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Bundling as an optimal selling mechanism for a multiple-good monopolist

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Abstract

Multiple objects may be sold by posting a schedule consisting of one price for each possible bundle and permitting the buyer to select the price–bundle pair of his choice. We identify conditions that must be satisfied by any price schedule that maximizes revenue within the class of all such schedules. We then provide conditions under which a price schedule maximizes expected revenue within the class of all incentive compatible and individually rational mechanisms in the n-object case. We use these results to characterize environments, mainly distributions of valuations, where bundling is the optimal mechanism in the two and three good cases.
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1. Introduction

It is not uncommon to sell a given number of indivisible objects by offering them in bundles, i.e., subcollections of objects. Bundling may be carried out by posting a schedule of prices, one price for each possible bundle, thus permitting the buyer to select the price–bundle pair of his choice. We consider a model with a seller with n indivisible objects, and a consumer with linear preferences (over goods and money) whose valuations for the objects are private information. Our goal is to identify environments in which bundling is optimal in that it maximizes the seller's

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expected revenue. (Henceforth the adjective optimal is used in this sense.) First, we study the revenue-maximizing price schedule within the class of all such schedules. We identify necessary conditions for optimality. Second, we investigate the optimality of price schedules within the class of all incentive compatible (henceforth IC) and individually rational (henceforth IR) mechanisms. We provide sufficient conditions for the optimality of bundling. These conditions can be expressed as a pair of functional inequalities. We illustrate how the conditions can be used to identify a class of environments for $n = 2$ and 3 in which bundling is the optimal mechanism.

It has long been known that when there is only one good, posting an appropriately selected, take-it-or-leave-it price generates the highest expected revenue among all feasible trading mechanisms. [1] This remarkable result implies that, despite the enormous class of incentive compatible, individual rational bilateral trading mechanisms, in the one-good case, the search for expected revenue-maximizing mechanisms can be restricted to a very simple class of institutions. Posting a price schedule, i.e. bundling, is the natural extension of the one-dimensional mechanism to the case $n > 1$. [2] In addition bundling is 'simple' in that randomization in the assignment of goods is not used; a buyer of certain type will buy a given bundle with probability zero or one. It is therefore valuable to understand when these deterministic posted price mechanisms are optimal.

In spite of its attractive characteristics, surprisingly little is known about the consequences of bundling for revenue. For instance, it is not known, even in the $n = 2$ case, what the revenue-maximizing price schedule is, or under what circumstances posting a price schedule is indeed the best trading institution. We show by example that, unlike in the one good case, revenue maximization may require the randomization of assignments even for $n = 2$. [3]

As an intermediate step in searching for environments where bundling is optimal with respect to all IC and IR mechanisms, we provide conditions that the best price schedule (within the class of all price schedules) must satisfy. We illustrate the usefulness of these conditions by providing sufficient conditions for when the optimal price schedule is submodular.

We next use our characterization of the best price schedule to explore our main goal. Our approach yields sufficient conditions for the optimality of the price schedule within the class of all IC and IR mechanisms. Those conditions are not easily interpretable but, nevertheless, prove very useful: we use them in the $n = 2$ and 3 cases (and they could potentially be used in other cases) to identify environments, mainly restrictions on the distribution of valuations, where bundling is indeed the revenue-maximizing institution.

Our approach is loosely based on the methodology we employed in [6]. Consider any specific trading institution. To such an institution corresponds an incentive compatible and individually rational direct mechanism. The direct mechanism is a solution to the seller's linear program if there is a feasible solution to the dual program such that dual and primal programs have the same value. We identify environments in which the proposed mechanism solves the primal program by constructing the relevant dual variables. The described approach could be applied, in principle, to any trading institution. We focus on price schedules (or bundling) because we believe they are simple, easily implemented institutions, and are natural extensions of the optimal mechanism in the one good case. Price schedules offer, in addition, a technical advantage: since the (sequentially

[1] See for example, [11,15].

[2] Preston McAfee et al. [9] show that posting prices both for individual goods and for bundles typically strictly dominates in terms of revenue the posting of prices for only the individual goods.

[3] As part of a 1988 piece, Preston McAfee and McMillan [8] describe an environment with $n = 2$ in which bundling is optimal. Our example shows that their claim is not accurate. A related counterexample was discovered independently and simultaneously by Thanassoulis [16].

rational) behavior of a buyer choosing among price–bundle pairs is straightforward, the implicit direct mechanism is immediate.

This paper is a contribution to the research on multidimensional mechanism design. McAfee and McMillan [8] examined the question of when deterministic mechanisms, i.e., mechanisms where the assignment is not random, are optimal in cases of multidimensional uncertainty. Armstrong [1] showed that under optimal mechanisms there is a set (of positive measure) of buyer types who never trade. Rochet [13] and Rochet and Choné [14] extended both of these papers and show that optimal mechanisms typically require 'bunching' (even in the case where goods are divisible). Bunching implies that buyer types virtually always pool into a set of positive measure of other buyer types. Rochet [13] also offers an example of discretely distributed buyer types in which deterministic mechanisms are suboptimal.

Our work contributes to the literature on bundling as a form of second degree price discrimination, and might also shed some light on a related problem. When $n = 1$, the optimal take-it-or-leave-it price in the seller's problem corresponds to the optimal reserve price in a standard auction with m buyers in an independent private values environment. In addition such auctions are optimal over the class of all IC and IR mechanisms. [4] Similarly, the optimal price schedule might play a role in the auctioning of n indivisible goods to m buyers.

The outline of the paper is as follows. In Section 2, the model and some notation is introduced. Section 3 provides some preliminary results that describe how buyer types self-select in response to price schedules. In Section 4, we provide necessary conditions for the optimality of a price schedule within the class of all price schedules. In Section 5, we provide a two-good example where the expected revenue generated by an optimal price schedule is strictly lower than that generated by another, more complicated mechanism. This leads to the question: under what conditions are price schedules optimal over all incentive compatible and individually rational mechanisms? In Section 6 we obtain sufficient conditions for the optimality of the price schedule within the class of all IC and IR mechanisms and show that they can be expressed as a pair of functional inequalities. In Section 7, we apply the results in Section 6 to the $n = 2$ and 3 cases.

2. Notation and preliminaries

A seller with n different objects attempts to maximize expected revenue by trading with a single buyer (that is, we assume zero marginal costs). The buyer's preferences over consumption and money transfers are given by

$$U(x, q, t) = x \cdot q - t,$$

where x is the vector of buyer's valuations, q is the quantity consumed of each good, and t is the monetary transfer made to the seller. Since the buyer has demand for at most one unit of each good, the vector q is an element of $\{0, 1\}^n$; x is assumed for simplicity to be in I^n where $I = [0, 1]$, and t is in $\mathbb{R}$.

Index the n goods by $i = 1, \ldots, n$ and let N represent the set of all available goods. Given a vector x in $\mathbb{R}^n$, x_i represents its ith component, x_{-i} the remaining components, and (y, x_{-i}) the vector where the ith component is y and the other components are x_{-i}. Similarly, for $J \subset N$, J^c denotes the bundle $N \setminus J$, x_J denotes the $|J|$-dimensional vector with components in J, I^J the $|J|$-cartesian product of I, and we may write (x_J, x_{J^c}) when convenient. Similar notation will be applied to other objects.

[4] See [11], for example.

The seller does not observe the buyer's valuation—the buyer's private information—but it is common knowledge that valuation x is distributed according to a prior density function $f(x)$. Assumption 1 is maintained throughout.

Assumption 1. The density $f(x)$ is a continuously differentiable, strictly positive function in I^n.

Additional requirements on f will be imposed at different points in our analysis. We list the requirements here for future reference.

Assumption 2. The density $f(x)$ satisfies

(a) $$f(x) = \Pi_{i=1}^{n} f_i(x_i),$$

(b) $$\forall i \forall x, \quad f(x) + x_i \frac{\partial f(x)}{\partial x_i} \geqslant 0.$$

Assumption 2a states that the buyer's valuations for the n goods are independently distributed. When Assumption 2a is invoked, it will be convenient to use the notation $f_i'(x_i) \equiv df_i(x_i)/dx_i$.

Given a function $f(x)$, $\nabla f(x)$ denotes the gradient of f evaluated at x. Note that Assumption 2b implies $(n+1)f(x)+x \cdot \nabla f(x) \geqslant 0$, which is an assumption invoked by McAfee and McMillan [8]. In the case where $n = 1$, the restriction implies that the 'virtual valuation function', $x - \frac{1-F(x)}{f(x)}$ crosses zero only once. This implies the uniqueness of the optimal take-it-or-leave-it price in that case and is an alternative assumption to the monotone hazard condition that is sometimes invoked.

In searching for an optimal mechanism, one may restrict attention to direct revelation mechanisms where buyers report their types truthfully. A direct revelation mechanism is a pair of functions

$$\begin{aligned} p &: I^n \longrightarrow I^n, \\ t &: I^n \longrightarrow \mathbb{R}, \end{aligned}$$

where $p_i(x)$, the ith component of $p(x)$, is the probability that the buyer will obtain good i when his valuation is x, and $t(x)$ is the transfer made by the buyer to the seller when valuations are x.[5] In addition, the buyer must have adequate incentives to reveal his information truthfully—incentive compatibility (IC)—and to participate in the mechanism voluntarily—individual rationality (IR). The buyer's expected payoff under the mechanism (p, t) when the buyer has valuation x and reports x' is $p(x') \cdot x - t(x')$. The equilibrium expected utility of a buyer of type x is denoted $\pi(x)$. Then, (p, t) must satisfy[6]

$$\begin{aligned} &\text{(IC)} \quad \forall x, \pi(x) \geqslant p(x') \cdot x - t(x') \; \forall x', \\ &\text{(IR)} \quad \forall x, \pi(x) \geqslant 0. \end{aligned}$$

We informally describe some readily available properties of IC and IR mechanisms—well known in one-dimensional problems—that have been noted and used in the literature in higher dimensional environments (see [12,1,3,5,7]). Graphically, a mechanism is IC if and only if the corresponding buyer's payoffs $\pi(x)$ are convex, with partial derivatives $\partial\pi(x)/\partial x_i$ between zero

[5] In order to compute expected payoffs, the functions p and t must be integrable.

[6] As stated, the constraints hold everywhere; it suffices that they hold almost everywhere in x and everywhere in x'.

and one. Furthermore, $\partial\pi(x)/\partial x_i$ represents the probability that the buyer of type x receives good i in equilibrium.

The preceding discussion completely characterizes IC mechanisms in terms of the buyer's expected-payoff function $\pi(x)$. Individual rationality requires in addition that π be non-negative. Define

$$C = \left\{\pi : I^n \to \mathbb{R}_+ \mid \pi(x) \text{ is increasing and convex}\right\}.$$

Thus, π is an incentive compatible, individually rational mechanism if and only if π belongs to C and $\nabla\pi(x) \in I^n$ almost everywhere (since the ith component of the gradient is the probability that good i is traded).

Given any $\pi \in C$, a buyer with type x receives a payoff $\pi(x) = \nabla\pi(x) \cdot x - t(x)$. Therefore, the seller's expected revenue when using the mechanism $\pi(\cdot)$ is

$$E[t(x)] = \int_{I^n} [\nabla\pi(x) \cdot x - \pi(x)]\, f(x)\, dx.$$

Given Assumption 1, we can apply integration by parts (as done in [8]) or the divergence theorem (as done in [14]) to obtain a representation of the seller's expected revenue in terms of $\pi(\cdot)$ alone:

$$\begin{aligned} E[t(x)] = &\sum_{i=1}^{n} \int_{I^{\{i\}^c}} \pi(1, x_{-i})\, f(1, x_{-i})\, dx_{-i} \\ &- \int_{I^n} \pi(x)\, [(n+1) f(x) + x \cdot \nabla f(x)]\, dx. \end{aligned} \tag{1}$$

The seller's expected revenue is a linear functional of the mechanism π employed in the transaction; we denote the linear functional by T, and the expected revenue of using the mechanism π by $\langle \pi, T\rangle$. The seller's problem is to maximize expected revenue over all IC and IR mechanisms:

$$\max\nolimits_{\pi \in C, \nabla\pi \leqslant \mathbf{1}} \langle \pi, T\rangle. \tag{2}$$

When there is only one good, maximum seller's revenue can be achieved with a mechanism that, depending on the buyer's reported valuation, either assigns the object for certain (i.e., with probability one), or not at all (i.e., with probability zero). Posting the good's price implements this mechanism; the potential buyer acquires the good if his/her valuation exceeds the posted price. With many goods there are additional issues to consider. The seller can post a price not only for each good but also for each combination of goods, i.e., for each bundle.

Definition 1. A bundle of goods is a set $J \subset N$.[7] A bundle J can also be represented by an n-dimensional vector of zeros and ones, $a^J = (a_1^J, a_2^J, \ldots, a_n^J)$ where a_i^J takes the value 1 if $i \in J$ and the value 0 otherwise.

Both representations of a given bundle are used in the paper.

Casual observation suggests that indeed sellers frequently set prices for different bundles leaving consumers the choice of what bundle to purchase. It may be profitable for the seller to set a price for a bundle that is higher than the sum of the prices of its components. In this case, the potential buyer has an incentive to bypass the bundle price, and acquire the bundle by purchasing the individual components.

[7] Note that the expression, $J \subset N$ includes the empty set.

The above discussion prompts the following definition.

Definition 2. A price schedule is a collection of prices $P = \{P_J\}_{J \subset N}$, one price per bundle; potential buyers select the bundle they prefer and pay the quoted price for that bundle (i.e., buyers cannot aggregate individual sub-bundles independently).

