The prodution possibility frontier

Discussion and derviation

A firm is described by its production technologies, methods for transforming input factors (such as labor and capital) into output commodities (such as coffee, computers, or linen). If the firm is operating efficiently, it is intuitive that it will not use any more of its input factors than it must, since ostensibly it has to pay for the use of labor and capital and overuse will cut profits.

Suppose, however, that the supply of labor and capital in the economy is fixed; the notion of "efficient production" is then not so intuitively defined, since what is efficiency when the factors to be used are fixed? However, the nature of production in an economy with more than one commodity is such that there is a tradeoff between production of various goods: resources may be allocated to production of any good, so it makes sense that this allocation of resources will maximize production *in some sense.*

Fix the labor supply at L and the capital stock at K, and let x and y be the goods to be produced. We are given production technologies

$$x = F\left(K_x, L_x\right), \qquad y = H\left(K_y, L_y\right)$$

Factor market clearing tells us that labor used in production must equal total labor supply — $L_x + L_y = L$ — and capital used in production must equal total capital stock — $K_x + K_y = K$. The question we now pose is this: *given* a level of x we would like to produce, what is the *maximum* amount of y which can be produced? Notice that this is not maximization of production in any proper sense, but rather the maximization of the production of one good subject to the production level of another; this is not altogether a ridiculous question since, in general, there are many combinations of K_x, L_x which will obtain the same overall level of x in production.

In lecture and in the text, a method which seems much different for obtaining the production possibility frontier appears. We'll now show that they are equivalent to this more intuitive notion of conditional maximization. To begin, we setup a constrained optimization problem,

$$\begin{aligned} \max_{K_y, L_y} \; & H\left(K_y, L_y\right) \\ \text{s.t. } & x = F\left(K_x, L_x\right) \\ & K = K_x + K_y \\ & L = L_x + L_y \end{aligned}$$

From this, we obtain a Lagrangian

$$\mathscr{L} = H\left(K_y, L_y\right) + \lambda\left(F\left(K_x, L_x\right) - x\right) + \mu_K\left(K_x + K_y - K\right) + \mu_L\left(L_x + L_y - L\right)$$

First-order conditions give us

$$\begin{aligned} \frac{\partial \mathscr{L}}{\partial K_x} &: \quad 0 = \lambda F_K\left(K_x, L_x\right) + \mu_K \\ \frac{\partial \mathscr{L}}{\partial L_x} &: \quad 0 = \lambda F_L\left(K_x, L_x\right) + \mu_L \\ \frac{\partial \mathscr{L}}{\partial K_y} &: \quad 0 = H_K\left(K_y, L_y\right) + \mu_K \\ \frac{\partial \mathscr{L}}{\partial L_y} &: \quad 0 = H_L\left(K_y, L_y\right) + \mu_L \end{aligned}$$

Substituting out the Lagrange multipliers μ_K, μ_L we find

$$\begin{aligned} 0 &= H_K\left(K_y, L_y\right) - \lambda F_K\left(K_x, L_x\right) \\ 0 &= H_L\left(K_y, L_y\right) - \lambda F_L\left(K_y, L_y\right) \end{aligned}$$

which is the same as

$$H_K(K_y, L_y) = \lambda F_K(K_x, L_x)$$
$$H_L(K_y, L_y) = \lambda F_L(K_y, L_y)$$

Dividing the second equation by the first, we get

$$\frac{H_L(K_y, L_y)}{H_K(K_y, L_y)} = \frac{F_L(K_x, L_x)}{F_K(K_x, L_x)}$$

By the factor market clearing conditions, this becomes

$$\frac{H_L(K - K_x, L - L_x)}{H_K(K - K_x, L - L_x)} = \frac{F_L(K_x, L_x)}{F_K(K_x, L_x)}$$

By definition, H_L is MPL_y and H_K is MPK_y (and similarly for F and x); recall that marginal products are given by the derivative of production with respect to a particular input factor. Then the above equation is

$$\frac{\text{MPL}_y}{\text{MPK}_y} = \frac{\text{MPL}_x}{\text{MPK}_x}$$

The marginal rate of technical substitution for some good i is $\text{RTS}_i = -\frac{\text{MPL}_i}{\text{MPK}_i}$. Then the above equation is equivalent to

$$\text{RTS}_y = \text{RTS}_x$$

This equality is evaluated subject to factor market clearing; this is precisely the formula given in the book and discussed in lecture!

An example

Consider a firm which has access to $K = 3$ units of capital and $L = 2$ units of labor. It owns the production technologies

$$x = K_x^{\frac{1}{2}} L_x^{\frac{1}{2}}$$
$$y = K_y^{\frac{2}{3}} L_y^{\frac{1}{3}}$$

What is the production possibility frontier?

