\ell}}^{\check{q}} G^i\left(q; b^i\right) dq\right].$$

When $q \in [0, \underline{q})$, $G^i(q; b^i) = 0$: there is no probability that the agent receives a quantity in this range. When $q \in [q^{\ell}, \check{q}]$, $G^i(q; b^i) = 1 - s$: there is no probability of being allocated in this interval (except for the probability-zero event $s_{-i} = s_i$). Then the only interval of interest is $(\underline{q}, q^{\ell})$. Since equilibrium quantities are linear in signal equilibrium probabilities must be linear in quantity, therefore

$$\int_{\underline{q}}^{q^{\ell}} G^i\left(q; b^i\right) dq = \frac{1}{2}(1+s)\left(q^{\ell} - \underline{q}\right) = \frac{1}{2}(1+s)\frac{\alpha_s}{\alpha_q}(1-s).$$

Putting all these pieces together gives

$$\begin{aligned} \text{LHS}(\check{q}; s) &= \check{q} - \left[\frac{1}{2}(1-s)\frac{\alpha_s}{\alpha_q}(1-s) + \left(\check{q} - q^{\ell}\right)(1-s)\right] \\ &= \check{q} - \left[\frac{1}{2}\left(\frac{\alpha_s}{\alpha_q}\right)(1-s) + 2\check{q} - Q\right](1-s) \\ &= \left[Q - \frac{1}{2}\left(\frac{\alpha_s}{\alpha_q}\right)(1-s)\right](1-s) + (2s-1)\check{q}. \end{aligned}$$

Right-hand side. As with the analysis of LHS$(\check{q}; s)$, I analyze RHS$(\check{q}; s)$ by splitting the integral into parts,

$$\begin{aligned}\text{RHS}\left(\check{q}; s\right) = &\int_0^{\underline{q}} \left(v^i\left(x; s\right) - b^i\left(\check{q}; s\right)\right) G_b^i\left(q; b^i\left(\cdot; s\right)\right) dq \\ &+ \int_{\underline{q}}^{q^\ell} \left(v^i\left(x; s\right) - b^i\left(\check{q}; s\right)\right) G_b^i\left(q; b^i\left(\cdot; s\right)\right) dq \\ &+ \int_{q^\ell}^{\check{q}} \left(v^i\left(x; s\right) - b^i\left(\check{q}; s\right)\right) G_b^i\left(q; b^i\left(\cdot; s\right)\right) dq.\end{aligned}$$

It will suffice to understand the behavior of G_b^i on each of these ranges. The analysis of each interval begins with the definitional equality

$$G^i\left(q; b\right) = 1 - \psi^{-i}\left(Q - q; b\right) \quad \Longrightarrow \quad G_b^i\left(q; b\right) = -\psi_p^{-i}\left(Q - q; b\right).$$

First, let $q \in (0, \underline{q})$. Then $G^i(q; b^i(\cdot; s)) = 0$. Since bids are symmetric, $G^i(q; b^i(\check{q}; s) + \varepsilon) = 0$ for all $\varepsilon > 0$, and for any such ε there exists $\delta > 0$ such that $G^i(q'; b^i(\check{q}) - \varepsilon) = 0$ for all $q' < q - \delta$. Since bids are continuous, as $\varepsilon \to 0$ it can be assumed that $\delta \to 0$. It follows that an order-ε change in b does not change G^i on this range, except possibly on a set with measure going to zero with ε. Thus $G_b^i(q; b^i(\check{q}; s)) = 0$ on this range.

Second, let $q \in (\underline{q}, q^\ell)$. On this range, the agent is competing against opponents who are in the strictly-decreasing portion of their bid functions. It follows that $-\psi_p^{-i}(Q - q; b) = -1/b_s^{-i}(Q - q; s_{-i})$. Earlier analysis showed $p_s(s) = b_q(\check{q}(s); s)\check{q}_s(s) + b_s(\check{q}(s); s)$ and $b_q = -\alpha_q b_s / \alpha_s$. Additionally, $b_q(\check{q}(s); s) = b_q(q(s'; s); s')$ for all $s' > s$. It follows that

$$p_s\left(s\right) = \left(1 - \frac{\alpha_q}{\alpha_s}\check{q}_s\right) b_s.$$