Given that IR must be satisfied, without loss of generality, for all price schedules, we set $P_{\emptyset} = 0$. Note that, as defined, price schedules are deterministic—purchasing bundle J implies obtaining all goods in J with probability one. This restriction is significant. Section 5 provides an example where deterministic price schedules are suboptimal.

Any price schedule P implicitly segments buyer types by grouping them according to the bundles they choose to consume. Employing the notation in Definition 1, the utility of a buyer of type $x \in I^n$ who acquires the bundle J at price P_J is $a^J \cdot x - P_J$.

Definition 3. Given a price schedule P and a bundle J, the market segment acquiring bundle J is

$$A_J = \{x \in I^n \mid a^J \cdot x - P_J \geqslant a^K \cdot x - P_K \ \forall K \subset N\}.$$

Note that A_J is the intersection of I^n with finitely many half spaces in R^n. If, given any two bundles J and K, $A_J \cap A_K \neq \emptyset$, then $A_J \cap A_K$ is a subset of the hyperplane $\{x \mid (a^J - a^K) \cdot x = P_J - P_K\}$ and has Lebesgue measure zero in I^n.

Fix a bundle J and its corresponding market segment A_J. For each $i \in J$,

$$B_J^i = \{x_{-i} \in I^{\{i\}^c} \mid (1, x_{-i}) \in A_J\}$$

represents the intersection of A_J with the boundary of I^n along the coordinate $x_i = 1$. If A_J has positive measure in $\mathbb{R}^n$, then B_J^i also has positive measure in $\mathbb{R}^{n-1}$.

For some results, we restrict attention to price schedules that satisfy a submodularity condition.

Definition 4. A price schedule P is submodular (*SM*) if

$$(SM) \quad \forall J, K \subset N, \quad P_{J \cup K} \leqslant P_J + P_K - P_{J \cap K}.$$

If *SM* is not satisfied, then a type of arbitrage incentive is present. Suppose that a buyer was allowed to buy *and sell* at the outstanding prices, P. If K and J overlap (that is, $K \cap J \neq \emptyset$) and the condition is violated, a buyer could form bundle $K \cup J$ more cheaply by buying K and J separately and then selling back the duplicated goods in $K \cap J$.

We emphasize that we do not impose *SM* as a constraint on the type of mechanisms the seller may use. (Such an imposition would correspond to mechanisms where the seller is unable to monitor the bundle acquired by the buyer and would require modifying the seller's program considered in this essay.) In Section 4, it is shown that in some environments the optimal price schedule must satisfy *SM* and we take advantage of the additional restrictions it implies. [8]

A final restriction concerns price schedules such that all bundles are purchased with positive probability.

[8] Condition *SM* does not appear to hold generally in optimal bundling mechanisms; we can show computationally, however, that in the case of independent and identically distributed valuations with $F^i(x_i) = x_i^{\alpha}$, $n = 3$ for $\alpha \leqslant 3$, optimal bundling mechanisms satisfy this condition. Computations suggest that *SM* is violated with $\alpha > 3$.

Definition 5. A price schedule P sells all bundles *(ABS)* if

$$(ABS) \quad \forall J \neq \emptyset, \quad \int_{A_J} f(x)\,dx > 0.$$

The condition *ABS* is typically invoked for technical reasons as it allows us to ignore some arguments that apply only on sets of zero measure.

3. Price schedules—some properties

We introduce here some technical observations, used in later sections in the proofs of our main results.

The first lemma illustrates that B^i_J corresponds to the projection of A_J on the boundary, $I^{\{i\}^c}$.

Lemma 1. *Let P be any price schedule and $\{A_J\}_{J \subset N}$ the corresponding market segments. For $J \subset N, i \in J$, if $(x_i, x_{-i}) \in A_J$ then $(x'_i, x_{-i}) \in A_J$ for all $x'_i > x_i$. For $i \notin J$, if $(x_i, x_{-i}) \in A_J$ then $(x'_i, x_{-i}) \in A_J$ for all $x'_i < x_i$.*

Proof. In the Appendix.

The lemma implies that for $i \in J$, if $(x_i, x_{-i}) \in A_J$ then $(1, x_{-i}) \in A_J$. Conversely, if x_{-i} is such that $x_{-i} \notin B^i_J$, then there does not exist any x_i such that $(x_i, x_{-i}) \in A_J$.

In general, the construction of the market segment A_J requires the comparison of utility obtained from purchasing J with the utility obtained from purchasing any other set K. The next lemma shows that if the price schedule satisfies *SM*, the number of relevant comparisons is much smaller since it implies that, for any J, we need only compare the purchase of J with any K such that either $K \subset J$ or $J \subset K$. The result also yields a type of independence of the set of valuations for the goods outside of the set J from the valuations for the goods in J.

Lemma 2. *Suppose the price schedule P satisfies SM and let $\{A_J\}_{J \subset N}$ be its corresponding market segments. Then,*

(i) *$x \in A_J$ if and only if $a^J \cdot x - P_J \geqslant a^K \cdot x - P_K$ for all K such that $K \subset J$ or $K \supset J$.*
(ii) *For all $K, J, K \not\subset J$ and $J \not\subset K$, $A_J \cap A_K$ has zero Lebesgue measure in $\mathbb{R}^{n-1}$.*
(iii) *Define*

$$\begin{aligned} A^J_J &= \{x_J \in I^J \mid (x_J, y) \in A_J \text{ for some } y\},\\ A^{J^c}_J &= \{y \in I^{J^c} \mid (x_J, y) \in A_J, \text{ for some } x_J \in A^J_J\},\\ D^i_J &= \{z \in I^{J/\{i\}} \mid (1, z) \in A^J_J\} \quad \text{where } i \in J. \end{aligned}$$

Let $x = (x_J, x_{J^c}) \in A_J$, $x' = (x'_J, x'_{J^c}) \in A_J$. Then $(x'_J, x_{J^c}) \in A_J$. Thus,
(a) *$A_J = A^J_J \times A^{J^c}_J$ for $J \neq N$; and*
(b) *$B^i_J = A^{J^c}_J \times D^i_J$ for $N \neq J \neq \{i\}$.*

Proof. In the Appendix.

4. Necessary conditions for optimal price schedules

Suppose the seller—perhaps due to industry regulations, convenience, or practice—is constrained to choosing a price schedule in order to sell his wares. How is the price schedule determined? Theorem 1 identifies a necessary condition for a price schedule to maximize expected revenue within this class. The remainder of the section offers some simple applications of the result.

In Section 6 we use the results of this section to identify sufficient conditions for the optimality of price schedules over *all* IC and IR mechanisms. In those environments the necessary conditions found in this section become sufficient as well.

Theorem 1. *Suppose f satisfies Assumption 1. Let P be a price schedule generating $\{A_J\}_{J\subset N}$. If P is optimal among all price schedules, then for all $J \neq \emptyset$ such that $\int_{A_J} f(x)\,dx > 0$, the following equation must hold*:

$$\int_{A_J} [(n+1)f(x) + x\cdot\nabla f(x)]\,dx - \sum_{i\in J}\int_{B_J^i} f(1,x_{-i})\,dx_{-i} = 0.$$

Proof. [9] We will state the seller's revenue $R(P)$ as a function of P and then compute the first order conditions. A P such that $P_J = 0$ for some $J \neq \emptyset$ cannot be optimal, since this would imply that the seller gains zero on buyers who purchase J and can always do better by charging a slightly higher price (see [1] for a fuller discussion).

For a given price schedule, P, the utility of a buyer of type x is given by $\max_{K\subset N}\{a^K\cdot x - P_K\}$. Thus, the revenue function is given by (utilizing the representation in Eq. (1))

$$\begin{aligned} R(P) = &\sum_{i=1}^{n}\int_{I^{\{i\}^c}} \max_{K\subset N}\{a^K\cdot(1,x_{-i}) - P_K\} f(1,x_{-i})\,dx_{-i} \\ &- \int_{I^n} \max_{K\subset N}\{a^K\cdot x - P_K\}\,[(n+1)f(x) + x\cdot\nabla f(x)]\,dx. \end{aligned}$$

For any set $B\subset\mathbb{R}^n$, let B^0 be its interior and ∂B its boundary.

Note that

$$\frac{\partial}{\partial P_J}\max_{K\subset N}\{a^K\cdot x - P_K\} = \begin{cases} -1 & \text{if } x\in(A_J)^0 \\ 0 & \text{if } x\notin A_J. \end{cases}$$

The derivative may be undefined on $\partial(A_J)\cap(I^n)^0$ but since this set has measure zero in I^n, and f is a density, we ignore this component in what follows.

The assumption that the measure of A_J is strictly positive implies that B_J^i has positive measure in $\mathbb{R}^{n-1}$ and $a^J\cdot(1,x_{-i}) - P_J > 0$, $x_{-i}\in B_J^i$. Therefore,

$$\frac{\partial}{\partial P_J}\max_{K\subset N}\{a^K\cdot(1,x_{-i}) - P_K\} = \begin{cases} -1 & \text{if } x_{-i}\in(B_J^i)^0, \\ 0 & \text{if } x_{-i}\notin B_J^i \end{cases}$$

and, again, is undefined on the ($\mathbb{R}^{n-1}$) measure zero boundary. Since these derivatives converge almost everywhere to a bounded, integrable function, we can apply the Lebesgue Bounded

[9] The present proof, suggested by Jean-Charles Rochet, is much simpler than our original proof.

Convergence Theorem [4, pp. 303–305] to take the derivative of $R(P)$ inside the integral and obtain the first order condition so the optimal selection of P_J must satisfy

$$\begin{aligned}0 = &-\sum_{i\in J}\int_{I^{\{i\}^c}} f(1,x_{-i})1_{(1,x_{-i})\in A_J}\,dx_{-i}\\ &+\int_{I^N}[(n+1)f(x)+x\cdot\nabla f(x)]1_{x\in A_J}\,dx.\end{aligned}$$

Since $1_{(1,x_{-i})\in A_J}=1$ if and only if $x_{-i}\in B^i_J$, the conclusion follows. $\square$

Theorem 1 states, for any market segment A_J (determined by an optimal price schedule P), the integral of $(n+1)f(x)+\nabla f(x)\cdot x$ on the interior of A_J must equal the integral of $f(x)$ restricted to the intersection of A_J with the 'outside boundary' of the set I^n.

When there is only one good, it is well known that the optimal price P must be a zero of the buyer's 'virtual valuation' function $x-\frac{1-F(x)}{f(x)}$ [10]. Theorem 1 generalizes this property. To see this, note that for $n=1$, the condition in Theorem 1 becomes

$$0=\int_P^1\{2f(x)+xf'(x)\}\,dx-f(1)=-\{Pf(P)-(1-F(P))\}.$$

We conclude the section with some applications of Theorem 1. The first order condition yielded by Theorem 1 provides an insight about how to compute the optimal price schedule when the prior density f is the uniform. In this case, $f_i'\equiv 0$. If the optimal price schedule satisfies *SM*, Lemma 2(iii) implies that

$$A_{\{i\}}=[P_{\{i\}},1]\times B^i_i.$$

Thus, the necessary condition determining the price of good i, call it $P_{\{i\}}$, can be expressed as

$$0=\int_{B^i_i}\left\{1-\int_{P_{\{i\}}}^1(n+1)\,dx_i\right\}dx_{-i}.$$

Solving this equation yields that the optimal price schedule (when f represents the uniform distribution) includes single good prices given by

$$P_{\{i\}}=\frac{n}{n+1}.$$

The next theorem shows that there exists a price schedule P, optimal among price schedules, that satisfies *SM*.

Theorem 2. *Suppose f satisfies Assumptions* 1 *and* 2. *Let* $n=2$. *If P satisfies ABS and is optimal among price schedules then P satisfies SM.*

Proof. Suppose $P_N>P_{\{1\}}+P_{\{2\}}$. Applying the definition of A_J (recalling $n=2$) yields

$$\begin{aligned}&A_N=\{x|x_i\geqslant P_N-P_j,\ j\neq i\},\\ &A_{\{1\}}=\{x|x_2\leqslant P_N-P_{\{1\}},\ x_1\geqslant\max\{P_{\{1\}},P_{\{1\}}-P_{\{2\}}+x_2\}\}.\end{aligned}$$

Applying Theorem 1 to the set $J = N$ yields,

$$
\begin{aligned}
0 &= \left[\int_{P_N - P_{\{1\}}}^{1} f_2(x_2) f_1(1) dx_2 + \int_{P_N - P_{\{2\}}}^{1} f_1(x_1) f_2(1) dx_1\right] \\
&\quad - \int_{A_N} \left[\sum_{i=1}^{2} \frac{x_i f_i'(x_i)}{f_i(x_i)} + \frac{3}{2}\right] f_1(x_1) f_2(x_2) dx_1 dx_2 \\
&= \sum_{i=1}^{2} \int_{P_N - P_{\{i\}}}^{1} f_{-i}(x_{-i}) \left\{ f_i(1) - \int_{P_N - P_{\{-i\}}}^{1} \left[\frac{x_i f_i'(x_i)}{f_i(x_i)} + \frac{3}{2}\right] f_i(x_i) dx_i \right\} dx_{-i}.
\end{aligned}
$$

Therefore, at least one element of the sum is non-positive. Suppose it is the element $i = 1$. Assumptions 1 and 2 imply $x_i f_i'(x_i) + \frac{3}{2} f_i(x_i) > 0,\ i = 1, 2$. Therefore, for all $z < P_N - P_{\{2\}}$,

$$
f_1(1) - \int_{z}^{1} \left[\frac{x_1 f_1'(x_1)}{f_1(x_1)} + \frac{3}{2}\right] f_1(x_1)\, dx_1 < 0. \tag{3}
$$