As above, we being by finding the marginal rate of technical substitution for both goods. We find

$$\text{MPK}_x = \frac{\partial x}{\partial K_x} = \frac{1}{2}\left(\frac{L_x}{K_x}\right)^{\frac{1}{2}} \qquad \text{MPK}_y = \frac{\partial y}{\partial K_y} = \frac{2}{3}\left(\frac{L_y}{K_y}\right)^{\frac{1}{3}}$$

$$\text{MPL}_x = \frac{\partial x}{\partial L_x} = \frac{1}{2}\left(\frac{K_x}{L_x}\right)^{\frac{1}{2}} \qquad \text{MPL}_y = \frac{\partial y}{\partial L_y} = \frac{1}{3}\left(\frac{K_y}{L_y}\right)^{\frac{2}{3}}$$

$$\mathrm{RTS}_x = -\frac{\mathrm{MPL}_x}{\mathrm{MPK}_x} = -\frac{K_x}{L_x} \qquad \mathrm{RTS}_y = -\frac{\mathrm{MPL}_y}{\mathrm{MPK}_y} = -\frac{1}{2}\left(\frac{K_y}{L_y}\right)$$

When we equate $\mathrm{RTS}_x = \mathrm{RTS}_y$, we find

$$-\frac{K_x}{L_x} = -\frac{1}{2}\left(\frac{K_y}{L_y}\right) \implies 2K_xL_y = K_yL_x$$

Factor market clearing conditions tell us $K_y = K - K_x = 3 - K_x$ and $L_y = L - L_x = 2 - L_x$. Plugging in,

$$\begin{aligned} & 2K_x(2 - L_x) = (3 - K_x)L_x \\ \iff \quad & 4K_x - 2K_xL_x = 3L_x - K_xL_x \\ \iff \quad & 4K_x = (3 + K_x)L_x \\ \iff \quad & L_x = \frac{4K_x}{3 + K_x} \end{aligned}$$

This gives us an explicit form for x,

$$x = K_x^{\frac{1}{2}}\left(\frac{4K_x}{3 + K_x}\right)^{\frac{1}{2}} = \left(\frac{4}{3 + K_x}\right)^{\frac{1}{2}} K_x$$

Although the form for y is slightly more involved, it is still directly computable

$$y = (3 - K_x)^{\frac{2}{3}}\left(2 - \frac{4K_x}{3 + K_x}\right)^{\frac{1}{3}} = (3 - K_x)^{\frac{2}{3}}\left(\frac{6 - 2K_x}{3 + K_x}\right)^{\frac{1}{3}}$$

At this point, we have a parametric form for the production possibility frontier,

$$(x, y) = \left(\left(\frac{4}{3 + K_x}\right)^{\frac{1}{2}} K_x, (3 - K_x)^{\frac{2}{3}}\left(\frac{6 - 2K_x}{3 + K_x}\right)^{\frac{1}{3}}\right)$$

To obtain the production possibility frontier we can vary K_x between 0 and $K = 3$ and see what points come out. In general, this should be a sufficient answer and will give more than enough information for plotting the production possibility frontier on a calculator.

However, often we have the ability to solve explicity for y in terms of x. While it's not obvious that we can do this here, it's worth the exercise to see how it can be done. From our equation for x, we know

$$x^2 = \left(\frac{4}{3 + K_x}\right) K_x^2 \implies 4K_x^2 - x^2K_x - 3x^2 = 0$$

Following the quadratic equation, this gives us values for K_x as the roots of the above polynomial,

$$K_x = \frac{x^2 \pm \sqrt{x^4 + 48x^2}}{8}$$

Since $\sqrt{x^4 + 48x^2} > x^2$, only the + root of this equation will be valid. We must then have

$$K_x = \frac{x^2 + x\sqrt{x^2 + 48}}{8}$$

When we substitute this into our equation for y, we get

$$y = \left(3 - \frac{x^2 + x\sqrt{x^2+48}}{8}\right)^{\frac{2}{3}} \left(\frac{6 - 2\frac{x^2+x\sqrt{x^2+48}}{8}}{3 + \frac{x^2+x\sqrt{x^2+48}}{8}}\right)^{\frac{1}{3}}$$

This form simplifies dramatically to

$$y = \left(\frac{24 - x^2 - x\sqrt{x^2+48}}{8}\right)^{\frac{2}{3}} \left(\frac{48 - 2x^2 - 2x\sqrt{x^2+48}}{24 + x^2 + x\sqrt{x^2+48}}\right)^{\frac{1}{3}}$$

which in turn is

$$y = \sqrt[3]{\frac{1}{32}} \left(24 - x^2 - x\sqrt{x^2+48}\right) \left(24 + x^2 + x\sqrt{x^2+48}\right)^{-\frac{1}{3}}$$