Hence

$$G_b^i\left(q; b^i\left(\cdot; s\right)\right) = -\frac{1 - \frac{\alpha_q}{\alpha_s}\check{q}_s}{p_s}.$$

Lastly, let $q \in (q^\ell, \check{q})$. Then following any infinitesimal change in bid these quantities are still won "on the gap," against the agents whose flats the deviation now beats. In this case,

$$G^i\left(q; b^i\left(\cdot; s\right)\right) = \Pr\left(b^{-i}\left(Q - q; s_{-i}\right) \geq b^i\left(q; s\right)\right).$$

On the gap, $b^{-i}(Q - q; s_{-i}) = p(s_{-i})$. Then $G^i(q; b^i(\cdot; s)) = 1 - p^{-1}(b)$. It follows that $G_b^i(q; b^i(\cdot; s)) = -1/p_s(s)$.

Recognizing that $b^i(\check{q}; s) = p(s)$ in any solution to this system, RHS may be rewritten as

$$\begin{aligned}\mathrm{RHS}\left(\check{q}; s\right) = &-\int_{\underline{q}}^{q^\ell} \left(v^i\left(x; s\right) - p\left(s\right)\right)\left(\frac{\alpha_s - \alpha_q \check{q}_s\left(s\right)}{\alpha_s p_s\left(s\right)}\right) dq \\ &-\int_{q^\ell}^{\check{q}} \left(v^i\left(x; s\right) - p\left(s\right)\right)\left(\frac{1}{p_s\left(s\right)}\right) dq.\end{aligned}$$

Since v^i is linear in q and none of the p terms depend on q, these integrals become

$$\begin{aligned}\mathrm{RHS}\left(\check{q}; s\right) = &-\left(\frac{\alpha_s - \alpha_q \check{q}_s\left(s\right)}{\alpha_s p_s\left(s\right)}\right)\frac{\alpha_s}{\alpha_q}\left(1-s\right) \\ &\times\left(\alpha_0 + s\alpha_s - \left(Q - \check{q}\left(s\right)\right)\alpha_q + \frac{1}{2}\left(\frac{\alpha_s}{\alpha_q}\right)\left(1-s\right)\alpha_q - p\left(s\right)\right) \\ &-\left(\frac{1}{p_s\left(s\right)}\right)\left(2\check{q}\left(s\right) - Q\right)\left(\alpha_0 + s\alpha_s - \frac{Q}{2}\alpha_q - p\left(s\right)\right).\end{aligned}$$

Differential equation. By construction, equilibrium requires $\mathrm{LHS}(\check{q}; s) = -\mathrm{RHS}(\check{q}; s)$. Omitting function arguments, this is

$$\begin{aligned}&\left(\left[Q - \frac{1}{2}\left(\frac{\alpha_s}{\alpha_q}\right)\left(1-s\right)\right]\left(1-s\right) + \left(2s-1\right)\check{q}\right)p_s \\ &= \left(\frac{\alpha_s}{\alpha_q} - \check{q}_s\right)\left(\alpha_0 + \frac{1}{2}\left(1+s\right)\alpha_s - \left(Q - \check{q}\right)\alpha_q - p\right)\left(1-s\right) \\ &\quad + \left(2\check{q} - Q\right)\left(\alpha_0 + s\alpha_s - \frac{Q}{2}\alpha_q - p\right). \qquad (7)\end{aligned}$$

To pin down the second dimension of this two-dimensional differential equation, recall that

$$b^i\left(q; s_i\right) = v^i\left(q; s_i\right) - s_{-i} p_s\left(s_{-i}\right).$$

At $\check{q}$, this is

$$p\left(s\right) = v^i\left(\check{q}\left(s\right); s\right) - s p_s\left(s\right).$$

Thus there is a second equation,

$$p = \alpha_0 + s\alpha_s - \check{q}\alpha_q - s p_s. \qquad (8)$$

Equations (7) and (8) completely define a differential system which may be solved computationally. While equation (8) may be easily solved in an integral form—$p = \int_0^s v^i(\check{q}; x)dx/s$—and substituted into equation (7) to obtain a second-order ordinary differential equation, the unavoidable presence of product terms makes explicit analysis intractible no matter the simplification applied.