Applying Theorem 1 to $A_{\{1\}}$, then yields

$$
\begin{aligned}
0 &= \int_{0}^{P_N - P_{\{1\}}} f_2(x_2) \\
&\quad \times \left\{ f_1(1) - \int_{\max\{P_{\{1\}}, P_{\{1\}} - P_{\{2\}} + x_2\}}^{1} \left[x_1 f_1'(x_1) + \frac{3}{2} f_1(x_1)\right] dx_1 \right\} dx_2 \\
&\quad - \int_{0}^{P_N - P_{\{1\}}} \int_{\max\{P_{\{1\}}, P_{\{1\}} - P_{\{2\}} + x_2\}}^{1} f_1(x_1) \left[x_2 f_2'(x_2) + \frac{3}{2} f_2(x_2)\right] dx_1\, dx_2 \\
&\leqslant \int_{0}^{P_N - P_{\{1\}}} f_2(x_2) \left\{ f_1(1) - \int_{\max\{P_{\{1\}}, P_{\{1\}} - P_{\{2\}} + x_2\}}^{1} \left[x_1 f_1'(x_1) + \frac{3}{2} f_1(x_1)\right] dx_1 \right\} dx_2 \\
&< 0.
\end{aligned}
$$

The first inequality follows from Assumption 2b. The second inequality from the fact that $P_N > P_{\{1\}} + P_{\{2\}}$ implies $\max\{P_{\{1\}}, P_{\{1\}} - P_{\{2\}} + x_2\} < P_N - P_{\{2\}}$ for $x_2 < P_N - P_{\{1\}}$ and applying (3). A contradiction. □

5. An example where price schedules are suboptimal

The following example illustrates that every price schedule may be dominated in terms of expected revenue by a mechanism involving random assignments. [10]

Let $f(x)$ be a constant on the region above the line joining the points $(0, 1)$ and $(1, 0.5)$ and zero elsewhere. Note that $f(x)$ is (weakly) increasing on the unit square and $\nabla f(x) = 0$ almost everywhere, so the McAfee and McMillan condition [8] is satisfied almost everywhere and a continuous approximation to this density would satisfy the condition everywhere. Note that a separate price for good 1 is never optimal if it is such that the line $x_1 = P_{\{1\}}$ intersects the line $x_1 + x_2 = P_N$ below the line $0.5x_1 + x_2 = 1$ since it must be strictly less than 1 and, in this case, will only draw buyers away from the more profitable bundle priced at P_N. If the intersection

[10] Preston McAfee and McMillan [8] claim that if $n = 2$ and if $3f(x) + \nabla f(x) \cdot x \geqslant 0$, then a price schedule maximizes expected revenue within the class of all IC and IR mechanisms. Our example indicates that their claim is not correct. Thanassoulis [16] independently discovered a related example.

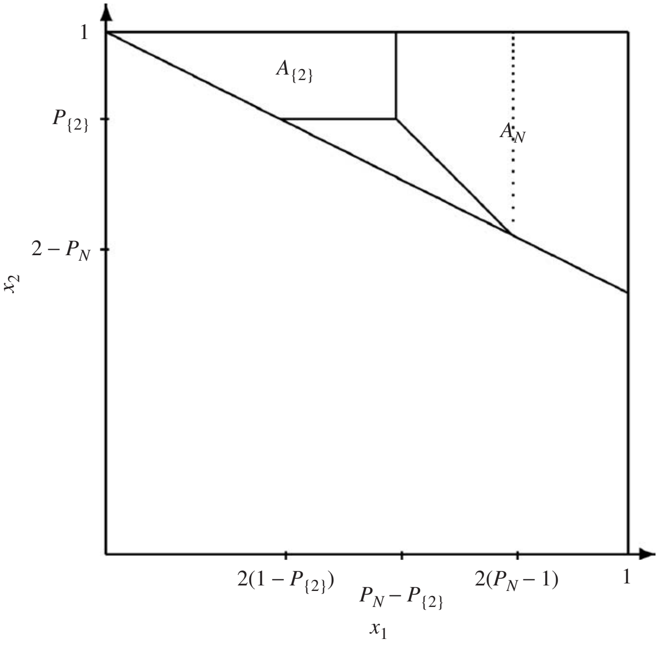

Fig. 1. Price schedules can be dominated.

is above this line, it is conceivable that a price for good one below $2(P_N - 1)$ could add more sales but, intuitively, it would have to be significantly below to add much and the costs from lost bundle sales are correspondingly large. For conciseness, we restrict attention here to two-price mechanisms. A formal analysis which shows that three price mechanisms are not optimal is provided in the Appendix.

A two-price schedule in this framework consists of the prices $(P_{\{2\}}, P_N)$ where $P_{\{2\}}$ is the price for good 2, and P_N is the price for the bundle. A typical two-price schedule is represented in Fig. 1. Buyer types who buy good 2 alone are in the set $A_{\{2\}}$. Types who buy the bundle are in A_N. The triangle below these regions represents types who do not trade. Note that as the figure is constructed, it is assumed both that $P_{\{2\}} < 1$ and that the intersection of the lines $x_2 = P_{\{2\}}$ and $x_2 = P_N - x_1$ lies in the support of buyer types. The former fact is shown below, the latter is true since if the intersection were below, the alternative randomized mechanism illustrated below is easily shown to dominate this mechanism.

Computing the area of a quadrilateral with two parallel sides, the probability mass of $A_{\{2\}}$ is given by [11]

$$\int_{A_{\{2\}}} dx = (1 - P_{\{2\}})(P_N - 1).$$

[11] To be probability density functions, all integrals should be multiplied by a factor of 4. For conciseness, the computations presented here ignore this inessential normalization.

The probability mass of A_N is given by

$$\int_{A_N} dx = \frac{1}{2}\left[(P_N + P_{\{2\}} - 2)(P_N - P_{\{2\}}) + (3 - 2P_N)(P_N - 1/2)\right].$$

Expected revenues from a given $P_{\{2\}}$, P_N are $P_{\{2\}} \int_{A_{\{2\}}} dx + P_N \int_{A_N} dx$. Using the above equations and differentiating with respect to $P_{\{2\}}$ yields a value that is strictly negative at $P_{\{2\}} = 1$ (unless $P_N = 1$ which is easily shown to be dominated) and the optimal $P_{\{2\}}$ as a function of P_N is given by

$$\hat{P}_{\{2\}} = (2P_N - 1)/(3P_N - 2).$$

Substituting this value for $P_{\{2\}}$ into the expected revenue function and maximizing over P_N gives via Mathematica, $\hat{P}_N = 1.26$, $\hat{P}_{\{2\}} = 0.85$ with expected revenues 1.16.

Suppose that instead of the price schedule, a buyer is offered the bundle at price $\hat{P}_N = 1.26$ or a stochastic bundle at price equal to 1 which consists of good two with probability one and good one with probability one-half. This mechanism can also be seen in Fig. 1. Observe that if a buyer type (x_1, x_2) is indifferent between the two choices, then $x_2 + x_1 - P_N = x_2 + 0.5x_1 - 1$, so a buyer of type (x_1, x_2'), $x_2' > x_2$ is also indifferent. Therefore, the set of buyers who choose the full bundle is given by the region above the bottom of the support and to the right of the dotted line. The remaining region represents buyers who choose the random bundle.

The probability mass of these regions are given by $(3 - 2P_N)(P_N - 1/2)/2$ for buyer types who get the full bundle and $(P_N - 1)^2$ for those who buy the random bundle. Therefore, profits from this mechanism are

$$(P_N - 1)^2 + \tfrac{1}{4}(3 - 2P_N)(2P_N - 1)P_N.$$

Evaluated at $P_N = 1.26$, i.e., the optimal two-price schedule, the expression above indicates profits of 1.19, higher than those obtained with the two-price schedule. Optimizing within this class of random mechanisms improves profits slightly to 1.192 and yields a bundle price of 1.28. Notice that this random mechanism is more efficient than the optimal price schedule because the latter never sells good 1 unless it is sold as part of the full bundle. The random mechanism offers at least a chance at good one. This increased efficiency also raises seller revenues. [12]

Compared to price schedules, random mechanisms may be very complicated to compute and difficult to implement. It is therefore of great interest to understand when attention can be restricted to price schedules. This is the subject of the following sections.

6. Revenue-maximizing price schedules

We now identify environments where a price schedule is the optimal mechanism within the class of all IC and IR mechanisms. We find conditions under which a price schedule is the solution to the optimization problem in (2). To that effect we use the following lemma. The lemma resembles a duality result from linear programming. Therefore, we use in its statement the abbreviations CSD,

[12] For computational ease, the density in the example is only continuously differentiable in a strict subset of the unit square with Lebesgue measure one. By continuity of expected revenue with respect to measures, there are continuously differentiable densities that yield similar results.

CSP, FD, and NN, that stand for complementary slackness in the dual, complementary slackness in the primal, feasibility of the dual, and the non-negativity of the dual operator, respectively. [13]

Lemma 3. *Let* $\hat{\pi} \in C$, $\nabla\hat{\pi} \leqslant 1$. *Suppose there exists a linear functional* ω *on* C *such that*

$$\begin{aligned}
&(\mathrm{CSD}) && \langle \hat{\pi}, T - \omega \rangle = 0, \\
&(\mathrm{CSP}) && \langle 1 \cdot x - \hat{\pi}, \omega \rangle = 0, \\
&(\mathrm{FD}) && \langle \pi, T - \omega \rangle \leqslant 0 \quad \forall \pi \in C, \\
&(\mathrm{NN}) && \langle 1 \cdot x - \pi, \omega \rangle \geqslant 0 \quad \forall \pi \in C, \quad \nabla\pi \leqslant 1.
\end{aligned}$$

Then $\hat{\pi} \in \arg\max_{\pi \in C,\, \nabla\pi \leqslant 1} \langle \pi, T \rangle$.

Proof. Using the four hypotheses, it follows that

$$\begin{aligned}
\langle \hat{\pi}, T \rangle &= \langle 1 \cdot x, \omega \rangle \\
&\geqslant \langle 1 \cdot x, \omega \rangle + \langle \pi, T - \omega \rangle \quad \forall \pi \in C \\
&= \langle \pi, T \rangle + \langle 1 \cdot x - \pi, \omega \rangle \quad \forall \pi \in C \\
&\geqslant \langle \pi, T \rangle \quad \forall \pi \in C, \quad \nabla\pi \leqslant 1.
\end{aligned}$$

The first line comes from CSP and CSD, the second line from FD, the third line from linearity, and the last line from NN. □

Let P be the revenue-maximizing price schedule within the class of such schedules. Suppose in addition that P satisfies *SM* and *ABS*. Let $\{A_J\}_{J \subset N}$ be the corresponding market segments, and $\hat{\pi}(x)$ be the expected utility of a buyer of type x in this mechanism.

Theorem 3 essentially shows that under certain conditions on the density f, P is globally optimal if FD and NN in Lemma 3 hold. After stating and proving Theorem 3, we discuss the case $n = 1$, and then offer an alternative representation of one of the theorem's conditions, later used in the applications.

We begin with some useful definitions. Suppose Assumptions 1 and 2 hold. The sets A_J^J, $A_J^{J^c}$, D_J^i are defined in Lemma 2. Define $K_N = 0$ and, for any bundle $J \neq N$, let

$$K_J = \frac{\int_{A_J^{J^c}} \sum_{i \notin J} \dfrac{x_i f_i'(x_i)}{f_i(x_i)} \prod_{k \notin J} f_k(x_k)\, dx_{J^c}}{\int_{A_J^{J^c}} \prod_{k \notin J} f_k(x_k)\, dx_{J^c}}. \tag{4}$$

Since A_J has positive measure, K_J is well-defined. Define the functions $t_J : A_J^J \to \mathbb{R}$ by $t_\emptyset \equiv 0$, and

$$t_J(x_J) = n + 1 + \sum_{i \in J} \frac{x_i f_i'(x_i)}{f_i(x_i)} + K_J. \tag{5}$$

[13] The duality theorem from linear programming yields both necessary and sufficient conditions. Our statement offers only sufficient conditions because we do not possess a complete characterization of the adjoint of the linear functional that corresponds to the constraint, $\nabla\pi \leqslant 1$.

Note that by definition of t_N and (1), the objective function $\langle \pi, T \rangle$ can be expressed as

$$\langle \pi, T \rangle = \sum_{i=1}^{n} \int_{I^{n-1}} \pi(1, x_{-i}) f(1, x_{-i})\, dx_{-i} - \int_{I^n} \pi(x) t_N(x)\, f(x)\, dx, \tag{6}$$

and that when P satisfies *SM* and $x \in A_J$, Assumption 2 and Lemma 2 imply

$$t_J(x_J) \int_{A_J^{J^c}} f(x)\, dx_{J^c} = \int_{A_J^{J^c}} t_N(x) f(x)\, dx_{J^c}. \tag{7}$$

Assumption 2 implies $t_N(\cdot) \geqslant 0$; therefore $t_J(\cdot) \geqslant 0$ for all J.