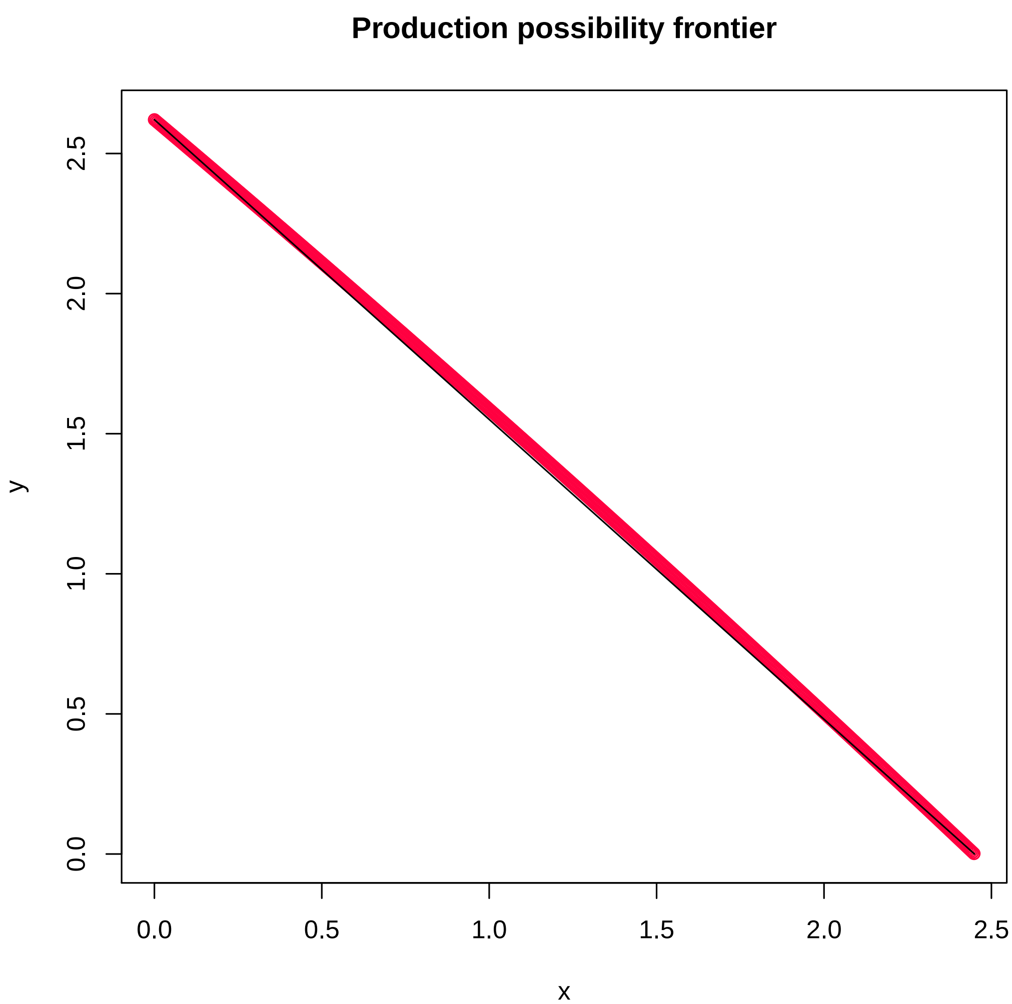

Figure 1: production possibility frontier; the black curve is a straight line, and serves to show that the PPF in this setup bows out slightly.

An exchange economy

Before discussing general equilibrium, we consider a simpler setup in which agents trade goods to one another and there is no production; this is referred to as an *exchange economy.* The most basic case of such an economy has two consumers and two goods: with fewer consumers there is no one to trade with, and with fewer goods there are no goods to trade for! In these setups, we are given agent utility functions and endowments, and are asked to find final consumption and prices.

As an example, consider two consumers, A and B, and two goods, x and y; utility functions are given by

$$u_A(x_A, y_A) = x_A^{\frac{2}{3}} y_A^{\frac{1}{3}} \qquad u_B(x_B, y_B) = x_B^{\frac{1}{3}} y_B^{\frac{2}{3}}$$

Endowments are $x_e^A, y_e^A, x_e^B, y_e^B$; this leads to budget constraints for the consumers,

$$p_x x_A + p_y y_A = p_x x_e^A + p_y y_e^A \qquad p_x x_B + p_y y_B = p_x x_e^B + p_y y_e^B$$

To obtain prices and consumption, we setup the agents' maximization problems and solve; we will then need to apply market clearing conditions to make sure consumption is in order. While maximizing these problems straight-up is not an issue, it would be simpler to do so without the exponents. We can abuse a neat mathematical fact to do this: maximizing any monotonic increasing function of utility is identical to maximizing utility. That is, rather than maximize utility itself, we can apply maximization to an increasing transformation of utility; if this transformation makes the maximization simpler, we're doing alright.