Before stating Theorem 3, two additional assumptions are necessary:

Assumption 3. Fix a density f and a price schedule P. Let $\{A_J\}$ and $\{B_J^i\}$ be the market segments and their boundaries corresponding to P and let $\{t_J\}$ be the functions defined using (4) and (5).[14] Then,

$$\begin{aligned}
&\forall\, \pi \in C, \nabla\pi \leqslant 1, \\
&\sum_{J \subset N} \left\{ \sum_{i \in J} \int_{B_J^i} \left[1 \cdot (1, x_{-i}) - \pi(1, x_{-i}) \right] f(1, x_{-i})\, dx_{-i} \right. \\
&\qquad \left. - \int_{A_J} [1 \cdot x - \pi(x)]\, t_J(x_J) f(x)\, dx \right\} \geqslant 0.
\end{aligned}$$

Assumption 4. Fix a density f and a price schedule P. Let $\{A_J\}$ be the market segments corresponding to P and let $\{t_J\}$ be the functions defined using (4) and (5). Then,

$$\forall \pi \in C, \quad \sum_{J \subset N} \int_{A_J} \pi(x)[t_N(x) - t_J(x_J)] f(x)\, dx \geqslant 0.$$

Assumptions 1 and 2 do not imply NN and FD in general, however, the next section identifies environments where, given Assumptions 1 and 2, there exists a price schedule P, optimal among all price schedules, that satisfies *SM*, *ABS* and Assumptions 3 and 4. It will then follow from Lemma 3 that P maximizes expected revenue over all IC and IR mechanisms. Assumptions 3 and 4 serve the role of ensuring that conditions NN and FD respectively are satisfied more generally. They are by no means transparent. Their role, here, is mainly to illustrate that the problem of checking the optimality of price schedules can be distilled to checking these conditions. We defer discussion of the assumptions until after the theorem.

Theorem 3. *Let the density f satisfy Assumptions* 1 *and* 2. *Suppose P is optimal among all price schedules and satisfies SM and ABS.*

1. *Then, there exists a linear functional ω such that conditions CSP and CSD hold.*
2. *If in addition Assumptions* 3 *and* 4 *hold at the optimal P, ω satisfies conditions FD and NN.*

[14] Note that all these objects depend on P. Note as well that Assumptions 3 and 4 require the inequalities to hold for a wide class of functions, π, not just the candidate optimal $\hat{\pi}$.

Proof. The proof proceeds by defining a candidate linear functional and then showing that, under the hypotheses of the theorem, all four conditions of Lemma 3 are satisfied where $\hat{\pi}(x) = \max_{J \subset N}(a^J \cdot x - P_J)$ is the equilibrium utility of a buyer of type x offered a price schedule P. Throughout, the market segments, $\{A_J\}$, the boundaries, $\{B^i_J\}$, and the functions, $\{t_J\}$, are determined by P.

We first prove an intermediate result that follows from Theorem 1. At the optimal P, for any $J \subset N$,

$$
\begin{aligned}
0 &= \sum_{i \in J} \int_{B^i_J} f(1, x_{-i})\, dx_{-i} - \int_{A^J_J} \int_{A^{J^c}_J} t_N(x) f(x)\, dx_{J^c}\, dx_J \\
&= \sum_{i \in J} \int_{A^{J^c}_J} \int_{D^i_J} \prod_{k \neq i} f_k(x_k) f_i(1)\, dx_{-i} - \int_{A^J_J} \int_{A^{J^c}_J} t_J(x_J) f(x)\, dx \\
&= \int_{A^{J^c}_J} \prod_{k \notin J} f_k(x_k)\, dx_{J^c} \left[\sum_{i \in J} \int_{D^i_J} \prod_{k \in J, k \neq i} f_k(x_k) f_i(1)\, dx_{J/i} \right. \\
&\quad \left. - \int_{A^J_J} t_J(x_J) \prod_{j \in J} f_j(x_j)\, dx_J \right].
\end{aligned}
$$

The first equality follows from Theorem 1 and (6). The second equality follows from Lemma 2, independence, and (7). The final equality collects the terms in $k \notin J$. *ABS* implies that the first term is strictly positive. The case for $J = \{i\}$ follows identically noting that $B^i_J = A^{J^c}_J$. Therefore,

$$
\forall J \neq \emptyset, \quad 0 = \sum_{i \in J} f_i(1) \left[\int_{D^i_J} \prod_{j \in J, j \neq i} f_j(x_j)\, dx_{J/i} \right] - \int_{A^J_J} t_J(x_J) \prod_{j \in J} f_j(x_j)\, dx_J, \tag{8}
$$

where it is to be understood that $\int_{D^i_J} \prod_{j \in J, j \neq i} f_j(x_j)\, dx_{J/i} = 1$ if $J = \{i\}$.

We now prove the theorem. The following expression defines the linear function ω. For any continuous function $\pi : I^n \to \mathbb{R}$,

$$
\langle \pi, \omega \rangle = \sum_{J \subset N} \left\{ \sum_{i \in J} \left[\int_{B^i_J} \pi(1, x_{-i}) f(1, x_{-i})\, dx_{-i} \right] - \int_{A_J} \pi(x) t_J(x_J) f(x)\, dx \right\}. \tag{9}
$$

To show CSP, $x \in A_J$ implies that $1 \cdot x - \hat{\pi}(x) = a^{J^c} \cdot x + P_J$. This implies that for $x \in A_J$, $1 \cdot x - \hat{\pi}(x)$ does not vary with x_J and, in particular, $1 \cdot (1, x_{-i}) - \hat{\pi}(1, x_{-i}) = 1 \cdot x - \hat{\pi}(x)$. Therefore, applying the definition in (9)

$$
\begin{aligned}
\langle 1 \cdot x - \widehat{\pi}, \omega \rangle = \sum_{J \subset N} &\int_{A^{J^c}_J} (a^{J^c} \cdot x + P_J) \prod_{j \notin J} f_j(x_j)\, dx_{J^c} \\
&\times \left(\sum_{i \in J} f_i(1) \int_{D^i_J} \prod_{j \in J, j \neq i} f_j(x_j)\, dx_{J/i} \right. \\
&\left. - \int_{A^J_J} t_J(x_J) \prod_{j \in J} f_j(x_j)\, dx_J \right) \\
= 0.&
\end{aligned}
$$

The first equality exploits Assumption 2a and Lemma 2(iii). The second equality applies Eq. (8).

To prove CSD, we show first that, for any i, $I^{\{i\}^c}$ and $\cup_{J,i\in J}\ B^i_J$ are equivalent. Note that *ABS* implies $P_{\{i\}} < 1$. Therefore, $(1, x_{-i}) \notin A_\emptyset$ for any i. Suppose there exists i and x_{-i} such that $x_{-i} \notin B^i_J$ for any J containing i. Then there is a $K \neq \emptyset$, $i \notin K$ such that

$$a^K \cdot (1, x_{-i}) - P_K > a^{K\cup\{i\}} \cdot (1, x_{-i}) - P_{K\cup\{i\}}.$$

But this implies

$$P_{K\cup\{i\}} > 1 + P_K > P_K + P_{\{i\}}$$

which violates *SM*. Thus $I^{\{i\}^c} = \cup_{J,i\in J}\ B^i_J$. Noting that $\cup_{i\in N} \cup_{J,i\in J}\ B^i_J = \cup_{J\subset N} \cup_{i\in J} B^i_J$ yields

$$\sum_{i=1}^{n} \int_{I^{\{i\}^c}} \pi(1, x_{-i}) f(1, x_{-i})\, dx_{-i} = \sum_{J\subset N} \sum_{i\in J} \int_{B^i_J} \pi(1, x_{-i}) f(1, x_{-i})\, dx_{-i}.$$

Therefore, using this expression in the definition of $\langle \pi, \omega\rangle$, we have (using (6))

$$\langle \pi, T - \omega\rangle = -\sum_{J\subset N} \int_{A_J} \pi(x)[t_N(x) - t_J(x_J)] f(x)\, dx_{J^c}\, dx_J. \tag{10}$$

Since $x \in A_J$ implies that $\hat{\pi}(x) = a^J \cdot x - P_J$ which does not vary with x_i, $i \in J^c$, and $x \in A_\emptyset$ implies $\hat{\pi}(x) = 0$, (10) becomes

$$-\sum_{J\subset N, J\neq\emptyset} \int_{A^J_J} \widehat{\pi}(x) \left\{ \int_{A^{J^c}_J} [t_N(x) - t_J(x_J)] f(x)\, dx_{J^c} \right\} dx_J = 0.$$

The equality follows using (7), *SM* and Assumption 2a.

FD follows immediately from Assumption 4 since this implies Expression (10) is non-positive for all $\pi \in C$.

The definition of ω implies $\langle 1 \cdot x - \pi, \omega\rangle$ is the same as the left side of the inequality in Assumption 3. Thus Assumption 3 implies NN is satisfied. □

Consider briefly the content of Theorem 3 in the special case of one good, i.e., $n = 1$. Assumption 4 is trivially satisfied because the only case to consider is $J = N = \{1\}$. Let P solve $0 = P - \frac{1-F(P)}{f(P)}$. The left side of the inequality in Assumption 3 can be written as

$$\begin{aligned}
&(1 - \pi(1)) f(1) - \int_P^1 [x - \pi(x)]\, t_N(x) f(x)\, dx \\
&\quad = \int_P^1 [1 - \nabla\pi(x)] \left[x - \frac{1 - F(x)}{f(x)}\right] f(x)\, dx,
\end{aligned}$$

by noting that $t_N(x) f(x) = 2f(x) + x f'(x) = \frac{d}{dx}\{x f(x) - (1 - F(x))\}$ and integrating by parts. Assumption 2 implies that $\{x - \frac{1-F(x)}{f(x)}\}$ is positive for all $x \geqslant P$ so the integrand is positive over the region of integration for all $\pi \in C$, $\nabla\pi \leqslant 1$. Thus Assumption 3 is satisfied as well. When $n = 1$, both *SM* and the independence of f are trivially satisfied and thus Assumptions 1 and 2 are sufficient to conclude the optimality of a price schedule. Theorem 3, therefore, specializes although in a somewhat weaker form (because it requires Assumption 2b), the known result for $n = 1$. Assumption 2b remains useful even for $n = 1$ because it implies that the requirement that the buyer expected utility function be convex, does not bind at the optimal solution. (Recall that $\pi \in C$ implies π is convex.) If Assumption 2b failed, so that $x - \frac{1-F(x)}{f(x)}$ becomes positive and then negative, setting $1 - \nabla\pi(x) = 0$ when it turns negative would satisfy Assumption 3 but would violate convexity. For the one-good case, without Assumption 2b, the solution involves the 'ironing' approach on $t_N(x)$ to eliminate the double-crossing. (See, for example, [14]) We conjecture that Assumption 2b plays a similar role in the general n-good case.

Assumptions 3 and 4 are only indirectly assumptions on the primitives of the environment (the f_is) since, in principle, testing whether they hold requires determining the optimal price schedule for the given f_is and then checking the conditions on the resulting market segments. The usefulness of Theorem 3, thus, relies to a large extent on the feasibility of verifying Assumptions 3 and 4. Admittedly, the assumptions do not possess a simple economic interpretation. However, the conditions *are* implied by a more familiar mathematical property. Inspection of Assumption 4 reveals that it is a type of covariance condition. Once the market segments have been constructed, if every feasible mechanism, π covaries positively with $t_N - t_J$ over A_J for all subsets J, then FD in Lemma 3 is satisfied. For some families of distributions, this feature follows readily. For example, if $F^i(x_i) = x_i^{\alpha}$, then

$$t_J(x_J) = n + 1 + n(\alpha - 1) \quad \forall J$$

so $t_N - t_J = 0$, and the condition follows directly. In other circumstances, more knowledge about the behavior of $t_N - t_J$ and the structure of the market segments will be required. Theorem 4 below offers an example of such an application.

We conclude this section with a lemma and corollary that play the role analogous to integration by parts in the one-dimensional case. Under Assumption 2b, these results offer a second covariance condition that implies Assumption 3. The next section uses the new condition to verify Assumption 3 in two different environments.

For $x \in A_J$, $i \in J$, define

$$T_J^i(x_i, x_{-i}) = f(1, x_{-i}) - \int_{x_i}^{1} t_J^i(v, x_{J/i}) f(v, x_{-i})\, dv \tag{11}$$

for some

$$t_J^i : A_J^J \to \mathbb{R}, \quad t_J^i(x_J) \geqslant 0, \quad \sum_{i \in J} t_J^i(x_J) = t_J(x_J).$$

Setting $t_J^i \equiv t_J$ for some $i \in J$, and $t_J^k \equiv 0$ for $k \neq i, k \in J$ illustrates that there always exists a collection of t_J^i's that satisfy the conditions in the definition. By construction and Assumption 2b, $T_J^i(x_i, x_{-i})$ is increasing in x_i and $T_J^i(1, x_{-i}) = f(1, x_{-i})$.

By Lemma 1, we can define the functions $x_i : I^{n-1} \to \mathbb{R}$ by

$$x_i(x_{-i}) = \min\{x_i \mid (x_i, x_{-i}) \in \cup_{J \subset N, i \in J} A_J\}. \tag{12}$$

Lemma 4. *Let Assumptions* 1 *and* 2 *hold. Suppose P is a price schedule satisfying SM and ABS and let $\{T_J^i, t_J^i\}$ be a collection of functions satisfying* (11) *and let $x_i(x_{-i})$ be defined as in* (12).

Then,

(i)
$$\langle \pi, \omega \rangle = \sum_{J \subset N} \sum_{i \in J} \int_{B_J^i} \left\{ \int_{x_i(x_{-i})}^1 [\pi(1, x_{-i}) - \pi(x)] t_J^i(x_J) f(x)\, dx_i \right.$$
$$\left. + \pi(1, x_{-i}) T_J^i(x_i(x_{-i}), x_{-i}) \right\} dx_{-i}.$$

If, *in addition*, P *is optimal among price schedules*,

(ii) $0 = \sum_{i \in J} \int_{D_J^i} T_J^i(x_i(x_{J/i}, x_{J^c}), x_{J/i}, x_{J^c})\, dx_{J/i}, \forall J \subset N, x_{J^c} \in A_J^{J^c}$; *and*

(iii) $T_J^i(x_i(x_{-i}),\ x_{-i}) = 0,\ \forall J = \{i\},\ \forall x_{-i} \in B_J^i$.

Proof. In the Appendix.

The following Corollary, based on Lemma 4(i), is useful to verify that Assumption 3 holds. It requires that the functions $T_J^i(x_i(\cdot), \cdot)$ covary positively with any increasing function over B_J^i.

Corollary 1. *Let Assumptions* 1 *and* 2 *hold. Suppose* P *satisfies SM and ABS*. *Let* $\{T_J^i, t_J^i\}$ *be a collection of functions satisfying* (11), *and let* $x_i(x_{-i})$ *be defined as in* (12). *If for all* $J \subset N$, *for all* $i \in J$, *for all* $\pi : I^{n-1} \to \mathbb{R}_+$, π *increasing*,
$$\int_{B_J^i} \pi(x) T_J^i(x_i(x), x)\, dx \geqslant 0,$$
then Assumption 3 *is satisfied*.

Proof. If $\pi \in C$, $\nabla\pi \leqslant 1$, then, for all $x \in I^n$, convexity implies
$$1 \cdot (1, x_{-i}) - \pi(1, x_{-i}) - (1 \cdot x - \pi(x)) \geqslant 0.$$
Since $t_J(x_J) \geqslant 0$ by Assumption 2, applying Lemma 4(i) gives
$$\langle 1 \cdot x - \pi, \omega \rangle \geqslant \sum_{J \subset N} \sum_{i \in J} \int_{B_J^i} (1 \cdot (1, x_{-i}) - \pi(1, x_{-i})) T_J^i(x_i(x_{-i}), x_{-i})\} \, dx_{-i}.$$
But $\pi \in C$, $\nabla\pi \leqslant 1$, implies $1 \cdot (1, x_{-i}) - \pi(1, x_{-i})$ is increasing so the conclusion follows. □

Note that Lemma 4(iii), under the hypotheses of Theorem 3, implies that for $J = \{i\}$
$$\int_{B_J^i} \pi(x_{-i}) T_J^i(x_i(x_{-i}), x_{-i})\, dx_{-i} = 0 \quad \forall \pi.$$
Therefore, we only need to check the hypothesis of Corollary 1 for J, $|J| > 1$. Of course, in the case $n = 1$, the only relevant bundle has cardinality one, so Assumption 3 follows directly from Assumptions 1 and 2.

The next section illustrates the usefulness of Theorem 3 by constructing the functions $\{T_J^i, t_J^i\}$ so that the hypothesis of Corollary 1 is satisfied.

7. Applications

Although the features of revenue-maximizing mechanisms are well-understood in the one-good case, far less is known in the very simple generalization to multiple goods. In this section, we use Theorem 3 to identify environments in which price schedules are optimal over all IC and IR mechanisms. We first provide simple sufficient conditions for the optimality of posted price mechanism in a general class of two-good models. Then we study a more specialized case for $n = 3$ with a uniform distribution of valuations.

Corollary 1 illustrates that Assumptions 3 and 4 can be confirmed by checking whether certain functions covary positively with increasing functions over the market segments or projections of the market sections. There exist a variety of results that inform us when two functions covary positively. For example, if a function $g : [a, b] \to \mathbb{R}$ integrates to zero and crosses zero once then the integral of the product of g with any increasing, positive function can be signed. This fact is used frequently in this section to prove that the covariance conditions are satisfied.

Theorem 4. *Let* $N = \{1, 2\}$, $(n = 2)$ *and let* f *satisfy Assumptions* 1 *and* 2. *Suppose* $\frac{x_i f_i'(x_i)}{f_i(x_i)}$ *is increasing for* $i = 1, 2$. *If* P *is optimal among price schedules and satisfies ABS, then it is optimal over all IR and IC mechanisms.*

Proof. By Theorem 2, P satisfies *SM*. Fig. 2 illustrates the typical form of A_J and B^i_J given *SM*:

$$\begin{aligned}
A_{\{i\}} &= \{(x_i, x_{-i}) \mid x_{-i} \leqslant P_N - P_{\{i\}}, x_i \geqslant P_{\{i\}}\},\\
A_N &= \{x \mid x_2 \geqslant P_N - P_{\{1\}}, x_1 \geqslant \max\{P_N - P_{\{2\}}, P_N - x_2\}\},\\
B^i_{\{i\}} &= \{x_{-i} \mid x_{-i} \in [0, P_N - P_{\{i\}}]\},\\
B^i_N &= \{x_{-i} \mid x_{-i} \in [P_N - P_{\{i\}}, 1]\}.
\end{aligned}$$

We first verify that Assumption 4 holds by showing each component in the summation is non-negative. This follows trivially for $J = \emptyset$ and $J = N$. For $J = \{1\}$, say, *SM* implies

$$\begin{aligned}
&\int_{A_{\{1\}}} \pi(x)\{t_N(x) - t_J(x_J)\} f(x)\, dx\\
&\quad = \int_{P_{\{1\}}}^1 \int_0^{P_N - P_{\{1\}}} \pi(x) \left[\frac{x_2 f_2'(x_2)}{f_2(x_2)} - K_{\{1\}}\right] f(x)\, dx\\
&\quad \geqslant \int_{P_{\{1\}}}^1 \left\{ \int_0^{P_N - P_{\{1\}}} \pi(x) f_2(x_2)\, dx_2 \right.\\
&\qquad \left. \times \int_0^{P_N - P_{\{1\}}} \left[\frac{x_2 f_2'(x_2)}{f_2(x_2)} - K_{\{1\}}\right] f_2(x_2)\, dx_2 \right\} f_1(x_1)\, dx_1\\
&\quad = 0.
\end{aligned}$$

The first equality follows by definition of $A_{\{1\}}$ and t_J, the inequality follows because π and $\frac{x_2 f_2'(x_2)}{f_2(x_2)} - K_{\{1\}}$ are both increasing in x_2 and, so, covary positively. The final equality follows by definition of $K_{\{1\}}$. A similar argument holds for $J = \{2\}$.

We now verify Assumption 3. We do so by defining the t^i_J's required by Corollary 1. Lemma 4 and Corollary 1 imply we need to check the result only for $J = N$. Lemma 5 shows that the proposed definition satisfies the conditions of (11). The spirit of the definitions can be seen in Fig. 2.

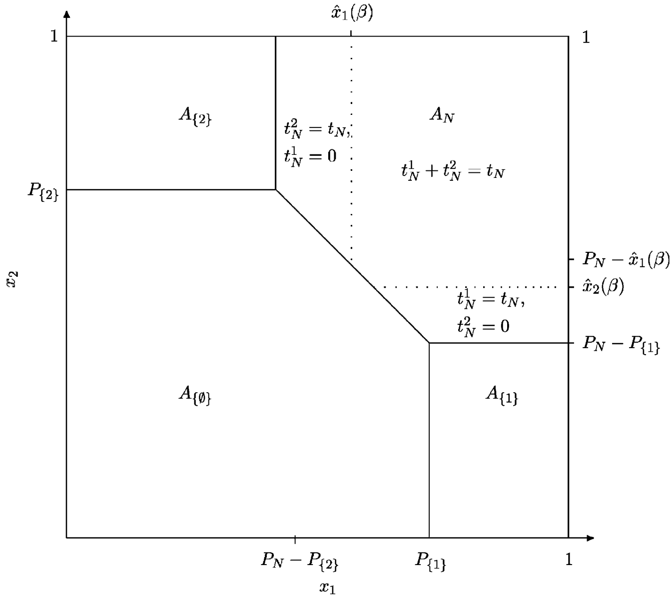

Fig. 2. Construction of t_N^i for Theorem 4.

Define

$$w^i(z;\beta) \equiv f_i(1) - \int_z^1 \left[\frac{v f_i'(v)}{f_i(v)} + 3/2 + (-1)^i \beta\right] f_i(v)\, dv. \tag{13}$$

Assumption 2 and $\beta \in (-1/2, 1/2)$ imply that w^i is strictly increasing in z. Since $w^i(1;\beta) > 0 > w^i(0;\beta)$, $\hat{x}_i(\beta)$ can be uniquely defined implicitly by

$$w^i(\hat{x}_i(\beta);\beta) = 0. \tag{14}$$

The next lemma characterizes the decomposition of t_J that is used to apply Corollary 1. The proof is in the Appendix.

Lemma 5. *For $\beta \in (-1/2, 1/2)$, define $\hat{x}_i(\beta)$ by* (14) *and define $t_N^i(x;\beta)$ by*

$$\begin{aligned} t_N^i(x;\beta) &= t_N(x), \quad x_{-i} \leqslant \hat{x}_{-i}(\beta), \\ &= \frac{x_i f_i'(x_i)}{f_i(x_i)} + 3/2 + (-1)^i \beta, \quad x_{-i} \geqslant \hat{x}_{-i}(\beta),\ \ x_i \geqslant \hat{x}_i(\beta), \\ &= 0, \quad x_i \leqslant \hat{x}_i(\beta). \end{aligned}$$

There exists $(\underline{\beta}, \overline{\beta}) \subset (-1/2, 1/2)$ such that for all $\beta \in (\underline{\beta}, \overline{\beta})$, $t_N^i(x;\beta)$ is defined for $x \in A_N(ae)$, $t_N^i(x;\beta) \geqslant 0$, and $t_N^1(x;\beta) + t_N^2(x;\beta) = t_N(x)$ for $x \in A_N(ae)$.

Thus, the conditions in (11) are satisfied and $T_N^i(x;\beta)$ is defined as in (11) using $t_N^i(\cdot;\beta)$. We prove that Assumption 3 is satisfied by showing the existence of a β such that the hypothesis of Corollary 1 holds for $T_N^i(x;\beta)$.

Part (i) of the next lemma confirms a single-crossing property and Part (ii) shows how to address the potential asymmetries of the f_is. The proof is in the Appendix.

Lemma 6. *For* i=1,2,

(i) $$T_N^i(x_i(x_{-i}), x_{-i}; \beta) \leqslant 0, \quad x_{-i} \leqslant \hat{x}_{-i}(\beta),$$
$$T_N^i(x_i(x_{-i}), x_{-i}; \beta) \geqslant 0, \quad x_{-i} \geqslant \hat{x}_{-i}(\beta),$$

(ii) $$\exists \beta \in (\underline{\beta}, \overline{\beta}) \text{ such that } \int_{P_N - P_{\{i\}}}^1 T_N^i(x_i(x_{-i}), x_{-i}; \beta)\, dx_{-i} = 0.$$

Thus, selecting a β satisfying Lemma 6(ii), for any increasing, positive π,

$$\begin{aligned}
&\int_{P_N - P_{\{2\}}}^1 \pi(x_1, 1) T_N^2(x_1, x_2(x_1); \beta)\, dx_1 \\
&\quad = \int_{P_N - P_{\{2\}}}^{\hat{x}_1(\beta)} \pi(x_1, 1) T_N^2(x_1, x_2(x_1); \beta)\, dx_1 + \int_{\hat{x}_1(\beta)}^1 \pi(x_1, 1) T_N^2(x_1, x_2(x_1); \beta)\, dx_1 \\
&\quad \geqslant \pi(\hat{x}_1(\beta), 1) \int_{P_N - P_{\{2\}}}^1 T_N^2(x_1, x_2(x_1); \beta)\, dx_1 \\
&\quad = 0.
\end{aligned}$$

The inequality follows from the single-crossing property of $T_N^2(x_1, x_2(x_1); \beta)$ in x_1 (Lemma 6(i)) and the monotonicity and non-negativity of π. The equality follows from Lemma 6(ii). A similar argument follows for T_N^1. Applying Corollary 1 and Theorem 3, the conclusion follows. □

The assumption that $\frac{x_i f_i'(x_i)}{f_i(x_i)}$ is increasing is unusual, and appears to arise specifically because of the multidimensional character of the problem. As far as we know, it is not a commonly imposed restriction. The conditions are strong but not empty. The class of distributions, $\Pi_{i=1}^n x_i^{\alpha_i}$, $\alpha_i \geqslant 0$, and the class of distributions, $K \Pi_{i=1}^n (e^{\alpha_i x_i} - 1)$, $\alpha_i \geqslant 0$, $K > 0$, both satisfy this condition and Assumption 2. [15]

Remark. Note that the program in Theorem 1, selecting an optimal price schedule among all such schedules, is typically not concave. Thus, in general the necessary conditions obtained need not be sufficient. Since Theorem 4 solves an optimization problem over a larger feasible set, it illustrates that, for $n = 2$, if f satisfies Assumptions 1 and 2, and if $\frac{x f_i'(x)}{f_i(x_i)}$ is increasing, the necessary conditions in conjunction with *ABS* are also sufficient. (The requirement that all bundles be sold with positive probability serves, in part, to rule out the possibility that the first order conditions are identifying minima to the seller's problem.)

Checking that Assumptions 3 and 4 are satisfied becomes progressively more challenging as n increases. If more structure is placed on the problem, however, it is still possible to verify both assumptions. We do so for $n = 3$ in the following theorem.

---

[15] The family of beta distributions also satisfy the assumption for $\beta \geqslant 1$. Neither the normal nor the Gamma distributions satisfy it. See [2, pp. 37–40].