The standard transformation with Cobb-Douglas utility is to take logarithms. We have

$$\ln u_A(x_A, y_A) = \frac{2}{3}\ln x_A + \frac{1}{3}\ln y_A \qquad \ln u_B(x_B, y_B) = \frac{1}{3}\ln x_B + \frac{2}{3}\ln y_B$$

To solve the consumers' problems, we setup a Lagrangian (we solve only A's problem, since B's solution will follow by rough symmetry),

$$\mathscr{L}_A = \frac{2}{3}\ln x_A + \frac{1}{3}\ln y_A + \lambda\left(p_x x_e^A + p_y y_e^A - p_x x_A - p_y y_A\right)$$

$$\frac{\partial \mathscr{L}_A}{\partial x_A}: \qquad 0 = \frac{2}{3x_A} - \lambda p_x$$

$$\frac{\partial \mathscr{L}_A}{\partial y_A}: \qquad 0 = \frac{1}{3y_A} - \lambda p_y$$

Substituting through for λ, we find

$$\frac{2y_A}{x_A} = \frac{p_x}{p_y} \quad \Longrightarrow \quad x_A = \frac{2p_y y_A}{p_x}$$

Similar logic will obtain, for consumer B,

$$\frac{y_B}{2x_B} = \frac{p_x}{p_y} \quad \Longrightarrow \quad x_B = \frac{p_y y_B}{2p_x}$$

Appealing to the agents' budget constraints, for A we find

$$p_x\left(\frac{2p_y y_A}{p_x}\right) + p_y y_A = p_x x_e^A + p_y y_e^A \quad \Longrightarrow \quad y_A = \frac{p_x x_e^A + p_y y_e^A}{3p_y}$$

Similarly, for B

$$p_x\left(\frac{p_y y_B}{2p_x}\right) + p_y y_B = p_x x_e^B + p_y y_e^B \quad \Longrightarrow \quad y_B = \frac{2\left(p_x x_e^A + p_y y_e^A\right)}{3p_y}$$

In turn, this gives us consumption of x as

$$x_A = \frac{2\left(p_x x_e^A + p_y y_e^A\right)}{3p_x} \qquad x_B = \frac{p_x x_e^B + p_y y_e^B}{3p_x}$$

This is a general feature of Cobb-Douglas utility: equilibrium budget shares are proportional to the exponent associated with the good in utility. As we see here, $\frac{2}{3}$ of agent A's spending power is dedicated to consumption of good x, and $\frac{1}{3}$ is dedicated to good y.

Lastly, we use market clearing conditions to pin down prices in terms of one another.

$$\begin{aligned}
x_e^A + x_e^B &= x_A + x_B \\
&= \frac{2\left(p_x x_e^A + p_y y_e^A\right)}{3p_x} + \frac{p_x x_e^B + p_y y_e^B}{3p_y} \\
&= \frac{p_x\left(2x_e^A + x_e^B\right) + p_y\left(2y_e^A + y_e^B\right)}{3p_x} \\
\iff \quad 3p_x x_e^A + 3p_x x_e^B &= p_x\left(2x_e^A + x_e^B\right) + p_y\left(2y_e^A + y_e^B\right) \\
\iff \quad p_x x_e^A + 2p_x x_e^B &= 2p_y y_e^A + p_y y_e^B \\
\iff \quad \frac{p_x}{p_y} &= \frac{2y_e^A + y_e^B}{x_e^A + 2x_e^B}
\end{aligned}$$

Substituting in to our demand equations, we find

$$\begin{aligned}
x_A &= \left(\frac{2}{3}\right) x_e^A + \left(\frac{2}{3}\right)\left(\frac{p_y}{p_x}\right) y_e^A \\
&= \left(\frac{2}{3}\right) x_e^A + \left(\frac{2}{3}\right)\left(\frac{x_e^A + 2x_e^B}{2y_e^A + y_e^B}\right) y_e^A \\
&= \frac{4x_e^A y_e^A + 2x_e^A y_e^B + 2x_e^A y_e^A + 4x_e^B y_e^A}{6y_e^A + 3y_e^B}
\end{aligned}$$

Descriptions of other consumption variables will follow analogously.

Since there is always one price free (or normalizable) in the description of equilibrium, we now have a full characterization:

$$x_A = \frac{6x_e^A y_e^A + 2x_e^A y_e^B + 4x_e^B y_e^A}{6y_e^A + 3y_e^B} \qquad x_B = \frac{2x_e^B y_e^A + x_e^A y_e^B + 3x_e^B y_e^B}{6y_e^A + 3y_e^B}$$

$$y_A = \frac{3y_e^A x_e^A + 2y_e^A x_e^B + y_e^B x_e^A}{3x_e^A + 6x_e^B} \qquad y_B = \frac{4y_e^A x_e^B + 2y_e^B x_e^A + 6y_e^B x_e^B}{3x_e^A + 6x_e^B}$$

$$\frac{p_x}{p_y} = \frac{2y_e^A + y_e^B}{x_e^A + 2x_e^B}$$

When calculating equilibrium parameters, it is often helpful to leave things a little simpler than this; generally if you can phrase everything in terms of the price ratio you are well-set to obtain numerical answers to a question.

As a follow-up question, what happens to prices, allocations, and utility (and what is the intuition) when:

- $x_e^A = x_e^B = y_e^A = y_e^B = 1$
- $x_e^A = x_e^B = 1$, $y_e^A = y_e^B = 2$
- $x_e^A = y_e^A = 1$, $x_e^B = y_e^B = 4$

General equilibrium

An equilibrium is described by **prices** $p \geq 0$[1] **and allocations** $\overline{x}$ such that

(a) Allocations are feasible (only positive quantities of goods and factors are consumed or utilised)

(b) Markets clear (for both consumption goods and productive factors)

(c) Agents are maximizing utility subject to their budget constraints

(d) Firms are maximizing profits

With this in mind, when we are asked to find equilibrium in an economy we are being asked to list the prices and allocations which support an equilibrium.