Theorem 5. *Let $N = \{1, 2, 3\}$, $(n = 3)$ and let $f_i = 1, i = 1, 2, 3$. The price schedule P, $P_{\{i\}} = 3/4$, $P_{\{i,j\}} \approx 1.14$, $j \neq i$, $P_N \approx 1.22$ is optimal among all such schedules. It is also optimal over all IR and IC mechanisms.*

Proof. Direct computation shows that the best price schedule is $P_{\{i\}} = 3/4$, $P_{\{i,j\}} \approx 1.14$, $j \neq i$, $P_N \approx 1.22$. Note that this satisfies *SM*, *ABS* and is symmetric. [16]

Assumption 4 holds because the uniform density implies $f_i'(x) = 0$ and thus $t_N(x) - t_J(x) = 0$ for all J and x.

It remains to verify that Assumption 3 holds. The proof proceeds as follows. First, the t_J^i's are defined and it is shown they satisfy (11). Second, it is verified that Assumption 3 holds for two and three good bundles. The symmetry of prices implies that if $(v, w, y) \in A_{\{1\}}$, then $(w, v, y) \in A_{\{2\}}$ and so on. Thus, we can restrict attention to the argument for one good and bundles containing it, say, good 3.

The next lemma exploits the symmetry of the optimal price schedule to construct the decomposition of $t_J(x_J)$ that is used to apply Corollary 1. The proof is in the Appendix.

Lemma 7. *Define*

$$S_J^i = \{x \in A_J \mid x_i \geqslant x_k, i \in J, k \neq i\}.$$

$$t_J^i(x_J) = t_J(x_J) 1_{S_J^i}.$$

Then $t_J^i(x)$ is defined for all $x \in A_J$, $t_J^i(x) \geqslant 0$, and $\sum_{i \in J} t_J^i(x) = t_J(x)$ for all $x \in A_J$.

Define

$$\underline{x}_i(x_{-i}) = \min\{x_i \mid (x_i, x_{-i}) \in S_J^i, i \in J\}.$$

Recall from Lemma 2 that, for $x \in A_J$, both $\underline{x}_i(\cdot)$ and $x_i(\cdot)$ are independent of components in J^c. Since $t_J^i(x_J) = 0$ for $x_i < \underline{x}_i(x_{J/i}, z)$, $z \in A_J^{J^c}$, and $t_J^i(x_J) = 4$ for $x_i > \underline{x}_i(x_{J/i}, z)$, $z \in A_J^{J^c}$, (because of the uniform independent density assumption) direct integration in (11) yields

$$T_J^i(x_i(x_{-i}), x_{-i}) = 4\underline{x}_i(x_{-i}) - 3, \quad x_{-i} \in B_J^i. \tag{15}$$

Two-good bundles: For (say) $J = \{1, 3\}$, *SM* and Lemma 2(i), imply that $B_{\{1,3\}}^3$ is bounded on the interior of $[0, 1] \times [0, 1]$ by $B_{\{3\}}^3$ and $B_{\{1,2\}}^3$. Using the definition of $B_{\{3\}}^3$ and the fact that $x \in A_{\{1,3\}} \cap A_N$ implies $x_2 = P_N - P_{\{1,3\}}$ yields

$$B_{\{1,3\}}^3 = \{(x_1, x_2) \mid x_1 \geqslant P_{\{1,3\}} - P_{\{3\}}, x_2 \leqslant P_N - P_{\{1,3\}}\},$$

and, applying the definition of $S_{\{1,3\}}^3$,

$$\begin{aligned} \underline{x}_3(x_{-3}) &= P_{\{1,3\}} - x_1, \quad x_1 \in [P_{\{1,3\}} - P_{\{3\}}, P_{\{1,3\}}/2] \\ &= x_1, \quad x_1 \in [P_{\{1,3\}}/2, 1] \end{aligned}$$

which does not vary with x_2. This implies that for $x_{-3} \in B_{\{1,3\}}^3$, $\underline{x}_3(x_{-3}) < P_{\{3\}}$ if and only if $x_1 < P_{\{3\}}$ and, thus,

$$T_{\{1,3\}}^3(x_3(x_{-3}), x_{-3}) < 0 \Leftrightarrow x_1 < P_{\{3\}}. \tag{16}$$

[16] The computed price schedule also satisfies the necessary conditions derived in Theorem 1.

Since $T^3_{\{1,3\}}(x_3(x_{-3}), x_{-3})$ does not vary with x_2 for $x_{-3} \in B^3_{\{1,3\}}$, Lemma 4(ii) along with symmetry implies

$$\forall x_2 \leqslant P_N - P_{\{1,3\}}, \quad \int_{P_{\{1,3\}}-P_{\{3\}}}^{1} T^3_{\{1,3\}}(x_3(x_{-3}), x_{-3})\, dx_1 = 0. \tag{17}$$

Thus, for all increasing π,

$$\begin{aligned}
&\int_{B^3_{\{1,3\}}} \pi(x_{-3}, 1) T^3_{\{1,3\}}(x_3(x_{-3}), x_{-3})\, dx_1\, dx_2 \\
&\quad \geqslant \int_0^{P_N - P_{\{1,3\}}} \pi(P_{\{3\}}, x_2, 1) \int_{P_{\{1,3\}}-P_{\{3\}}}^{1} T^3_{\{1,3\}}(x_3(x_{-3}), x_{-3})\, dx_1\, dx_2 \\
&\quad = 0.
\end{aligned}$$

The inequality follows from the definition of $B^3_{\{1,3\}}$, (16) and the restriction to π increasing. The equality follows from (17). The symmetric argument shows the same inequality for $J \in \{\{2,3\}, \{1,2\}\}$.

Three-good bundle: Fig. 3 in the Appendix represents the set $B^3_N \cap \{(x_1, x_2) \mid x_1 \geqslant x_2\}$ in (x_1, x_2) space. By *SM* and Lemma 2(i), B^3_N is bounded on the interior of $[0,1] \times [0,1]$ by the sets $B^3_{\{1,3\}}$, $B^3_{\{3\}}$, $B^3_{\{2,3\}}$. Thus,

$$B^3_N = \{(x_1, x_2) \mid x_1 \geqslant P_N - P_{\{2,3\}}, x_2 \geqslant \max\{P_N - P_{\{1,3\}}, P_N - P_{\{3\}} - x_1\}\}.$$

We use the following lemma, shown in the Appendix. For $i = 1, 2$, $j \neq i$, define the function,

$$G^i(x_i) \equiv \int_{\max\{x_i, P_N - P_{\{3\}} - x_i\}}^{1} T^3_N(x_3(x_{-3}), x_{-3})\, dx_j. \tag{18}$$

Lemma 8. *For* $i = 1, 2$,

(i) $T^3_N(x_3(x_{-3}), x_{-3}) > 0 \Leftrightarrow \max\{x_1, x_2\} > P_{\{3\}}$.

(ii) $\exists a < P_{\{3\}}$ *such that* $G^i(x_i) > 0 \Leftrightarrow x_i > a$.

(iii) $$\int_{P_N - P_{\{1,3\}}}^{1} G^i(x_i)\, dx_i = 0.$$

Let π be any increasing, positive function. We can now apply the following inequalities:

$$\begin{aligned}
&\int_{B^3_N} \pi(x_{-3}, 1) T^3_N(x_3(x_{-3}), x_{-3}) 1_{\{x_1 \geqslant x_2\}}\, dx_1\, dx_2 \\
&\quad = \int_{P_N - P_{\{1,3\}}}^{1} \int_{\max\{x_2, P_N - P_{\{3\}} - x_2\}}^{1} \pi(x_{-3}, 1) T^3_N(x_3(x_{-3}), x_{-3})\, dx_1\, dx_2 \\
&\quad \geqslant \int_{P_N - P_{\{1,3\}}}^{1} \int_{\max\{x_2, P_N - P_{\{3\}} - x_2\}}^{1} \pi(P_{\{3\}}, x_2, 1) T^3_N(x_3(x_{-3}), x_{-3})\, dx_1\, dx_2 \\
&\quad = \int_{P_N - P_{\{1,3\}}}^{1} \pi(P_{\{3\}}, x_2, 1) G^2(x_2)\, dx_2
\end{aligned}$$

$$
\begin{aligned}
&= \int_{P_N - P_{\{1,3\}}}^{a} \pi(P_{\{3\}}, x_2, 1) G^2(x_2)\, dx_2 + \int_a^1 \pi(P_{\{3\}}, x_2, 1) G^2(x_2)\, dx_2 \\
&\geqslant \pi(P_{\{3\}}, a, 1) \int_{P_N - P_{\{1,3\}}}^{1} G^2(x_2)\, dx_2 \\
&= 0.
\end{aligned}
$$

The equality follows by applying the characterization of B_N^3. The first inequality follows because π is positive and increasing in x_1 and applying Lemma 8(i). The next equality applies the definition of $G^2(\cdot)$ in (18). The second inequality follows from Lemma 8(ii) and because $\pi(P_{\{3\}}, x_2, 1)$ is positive and increasing in x_2. The final equality follows from Lemma 8(iii).

A symmetric argument shows

$$
\int_{B_N^3} \pi(x_1, x_2, 1) T_N^3(x_3(x_{-3}), x_{-3})) 1_{\{x_2 \geqslant x_1\}}\, dx_2\, dx_1 \geqslant 0.
$$

Thus, for all increasing π,

$$
\int_{B_N^3} \pi(x_{-3}, 1) T_N^3(x_3(x_{-3}), x_{-3})\, dx_{-3} \geqslant 0.
$$

Applying the same arguments to B_N^2, B_N^1 yields the conditions required by Corollary 1 to show that Assumption 3 holds for $J = N$. Combining with the argument for $|J| = 2$ and applying Corollary 1 we have Assumption 3 is satisfied and Theorem 3 yields the conclusion. □

8. Conclusion

We conclude with a brief discussion of the possibilities for weakening some of the conditions invoked in Theorems 4 and 5.

The identified environments are quite restrictive even in the $n = 2$ case. We believe this is no accident. In a companion paper, [7], we note that the set of IC and IR mechanisms is convex and has extreme points. Since the seller's objective functional is linear, the solution set will always contain an extreme point of the feasible set. In the case of $n = 1$, the extreme points are simply the set of take-it-or-leave-it prices. Thus, the well-known result for $n = 1$ is immediate. The set of extreme points when $n > 1$ is far richer and includes mechanisms with significant randomization in the allocation of objects.

Some conditions arise as a consequence of the strategy of proof. Assumption 2b also appears (in various forms) in many single-dimensional applications. In the one good, one buyer case, it is known not to be required, however, it is often invoked to simplify the analysis. It implies the monotonicity of the virtual valuation function. In the context of our approach, it allows us to ignore the convexity constraint on the utility functions that comes from incentive compatibility because the requirement that utility be convex is not binding at a solution.

The example in Section 5 suggests that negative covariance of valuations poses problems, so, it may be possible to weaken Assumption 2a (independence). A potential conjecture to explore is whether this can be weakened to the requirement that f satisfy affiliation. One hurdle to such an extension is that sufficient conditions for multivariate functions to covary positively against affiliated densities (and therefore, to check Assumptions 3 and 4) require the domain of the functions to be sublattices—a condition that is not typically satisfied by market segments even when *SM* holds.

The requirement that $\frac{x_i f_i'(x_i)}{f_i(x_i)}$ be increasing is the most unusual condition. It does not arise in the one good case. However, ensuring that Assumption 4 in Theorem 3 is satisfied relies critically on this restriction and its role is clearly tied to the multiple-good problem. We have constructed examples which satisfy Assumptions 1 and 2 but not this final restriction and which appear to show that price schedules can be dominated. However, the comparisons lead to differences in the order of the sixth digit and we do not have that much confidence in these results.

Finally, the extension to $n > 2$ brings forth additional difficulties. To verify that Assumption 4 is satisfied, additional restrictions on the family of distributions may be necessary. A density such that $\frac{x_i f_i'(x_i)}{f_i(x_i)}$ is increasing need not suffice because sets such as $A_J^{J^c}$ are not generally sublattices and the covariance argument used in the proof of Theorem 4 no longer can be applied. If we restrict attention, however, to distributions of the form $F_i(x_i) = x_i^{\alpha}$, then it can be shown that Assumption 4 is always satisfied.

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Appendix A.

A.1. Proofs

Proof of Lemma 1. $(x_i, x_{-i}) \in A_J$ implies that $a_J \cdot x - P_J \geqslant a_K \cdot x - P_K$ for all K. If $i \in K$, then raising x_i to x_i' raises both the left and right side without affecting the inequality. If $i \notin K$, then raising x_i to x_i' increases the left side but the right remains fixed. The argument for the second statement is similar. □

Proof of Lemma 2. (i) Necessity follows from the definition of A_J. To show sufficiency, suppose x, J satisfy the hypotheses of Part (i) but $x \notin A_J$. Then, there exists a set K such that

$$\begin{aligned} a^J \cdot x - P_J &\geqslant a^{J\cap K} \cdot x - P_{J\cap K},\\ a^J \cdot x - P_J &\geqslant a^{J\cup K} \cdot x - P_{J\cup K},\\ a^K \cdot x - P_K &> a^J \cdot x - P_J. \end{aligned}$$

Summing the inequalities and using $a^J + a^K = a^{J\cap K} + a^{K\cup J}$ yields $P_{K\cup J} > P_J + P_K - P_{J\cap K}$ which violates *SM*.

(ii) Now suppose that $x \in A_J \cap A_K$, $K \cap J \notin \{J, K\}$. Applying the same argument as above implies

$$\begin{aligned} a^J \cdot x - P_J &\geqslant a^{J\cap K} \cdot x - P_{J\cap K},\\ a^J \cdot x - P_J &\geqslant a^{J\cup K} \cdot x - P_{J\cup K},\\ a^K \cdot x - P_K &= a^J \cdot x - P_J. \end{aligned}$$

Summing the three inequalities and applying *SM* implies that all three must hold with equality. But this means that $A_J \cap A_K$ is the intersection of at least three linearly independent hyperplanes (we could have $K \cap J = \emptyset$) which has zero measure in $\mathbb{R}^{n-1}$.