Solving for equilibrium involves a lot of algebra, a bit of calculus, and patience. There are tricks to solving particular models, but they are ad hoc and not generally applicable so we won't discuss them here. The only real way to get comfortable with general equilibrium is practice, so let's proceed with an example problem.

There are C capitalists and W workers in an economy. All agents are endowed with 1 unit of labor, and capitalists also have 1 unit of capital apiece. We have production functions for goods b and g,

$$g = K_g + L_g, \qquad b = K_b^{\frac{1}{2}} L_b^{\frac{1}{2}}$$

Agent utility is given by

$$u_c(b_c, g_c) = b_c^{\frac{1}{3}} g_c^{\frac{2}{3}}, \qquad u_w(b_w, g_w, \ell_w) = b_w^{\frac{2}{3}} g_w^{\frac{1}{3}}$$

So capitalists enjoy g slightly more than b, and workers enjoy b slightly more than g (relatively speaking; in an absolute sense, this depends on where they are in their margins).

Equilibrium is described by prices and quantities. We therefore work to find prices on all goods — p_K, p_L, p_b, p_g — and consumption/production levels — b, b_c, b_w, g, g_c, g_w of commodities and K, K_b, K_g, k_c, $L, L_b, L_g, \ell_c, \ell_w$ of productive factors — that solve the definition of equilibrium listed above.

We begin looking for equilibrium by applying a few intuitive arguments. Since within each type all agents are identical, this equilibrium should have a symmetric structure; all workers have the same consumption and labor supply, and all capitalists have the same consumption and labor supply. With this in mind, the factor market clearing conditions are

$$W\ell_w + C\ell_c = L_b + L_g, \qquad Ck_c = K_b + K_g$$

and commodity market clearing conditions are

$$b = Cb_c + Wb_w, \qquad g = Cg_c + Wg_w$$

Notice that labor and capital are both supplied inelastically in this model! That is, agents realize some income from wages (and, if they are capitalists, from rents) but are otherwise unaffected; utility does not decrease when working or renting out capital. Since a greater budget is better and allows more purchasing options, agents will provide as much of each productive factor as they have at their disposal. That is, $\ell_c = k_c = 1$ and $\ell_w = 1$; so we can now restate the factor market clearing conditions as

$$L_b + L_g = W + C, \qquad K_b + K_g = C$$

[1]Aside: there are some fairly general conditions under which all prices must be *strictly* greater than 0; although I can by no means guarantee this, my intuition is that all questions we will see in this class will have this feature.

The worker's problem

We need to start somewhere, so we begin with the worker's problem; we turn to calculus to help uncover relationships between variables. The problem of an individual worker is

$$\max_{b_w, g_w, \ell_w} b_w^{\frac{2}{3}} g_w^{\frac{1}{3}}, \text{ s.t. } p_b b_w + p_g g_w = p_L \ell_w$$

We have already reasoned that $\ell_w = 1$, so we can rephrase the optimization as

$$\max_{b_w, g_w} b_w^{\frac{2}{3}} g_w^{\frac{1}{3}}, \text{ s.t. } p_b b_w + p_g g_w = p_L$$

Now, we *could* solve this problem using a Lagrangian or through direct substitution; but, since we know that utility in Cobb-Douglas form has consumption's share of budget proportional to the exponent (see the previous exercise with an exchange economy) we know

$$b_w = \frac{2p_L}{3p_b} \qquad g_w = \frac{p_L}{3p_g}$$

The capitalist's problem

The capitalist's problem is given by

$$\max_{b_c, g_c, k_c, \ell_c} b_c^{\frac{1}{3}} g_c^{\frac{2}{3}}, \text{ s.t. } p_b b_c + p_g g_c = p_K k_c + p_L \ell_c$$

We have already reasoned through $k_c = 1$ and $\ell_c = 1$, so we can simplify the optimization to

$$\max_{b_c, g_c} b_c^{\frac{1}{3}} g_c^{\frac{2}{3}}, \text{ s.t. } p_b b_c + p_g g_c = p_K + p_L$$

As is the case in the worker's problem, this is a Cobb-Douglas form so we know that consumption levels will be

$$b_c = \frac{p_K + p_L}{3p_b} \qquad g_c = \frac{2(p_K + p_L)}{3p_g}$$

The firm's problem

With access to two production technologies, the firm solves

$$\max p_b K_b^{\frac{1}{2}} L_b^{\frac{1}{2}} + p_g (K_g + L_g) - p_K (K_b + K_g) - p_L (L_b + L_g)$$

We can apply some intuition to a few price relations:

- Suppose $p_g > p_K$. Then for every unit of g produced using K_g as input, positive profit is obtained; the firm's profit-maximizing consumption of K_g is then infinite! This will certainly violate the capital market clearing condition, so we cannot have $p_g > p_K$; it must be that $p_g \leq p_K$.
- Suppose $p_g > p_L$. Similar logic to the above will hold; it must be that $p_g \leq p_L$.
- Suppose $p_g < p_K$ and $p_g < p_L$. Then the firm realizes a loss for every unit of g produced; it will then choose $g = 0$. But by the equation for demand, $g = 0$ only if $p_w = +\infty$; this contradicts the notion that $p_g < p_K$ and $p_g < P_L$! So we cannot have both $p_g < p_K$ and $p_g < p_L$: at least one of p_K or p_L equals p_g.

This method of obtaining restrictions on prices works only because we have a production function for b which permits a neat, linear comparison. With regards to the production of b, we cannot apply any such logic; it is necessary to use calculus to determine tradeoffs between production and expenditure. We see

$$\begin{aligned} \frac{\partial}{\partial K_x}: &\quad 0 = p_b \left(\frac{L_b}{4K_b}\right)^{\frac{1}{2}} - p_K \\ \frac{\partial}{\partial L_x}: &\quad 0 = p_b \left(\frac{K_b}{4L_b}\right)^{\frac{1}{2}} - p_L \end{aligned}$$

From these equations, we obtain

$$p_L = \left(\frac{K_b}{L_b}\right) p_K \quad \Longleftrightarrow \quad \frac{p_L}{p_K} = \frac{K_b}{L_b} \tag{1}$$

Putting it all together

From the demand conditions for the agents together with the commodity market clearing conditions, we know

$$Wb_w = \frac{2Wp_L}{3p_b} \qquad Cb_c = \frac{Cp_L + Cp_K}{3p_b}$$

$$Wg_w = \frac{Wp_L}{3p_g} \qquad Cg_c = \frac{2\left(Cp_L + Cp_K\right)}{3p_g}$$

Appealing to our earlier discussion of the firm's problem, we reduce prices to three cases:

- $p_g < p_L$. Then $p_g = p_K$. Further, $L_g = 0$ so $L_b = W + C$. Solving through aggregate demand for b,

$$\begin{aligned} Wb_w + Cb_c &= W\left(\frac{2p_L}{3p_b}\right) + C\left(\frac{p_L}{3p_b} + \frac{p_K}{3p_b}\right) \\ &= \frac{W}{3}\left(\frac{K_b}{L_b}\right)^{\frac{1}{2}} + \frac{C}{6}\left[\left(\frac{K_b}{L_b}\right)^{\frac{1}{2}} + \left(\frac{L_b}{K_b}\right)^{\frac{1}{2}}\right] \\ &= \frac{W}{3}\left(\frac{K_b}{W+C}\right)^{\frac{1}{2}} + \frac{C}{6}\left[\left(\frac{K_b}{W+C}\right)^{\frac{1}{2}} + \left(\frac{W+C}{K_b}\right)^{\frac{1}{2}}\right] \end{aligned}$$

From market clearing, we know $b = K_b^{\frac{1}{2}} L_b^{\frac{1}{2}}$. Then we have

$$\begin{aligned} & K_b^{\frac{1}{2}}(W+C)^{\frac{1}{2}} = \frac{W}{3}\left(\frac{K_b}{W+C}\right)^{\frac{1}{2}} + \frac{C}{6}\left[\left(\frac{K_b}{W+C}\right)^{\frac{1}{2}} + \left(\frac{W+C}{K_b}\right)^{\frac{1}{2}}\right] \\ \Longleftrightarrow\quad & (W+C)^{\frac{1}{2}} = \frac{W}{3\sqrt{W+C}} + \frac{C}{6\sqrt{W+C}} + \frac{C\sqrt{W+C}}{6K_b} \\ \Longleftrightarrow\quad & 1 = \frac{W}{3(W+C)} + \frac{C}{6(W+C)} + \frac{C}{6K_b} \\ \Longleftrightarrow\quad & \frac{C}{6K_b} = 1 - \left(\frac{2W+C}{6(W+C)}\right) \\ \Longleftrightarrow\quad & \frac{C}{6K_b} = \frac{4W+5C}{6W+6C} \\ \Longleftrightarrow\quad & K_b = \frac{C(W+C)}{4W+5C} \end{aligned}$$

From factor market clearing, we know then that

$$\begin{aligned} K_g &= C - \left(\frac{C(W+C)}{4W+5C}\right) \\ &= \frac{C(4W+5C) - C(W+C)}{4W+5C} \\ &= \frac{C(3W+4C)}{4W+5C} \end{aligned}$$

Since $L_g = 0$, this gives us g directly.

As a final check, we have assumed that $p_K < p_L$. Then by equation (1), we need

$$\frac{K_b}{L_b} > 1$$

Verifying against our determined levels of capital and labor,

$$\frac{K_b}{L_b} = \frac{C(W+C)}{(4W+5C)(W+C)} = \frac{C}{4W+5C} < 1$$

So the prices arising in this equilibrium contradict our assumptions; this cannot be an equilibrium, and we will never see $p_g < p_L$!