(iii) Let $\tilde{x} = (x'_J, x_{J^c})$. Part (i) implies if $\tilde{x} \notin A_J$ there must exist K, $K \subset J$ or $J \subset K$ such that $a^K \cdot \tilde{x} - P_K > a^J \cdot \tilde{x} - P_J$.

Suppose $a^K \cdot \tilde{x} - P_K > a^J \cdot \tilde{x} - P_J$ for $K \subset J$. $x' \in A_J$ implies

$$a^J \cdot x' - P_J = a^J \cdot \tilde{x} - P_J \geqslant a^K \cdot x' - P_K = a^K \cdot \tilde{x} - P_K,$$

a contradiction.

Observe that $J \subset K$, implies $a^K \cdot \tilde{x} - a^J \cdot (x' - x) = a^K \cdot x$ and $a^J \cdot \tilde{x} - a^J \cdot (x' - x) = a^J \cdot x$. Therefore, $a^J \cdot \tilde{x} - P_J < a^K \cdot \tilde{x} - P_K$ implies $a^J \cdot x - P_J < a^K \cdot x - P_K$ which contradicts the hypothesis that $x \in A_J$.

The cartesian product representation of the sets A_J now follows. $\square$

Proof of Lemma 4. (i) By definition of ω in (9) and t^i_J, in (11)

$$\begin{aligned}
\langle \pi, \omega \rangle &= \sum_{J \subset N} \sum_{i \in J} \int_{B^i_J} \pi(1, x_{-i}) f(1, x_{-i})\, dx_{-i} - \sum_{J \subset N} \sum_{i \in J} \int_{A_J} \pi(x) t^i_J(x_J) f(x)\, dx \\
&= \sum_{J \subset N} \sum_{i \in J} \int_{B^i_J} \pi(1, x_{-i}) f(1, x_{-i})\, dx_{-i} - \sum_{J \subset N} \sum_{i \in J} \int_{B^i_J} \int_{x_i(x_{-i})}^1 \pi(x) t^i_J(x_J) f(x)\, dx \\
&= \sum_{J \subset N} \sum_{i \in J} \left\{ \int_{B^i_J} \left[\pi(1, x_{-i}) f(1, x_{-i}) - \int_{x_i(x_{-i})}^1 \pi(x) t^i_J(x_J) f(x)\, dx_i \right] dx_{-i} \right\} \\
&= \sum_{J \subset N} \sum_{i \in J} \left\{ \int_{B^i_J} [\pi(1, x_{-i})(f(1, x_{-i}) - T^i_J(x_i(x_{-i}), x_{-i})) \right. \\
&\quad \left. - \int_{x_i(x_{-i})}^1 \pi(x) t^i_J(x_J) f(x)\, dx_i + \pi(1, x_{-i}) T^i_J(x_i(x_{-i}), x_{-i})]\, dx_{-i} \right\} \\
&= \sum_{J \subset N} \sum_{i \in J} \left\{ \int_{B^i_J} \int_{x_i(x_{-i})}^1 (\pi(1, x_{-i}) - \pi(x)) t^i_J(x_J) f(x)\, dx_i \right. \\
&\quad \left. + \pi(1, x_{-i}) T^i_J(x_i(x_{-i}), x_{-i})\, dx_{-i} \right\}.
\end{aligned}$$

The second equality follows from the definition of $x_i(\cdot)$ and Lemmas 1 and 2. The next equality collects all terms in the summation in i and the next one adds and subtracts $T^i_J(x_i(x_{-i}), x_{-i})$ $\pi(1, x_{-i})$. The final equality follows by (11) using the fact that $\pi(1, x_{-i})$ does not vary in x_i.

(ii) Applying Eq. 8, Lemma 2 and the definition of t^i_J yields for all J, $x \in A_J$,

$$\begin{aligned}
0 &= \prod_{k \notin J} f_k(x_k) \sum_{i \in J} \left[\int_{D^i_J} \prod_{j \in J, j \neq i} f_j(x_j) f_i(1)\, dx_{J/i} - \int_{A^J_J} t^i_J(x_J) \prod_{j \in J} f_j(x_j)\, dx_J \right] \\
&= \sum_{i \in J} \int_{D^i_J} \prod_{j \neq i} f_j(x_j) \left[f_i(1) - \int_{x_i(x_{-i})}^1 t^i_J(x_J) f_i(x_i)\, dx_i \right] dx_{J/i} \\
&= \sum_{i \in J} \int_{D^i_J} T^i_J(x_i(x_{-i}), x_{-i})\, dx_{J/i}.
\end{aligned}$$

The second equality follows from the definition of $x_i(\cdot)$ and Lemmas 1 and 2. The next equality uses the definition of T^i_J.

(iii) If $|\ J\ |= 1$, then the integration operation vanishes and $x_{-i} \in B^i_J$ implies $T_i^J(x_i(x_{-i}), x_{-i}) = 0$. $\square$

Proof of Lemma 5. The proof requires three intermediate results.

(i) $\beta \in (-1/2, 1/2)$ implies $\hat{x}_1(\cdot)$ (resp. $\hat{x}_2(\cdot)$) is continuous and strictly decreasing (increasing).
(ii) $\exists(\underline{\beta}, \overline{\beta}) \subset (-1/2, 1/2)$ such that $\beta \in (\underline{\beta}, \overline{\beta})$ implies $\hat{x}_i(\beta) \geqslant P_N - P_{\{-i\}}$, $i = 1, 2$.
(iii) $\forall\beta \in (\underline{\beta}, \overline{\beta})$, $\hat{x}_i(\beta) \leqslant P_N - \hat{x}_{-i}(\beta)$.

(Part i): Total differentiate (13) to obtain

$$\hat{x}_i'(\beta) = (-1)^i \frac{1 - F^i(\hat{x}_i(\beta))}{\hat{x}_i(\beta) f^{i'}(\hat{x}_i(\beta)) + (3/2 + (-1)^i\beta) f_i(\hat{x}_i(\beta))}.$$

Assumption 2b implies that the denominator is strictly positive so the implicit function theorem implies $\hat{x}_i(\cdot)$ is continuous. $f_i(1) > 0$ implies $\hat{x}_i(\beta) < 1$ so the sign of the numerator is $(-1)^i$.

(Part ii): We show first that we cannot have $\hat{x}_i(\beta) \leqslant P_N - P_{\{-i\}}$ for both $i = 1, 2$. The first order condition from Theorem 1 and the definition of $t_N(x)$ imply

$$\begin{aligned} 0 = &\sum_{i=1, j\neq i}^{2} \int_{P_N - P_{\{i\}}}^{1} f_j(x_j) \\ &\times \left\{ f_i(1) - \int_{\max\{P_N - P_{\{j\}}, P_N - x_j\}}^{1} \left[x_i f^{i'}(x_i) + (3/2 + (-1)^i\beta) f_i(x_i) \right] dx_i \right\} dx_j \\ = &\sum_{i=1, j\neq i}^{2} \int_{P_N - P_{\{i\}}}^{P_{\{j\}}} f_j(x_j) w^i(P_N - x_j; \beta)\, dx_j + \int_{P_{\{j\}}}^{1} f_j(x_j) w^i(P_N - P_{\{j\}}; \beta)\, dx_j. \end{aligned}$$

Theorem 2 implies that $P_N - P_{\{i\}} < P_{\{j\}}$ so the first term has positive measure and for x_j in this range, $P_N - x_j > P_N - P_j$. Since $w^i(\cdot; \beta)$ is strictly increasing and $w^i(\hat{x}_i(\beta); \beta) = 0$, if $\hat{x}^i(\beta) < P_N - P_{\{j\}}$, then both terms are non-negative and the first term is strictly positive yielding a contradiction. Therefore, suppose that $\hat{x}_1(\beta) > P_N - P_{\{2\}}$ while $\hat{x}_2(\beta) \leqslant P_N - P_{\{1\}}$. Since $\hat{x}_1(\cdot)$ is decreasing and continuous and $\hat{x}_1(1/2) = 0$ we can raise β to $\overline{\beta} < 1/2$ such that $\hat{x}_1(\overline{\beta}) = P_N - P_{\{2\}} > 0$ (since $A_{\{2\}}$ has positive measure). By the above argument, this implies $\hat{x}_2(\overline{\beta}) > P_N - P_{\{1\}}$. To find $\underline{\beta} > -1/2$ now reduce β so that $\hat{x}_2(\underline{\beta}) = P_N - P_{\{1\}} > 0$.

(Part iii): The following inequality is used in the proof. For $J = \{i\}$, $j \neq i$, $z \leqslant P_J$ and $x_j \geqslant P_N - P_J$, Eq. (8) implies using the definition of t_J and K_J,

$$\begin{aligned} 0 &= f_j(x_j) \left\{ f_i(1) - \int_{P_J}^{1} \left[\frac{v f_i'(v)}{f_i(v)} + 3 + E\left[\frac{w f_j'(w)}{f_j(w)} \middle| w \leqslant P_N - P_J \right] \right] f_i(v)\, dv \right\} \\ &\geqslant f_j(x_j) \left\{ f_i(1) - \int_{P_J}^{1} \left[\frac{v f_i'(v)}{f_i(v)} + 3 + \frac{x_j f_j'(x_j)}{f_j(x_j)} \right] f_i(v)\, dv \right\} \\ &\geqslant f_j(x_j) \left\{ f_i(1) - \int_{z}^{1} \left[\frac{v f_i'(v)}{f_i(v)} + 3 + \frac{x_j f_j'(x_j)}{f_j(x_j)} \right] f_i(v)\, dv \right\} \\ &= f_j(x_j) f_i(1) - \int_{z}^{1} t_N(v, x_j) f(v, x_j)\, dv. \end{aligned} \tag{19}$$

The first inequality follows because $\frac{x_j f_j'(x_j)}{f_j(x_j)}$ increasing and $x_j \geqslant P_N - P_J$ implies $E\left[\frac{w f_j'(w)}{f_j(w)} \mid w \leqslant P_N - P_J\right] \leqslant \frac{x_j f_j'(x_j)}{f_j(x_j)}$. The second inequality follows by Assumption 2b and $z \leqslant P_J$. The last line comes by definition of t_N.

Suppose that $\hat{x}_i(\beta) > P_N - \hat{x}_j(\beta)$, $j \neq i$. The first order condition from Theorem 1 applied to A_N can be written as

$$\begin{aligned}
0 &= \sum_{i,j\neq i}\left\{\int_{P_N-P_{\{i\}}}^{P_N-\hat{x}_i(\beta)}\left[f_i(1)-\int_{P_N-x_j}^{1} t_N(x) f_i(x_i)\,dx_i\right] f_j(x_j)\,dx_j\right\}\\
&\quad+\sum_{i,j\neq i}\int_{P_N-\hat{x}_i(\beta)}^{\hat{x}_j(\beta)}\left[f_i(1)-\int_{P_N-x_j}^{1}(x_i f_i'(x_i)+(3/2+(-1)^i\beta)f_i(x_i))\,dx_i\right] f_j(x_j)\,dx_j\\
&\quad+\sum_{i,j\neq i}\int_{\hat{x}_j(\beta)}^{1}\left[f_i(1)-\int_{P_N-\hat{x}_j(\beta)}^{1}(x_i f_i'(x_i)+(3/2+(-1)^i\beta)f_i(x_i))\,dx_i\right] f_j(x_j)\,dx_j\\
&\leqslant \sum_{i=1}^{2}\int_{P_N-\hat{x}_i(\beta)}^{\hat{x}_j(\beta)} w^i(P_N-x_j;\beta) f_j(x_j)\,dx_j+\int_{\hat{x}_j(\beta)}^{1} w^i(P_N-\hat{x}_j(\beta);\beta) f_j(x_j)\,dx_j\\
&< 0
\end{aligned}$$

which is a contradiction. The equality follows because the limits of integration divide A_N into three sections, disjoint except for a measure zero intersection. Two sections are reflected in the limits of integration in the first term of the summation. The third section is repeated in the second and third lines but adding each of the integrands yields $t(x)$. (We exploit the hypothesis that $\hat{x}_i(\beta) > P_N - \hat{x}_j(\beta)$. The expression assumes that $P_N - P_{\{i\}} \leqslant P_N - \hat{x}_i(\beta)$. If this does not hold, then the first line vanishes and the lower limit of integration in the second line becomes $P_N - P_{\{i\}}$. The rest of the argument remains the same.) The first inequality follows since (19) implies the first line is non-positive and by applying the definition of $w^i(\cdot;\beta)$ in (13). The second inequality follows because $w^i(x;\beta)$ is strictly increasing in x and is zero at $x = \hat{x}_i(\beta)$ and in both terms of the fourth line, $P_N - x_j < \hat{x}_i(\beta)$ (because $x_j \geqslant P_N - \hat{x}_i(\beta)$) and $P_N - \hat{x}_j(\beta) < \hat{x}_i(\beta)$ (by hypothesis).