Why might this be the case? Consider the fact that while only one type of agent is supplied with capital, both types of agents are supplied with labor. Although this is by no means a mathematical argument, it follows that labor will, in general, be the less dear factor of production as it is supplied inelastically by all members of the economy. Since $p_g < p_L$ implies $p_K < p_L$, relative dearness of capital violates the pricing assumption above.

- $p_g = p_K$, $p_g = p_L$. From equation (1) it follows that $K_b = L_b$; the total production of b is then $b = K_b = L_b$. We know that aggregate demand for b is

$$\begin{aligned} Wb_w + Cb_c &= W\left(\frac{2p_L}{3p_b}\right) + C\left(\frac{p_L + p_K}{3p_b}\right) \\ &= \frac{2}{3}(W+C)\left(\frac{p_L}{p_b}\right) \\ &= \frac{2}{3}(W+C)\left(\frac{1}{2}\right) \\ &= \frac{1}{3}(W+C) \end{aligned}$$

Then $K_b = L_b = \frac{1}{3}(W+C)$. Since $C = K_g + K_b$, we have then that

$$K_g = \frac{2C - W}{3}$$

It follows that for this equilibrium to be valid, we must have $W < 2C$ (otherwise K_g is negative!)

As we found K_g above, we find L_g according to clearing constraints,

$$L_g = (C+W) - L_b = \frac{2}{3}(W+C)$$

Total g production is then

$$g = \frac{4C+W}{3}$$

- $p_g < p_K$. Then $p_g = p_L$. Further, $K_g = 0$ so $K_b = C$. Turning again to aggregate demand for b, we find

$$\begin{aligned} Wb_w + Cb_c &= W\left(\frac{2p_L}{3p_b}\right) + C\left(\frac{p_L}{3p_b} + \frac{p_K}{3p_b}\right) \\ &= \frac{W}{3}\left(\frac{K_b}{L_b}\right)^{\frac{1}{2}} + \frac{C}{6}\left[\left(\frac{K_b}{L_b}\right)^{\frac{1}{2}} + \left(\frac{L_b}{K_b}\right)^{\frac{1}{2}}\right] \\ &= \frac{W}{3}\left(\frac{C}{L_b}\right)^{\frac{1}{2}} + \frac{C}{6}\left[\left(\frac{C}{L_b}\right)^{\frac{1}{2}} + \left(\frac{L_b}{C}\right)^{\frac{1}{2}}\right] \end{aligned}$$

From market clearing, we know $b = K_b^{\frac{1}{2}} L_b^{\frac{1}{2}}$. Then we have

$$\begin{aligned} K_b^{\frac{1}{2}} L_b^{\frac{1}{2}} &= \frac{W}{3}\left(\frac{C}{L_b}\right)^{\frac{1}{2}} + \frac{C}{6}\left[\left(\frac{C}{L_b}\right)^{\frac{1}{2}} + \left(\frac{L_b}{C}\right)^{\frac{1}{2}}\right] \\ \iff \quad C^{\frac{1}{2}} L_b &= \frac{WC^{\frac{1}{2}}}{3} + \frac{C^{\frac{3}{2}}}{6} + \frac{L_b C^{\frac{1}{2}}}{6} \\ \iff \quad L_b &= \frac{W}{3} + \frac{C}{6} + \frac{L_b}{6} \\ \iff \quad L_b &= \frac{2W + C}{5} \end{aligned}$$

It follows that

$$\begin{aligned} L_g &= L - L_b \\ &= (W + C) - \frac{2W + C}{5} \\ &= \frac{3W + 4C}{5} \end{aligned}$$

Since $K_g = 0$, we know $g = \frac{3W+4C}{5}$.

Having assumed $p_K > p_L$, we need $\frac{K_b}{L_b} < 1$. This is checked as

$$\begin{aligned} \frac{K_b}{L_b} &= \frac{C}{\frac{2W+C}{5}} \\ &= \frac{5C}{2W + C} \end{aligned}$$

$$\begin{aligned} & \frac{K_b}{L_b} < 1 \\ \iff \quad & 5C < 2W + C \\ \iff \quad & W > 2C \end{aligned}$$

Then this equilibrium is valid when $W > 2C$. That is, when the number of workers is sufficiently large relative to the number of capitalists, we will see the price of capital rise above the price of labor.

How can we summarize these results? We tabulate what we've found on the basis of the necessary relationships between W and C to support different equilibria.

	$W < 2C$	$2C < W$
p_K	p_K	p_K
p_L	p_K	$\left(\frac{5C}{2W+C}\right) p_K$
p_b	$2p_K$	$2\left(\frac{5C}{2W+C}\right)^{\frac{1}{2}} p_K$
p_g	p_K	$\left(\frac{5C}{2W+C}\right) p_K$
L	$C+W$	$C+W$
L_b	$\left(\frac{C+W}{3}\right)$	$\left(\frac{2W+C}{5}\right)$
L_g	$\left(\frac{2C+2W}{3}\right)$	$\left(\frac{3W+4C}{5}\right)$
K	C	C
K_b	$\left(\frac{C+W}{3}\right)$	C
K_g	$\left(\frac{2C-W}{3}\right)$	0

	$W < 2C$	$2C < W$
b	$\left(\frac{C+W}{3}\right)$	$\left(\frac{2WC+C^2}{5}\right)^{\frac{1}{2}}$
b_c	$\left(\frac{1}{3}\right)$	$\frac{1}{6}\left(\frac{2W+6C}{\sqrt{5C(2W+C)}}\right)$
b_w	$\left(\frac{1}{3}\right)$	$\frac{1}{3}\left(\frac{5C}{2W+C}\right)^{\frac{1}{2}}$
g	$\left(\frac{4C+W}{3}\right)$	$\left(\frac{3W+4C}{5}\right)$
g_c	$\left(\frac{4}{3}\right)$	$\frac{2}{3}\left(\frac{2W+6C}{5C}\right)$
g_w	$\left(\frac{1}{3}\right)$	$\frac{1}{3}$
u_c	$\left(\frac{\sqrt[3]{16}}{3}\right)$	a mess
u_w	$\left(\frac{1}{3}\right)$	$\frac{1}{3}\left(\frac{5C}{2W+C}\right)^{\frac{1}{3}}$

That is a more-than-thorough answer to the question. In general, it will be clear what is being asked for in the question — prices, quantities, etc. — so everything should not need to be tabulated. But a full description of the economy requires all variables to be accounted for.

It is obviously tough to keep track of all cases, equations, and variables. The best tip here is to keep organized and consider cases one at a time. As far as I can tell, there are no real tricks to solving general equilibrium problems, only a willingness to work through the algebra. If you work back through the question above, you can see that if we try to use the market clearing condition for good g we wind up mired in a morass of square roots; we can get the same answer by going this direction, but it is not as clean as using the market clearing condition for b ("clean" here in the relative sense). How can you recognize that this is the case? Frankly, by solving the question. If you get stuck or start getting forms that look unholy, try approaching a different clearing condition. As I was working through this problem, I initially solved for g until it proved too tedious. Solving for b was much simpler.

The advice for solving general equilibrium problems is then twofold: be patient, and don't be so attached to your approach that you're unwilling to try a different tack.

A note on maximization

It was mentioned in section that maximizing a function f is the same as maximizing some strictly increasing function g of f. To make this look a little more concrete, suppose x^* is such that

$$f(x^*) = \max_x f(x)$$

That is, x^* maximizes f.

Let g be a strictly increasing function, and suppose that there is some x' such that

$$g \circ f(x') > g \circ f(x^*)$$

Now, since g is strictly increasing it is uniquely-valued and hence invertible. Moreover, this inverse is also strictly increasing (for an illustration of this — but not a proof — think of the graphs of $y = x$ (which is its own inverse) or of $y = x^2$ and $y = \sqrt{x}$). If $a > b$ and h is a strictly increasing function, then $h(a) > h(b)$; we apply this logic to see that

$$g^{-1} \circ g \circ f(x') > g^{-1} \circ g \circ f(x^*)$$

Which will hold if and only if

$$f(x') > f(x^*)$$

But this contradicts the statement that x^* solves $\max_x f(x)$! So we cannot have that $g \circ f(x') > g \circ f(x^*)$, and x^* must also maximize $g \circ f(x)$.

Where is this useful? Often it is helpful to transform utility functions to find optimum consumption (think about why we, in general, *cannot* do this with production functions in any regular way). Suppose we are given a Cobb-Douglas utility function, $u(x, y) = x^\alpha y^{1-\alpha}$. Although by now we are all familiar with solving this optimization, it's kind of messy; this is even more true as we add consumption goods. In this case, we note that ln is an increasing function, so we can just as well optimize $\ln \circ u(x, y)$; that is, we can solve

$$\max_{x,y} \ln \circ u(x, y) = \max_{x,y} \alpha \ln x + (1 - \alpha) \ln y$$

This problem is much simpler to optimize on paper, since consumption goods no longer affect each other's optimization.

Further, we can use this trick to see that some utility functions are more familiar than we might think. Suppose $u(x, y) = x^m y^n$ for some $m, n \in \mathbb{N}$. Again, this optimization is a little messy on paper; we *could* apply the logarithm trick from above, but to appeal to intuition we'll do something slightly different. Notice that $f(u) = u^{\frac{1}{m+n}}$ is a strictly increasing function. Then maximizing the utility above is equivalent to solving

$$\max_{x,y} f \circ u(x, y) = \max_{x,y} x^{\frac{m}{m+n}} y^{\frac{n}{m+n}} = \max_{x,y} x^{\frac{m}{m+n}} y^{1-\frac{m}{m+n}}$$

So solving this utility form is equivalent to solving a Cobb-Douglas utility form! Since we know that Cobb-Douglas consumption expenditure is proportional to the exponent on the good, we don't even need to perform any real calculus here and can safely plug into a known formula.