The t_N^is are seen to satisfy (11) as follows. The set of points

$$\{x_1 \leqslant \hat{x}_1(\beta)\} \cup \{x_1 \geqslant \hat{x}_1(\beta), x_2 \geqslant \hat{x}_2(\beta)\} \cup \{x_2 \leqslant \hat{x}_2(\beta)\}$$

covers the set A_N. By construction, the intersection of the first two sets and the intersection of the last two sets has measure zero. Suppose $x_2 < \hat{x}_2(\beta)$. Since $x_1 + x_2 \geqslant P_N$, Result (iii) implies

$$x_1 \geqslant P_N - x_2 > P_N - \hat{x}_2(\beta) \geqslant \hat{x}_1(\beta)$$

so the intersection of the first and last set has zero measure as well. Therefore, whenever, $t_N^i(x;\beta) = 0$, $t_N^{-i}(x;\beta) = t_N(x)$. The definitions of t_N^i now yield $t_N^1(x;\beta) + t_N^2(x;\beta) = t_N(x)$ for all $x \in A_N$. Assumption 2 and $\beta \in (-1/2, 1/2)$ imply $t_N^i(\cdot;\beta) \geqslant 0$. $\square$

Proof of Lemma 6. (Part i) Consider $i = 2$. The proof of Lemma 5, (19) implies for $x_1 \in [P_N - P_{\{2\}}, \hat{x}_1(\beta)]$,

$$\begin{aligned} 0 &\geqslant \left\{ f_2(1) - \int_{P_N - x_1}^{1} t_N^2(x_1, x_2; \beta) f_2(x_2)\, dx_2 \right\} f_1(x_1) \\ &= T_N^2(x_1, x_2(x_1); \beta). \end{aligned}$$

The inequality follows because in this region, $t_N^2 = t_N$ and $x_2(x_1) = P_N - x_1 \leqslant P_{\{2\}}$.

For $x_1 \in (\hat{x}_1(\beta), P_N - \hat{x}_2(\beta)]$,

$$\begin{aligned} T_N^2(x_1, x_2(x_1); \beta) &= \left\{ f_2(1) - \int_{P_N - x_1}^{1} t_N^2(x_1, x_2; \beta) f_2(x_2)\, dx_2 \right\} f_1(x_1) \\ &= w^2(P_N - x_1; \beta) f_1(x_1) \\ &\geqslant w^2(\hat{x}_2(\beta); \beta) f_1(x_1) \\ &= 0. \end{aligned}$$

The second equality applies (13) and the definition of t_N^2. The inequality follows since $P_N - x_1 \geqslant \hat{x}_2(\beta)$ and $w^2(\cdot; \beta)$ is strictly increasing. The equality is by definition of $\hat{x}_2(\beta)$.

Finally, for $x_1 \geqslant P_N - \hat{x}_2(\beta)$,

$$\begin{aligned} T_N^2(x_1, x_2(x_1); \beta) &= \left\{ f_2(1) - \int_{\max\{P_N - x_1, P_N - P_{\{1\}}\}}^{1} t_N^2(x_1, x_2; \beta) f_2(x_2)\, dx_2 \right\} f_1(x_1) \\ &= \left\{ f_2(1) - \int_{\hat{x}_2(\beta)}^{1} t_N^2(x_1, x_2; \beta) f_2(x_2)\, dx_2 \right\} f_1(x_1) \\ &= 0. \end{aligned}$$

The second equality follows because Lemma 5 implies $\hat{x}_2(\beta) \geqslant \max\{P_N - x_1, P_N - P_{\{1\}}\}$ and for $x_2 \leqslant \hat{x}_2(\beta)$, $t_N^2(x) = 0$. The final equality follows by definition of $\hat{x}_2(\beta)$. The same argument follows for $i = 1$.

(Part ii): Since $T_J^i(\cdot; \beta)$ satisfies (11), applying Lemma 4(ii) to $J = N$ gives

$$\begin{aligned} 0 &= \sum_{i=1, j \neq i}^{2} \int_{P_N - P_{\{i\}}}^{1} T_N^i(x_i(x_j), x_j; \beta)\, dx_j \\ &= \sum_{i=1, j \neq i}^{2} \left\{ \int_{P_N - P_{\{i\}}}^{\hat{x}_j(\beta)} T_J^i(P_N - x_j, x_j; \beta)\, dx_j + \int_{\hat{x}_j(\beta)}^{P_N - \hat{x}_i(\beta)} T_J^i(P_N - x_j, x_j; \beta)\, dx_j \right\} \\ &\equiv \sum_{i=1}^{2} \{F_i(\beta) + G_i(\beta)\}. \end{aligned}$$

The second equality uses the implication from Part (i) that $T_J^i(x_j(x_j), x_j; \beta) = 0$ for $x_j \geqslant P_N - \hat{x}_i(\beta)$ and for $x_j \leqslant P_N - \hat{x}_i(\beta)$, $x_i(x_j) = P_N - x_j$. Suppose that for some i, $F_i(\beta) + G_i(\beta)$ is strictly negative (and therefore $F_j(\beta) + G_j(\beta) > 0$, $j \neq i$) for all $\beta \in (\underline{\beta}, \overline{\beta})$. Part (i) implies that $F_i(\beta)$ is non-positive and $G_i(\beta)$ is non-negative. Note that $F_i(\beta) + G_i(\beta)$ varies continuously with β. Let β increase or decrease as necessary so that $\hat{x}_j(\beta)$ approaches $P_N - P_{\{i\}}$. Since the measure of $(P_N - P_i, \hat{x}_j(\beta)]$ goes to zero, $F_i(\beta)$ goes to zero. But $G_i(\beta)$ is non-negative by Part (i) and, by assumption, $F_j(\beta) + G_j(\beta)$ is strictly positive. This yields a contradiction. □

Proof of Lemma 7. To ensure that the defined $t^i_J(x_J)$ satisfy (11), we must show that for each J, $S^i_J \cap S^j_J$, $i \neq j \in J$ has measure zero in $\mathbb{R}^n$, and that $\{S^i_J\}_{i \in J}$ covers A_J. The first requirement follows from the definitions. Thus, it only remains to show that there does not exist an $x \in A_J$ with component $k \notin J$ such that $x_k > x_i$, $i \in J$. Suppose there is such an element. Define $K = J \cup k$ and for any set J, let 1^J and 0^J denote the $\mid J \mid$ vector of all ones or zeroes, respectively. $x \in A_J$ implies $a^J \cdot x \geqslant P_J$, which, in turn implies

$$x_k > \max_{i \in J} x_i \geqslant \frac{P_J}{\mid J \mid}. \tag{20}$$

Symmetry and *SM* yield

$$\left(\frac{P_J}{\mid J \mid} 1^J, 0^{N/J}\right) \in A_J \cap A_\emptyset.$$

(Consider, for example, $J = N$. Since $\frac{P_{1,2,3}}{3} < \frac{P_{i,j}}{2} < P_{\{i\}}$, an agent with type $(\frac{P_{1,2,3}}{3}, \frac{P_{1,2,3}}{3}, \frac{P_{1,2,3}}{3})$ would be indifferent between not buying any bundle and buying the whole bundle and would strictly prefer not to buy any other bundle.) Since by hypothesis, $(x_J, x_{J^c}) \equiv (x_J, x_k, x_{K^c}) \in A_J$, Lemma 2 implies

$$\tilde{x} \equiv \left(\frac{P_J}{\mid J \mid} \mathbf{1}^J, x_k, x_{K^c}\right) \in A_J. \tag{21}$$

By construction, $\tilde{x} \cdot a^J - P_J = 0$, a buyer with valuation $\tilde{x}$ gains exactly zero utility from purchasing the bundle J. If an agent of type $\tilde{x}$ bought K instead, he would receive

$$\begin{aligned} a^K \cdot \tilde{x} - P_K &= \mid J \mid \frac{P_J}{\mid J \mid} + x_k - P_K \\ &> \mid J \mid \frac{P_J}{\mid J \mid} + \frac{P_K}{\mid K \mid} - P_K \\ &\geqslant \mid K \mid \frac{P_K}{\mid K \mid} - P_K \\ &= 0, \end{aligned}$$

where the first inequality follows from (20) and the second because *SM* implies $\frac{P_J}{|J|} \geqslant \frac{P_K}{|K|}$. Thus, the buyer of type $\tilde{x}$ does strictly better buying K than J, a contradiction to the conclusion in (21). □

Proof of Lemma 8. We first characterize $\underline{x}_3(x_1, x_2)$ on $B = B^3_N \cap \{(x_1, x_2) \mid x_1 \geqslant x_2\}$ shown in Fig. 3. Observe that *SM* implies that $P_{\{3\}} > \frac{P_N}{3} > \frac{P_N - P_{\{3\}}}{2} > P_N - P_{\{1,3\}}$. Symmetry implies that $\underline{x}_3(v, w) = \underline{x}_3(w, v)$ and $T^3_N(x_3(v, w), v, w) = T^3_N(x_3(w, v), w, v)$ and, therefore, $G^1(v) = G^2(v)$ for all $v \in [P_N - P_{\{1,3\}}, 1]$. Thus, if we show the desired results on this region, they follow with the appropriate permutation of variables on the complement.

The manifold,

$$\{(x_1, x_2, x_3) \mid x_1 \in [P_N/3, P_{\{1,3\}}/2), x_3 = x_1, x_2 = P_N - 2x_1\}$$

belongs to $A_N \cap A_\emptyset$. This follows because the two endpoints, $(P_N/3, P_N/3, P_N/3)$ and $(P_{\{1,3\}}/2, P_N - P_{\{1,3\}}, P_{\{1,3\}}/2)$ are in $A_N \cap A_\emptyset$ and $A_N \cap A_\emptyset$ is a convex set. (The first inclusion follows from the argument for S^3_N in the proof of Lemma 7, the second because the full bundle and the

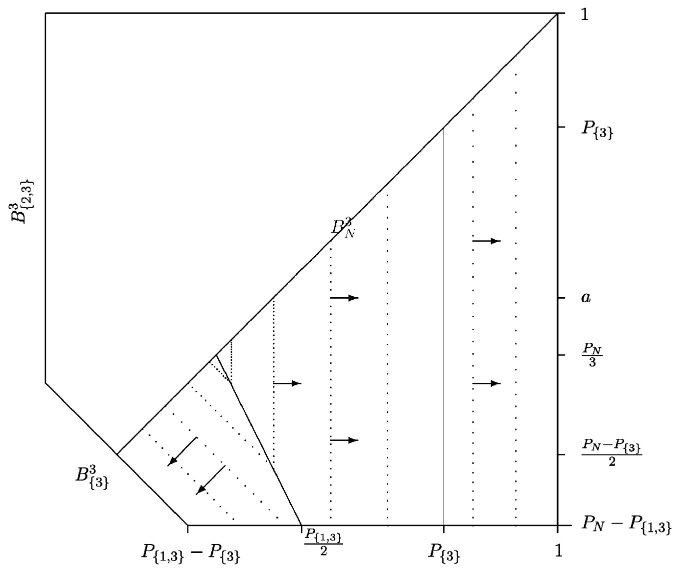

Fig. 3. Level Sets of $\underline{x}_3(x_1, x_2)$ on $B^3_N \cap \{(x_1, x_2) \mid x_1 \geqslant x_2\}$.

bundle $\{1, 3\}$ give exactly zero utility while *SM* implies $\frac{P_{\{1,3\}}}{2} < P_{\{3\}}$ so any single good bundle gives strictly negative utility.) Since A_N is increasing in x_i, $i = 1, 2, 3$,

$$\{x \mid x_3 \geqslant x_1, x_1 \geqslant P_N/3, x_2 \geqslant \max\{P_N - 2x_1, P_N - P_{\{1,3\}}\}\} \subset A_N.$$

For any x such that $x_3 = x_1, x_1 \leqslant P_{\{1,3\}}/2, x_2 < P_N - 2x_1$, we have $x_1 + x_2 + x_3 < P_N$, so buying the full bundle yields strictly negative utility. Thus, such points cannot be in A_N and the lower bound of S^3_N in this region must be contained in the manifold $A_N \cap A_\emptyset$. Combining these arguments yield

$$\begin{aligned} \underline{x}_3(x_{-3}) &= x_1, \quad x_1 \geqslant x_2, x_1 \geqslant P_N/3, x_2 \geqslant P_N - 2x_1, \\ &= P_N - x_1 - x_2, \quad x_1 \geqslant x_2, x_1 \leqslant P_{\{1,3\}}/2, x_2 \leqslant P_N - 2x_1. \end{aligned} \tag{22}$$

The thick line with slope -2 divides the two regions. The level sets of $\underline{x}_3(x_{-3})$ are illustrated by the dotted lines in Fig. 3. The arrows denote the direction of increase.

(Part i): Applying (22), $x_1 > P_{\{3\}} = \frac{3}{4}$ implies $T^3_N(x_3(x_{-3}), x_{-3}) = 4x_1 - 3 > 0$.

Similarly, $x_1 \leqslant P_{\{3\}}$ implies either $T^3_N(x_3(x_{-3}), x_{-3}) = 4x_1 - 3 \leqslant 0$ or

$$\begin{aligned} T^3_N(x_3(x_{-3}), x_{-3}) &= 4(P_N - x_1 - x_2) - 3 \\ &\leqslant 4(P_N - (P_N - P_{\{3\}})) - 3 \\ &= 4P_{\{3\}} - 3 \\ &= 0. \end{aligned}$$

The inequality follows because $x_{-3} \in B^3_N$ implies $x_1 + x_2 \geqslant P_N - P_{\{3\}}$ otherwise $x_1 + x_2 + 1 - P_N < 1 - P_{\{3\}}$ and a buyer with type $(1, x_{-3})$ would do better buying good 3 alone.

(Part ii): The proof proceeds by showing that $G^2(x_2)$ is positive for x_2 in $[P_{\{3\}}, 1]$, non-positive for x_2 in $[P_N - P_{\{1,3\}}, \frac{P_N - P_{\{3\}}}{2}]$ and quasi-convex over $[\frac{P_N - P_{\{3\}}}{2}, P_{\{3\}}]$. Since this implies that $G^2(x_2)$ crosses 0 at one point, say, a, the conclusion then follows.

Part (i) implies that $G^2(x_2) > 0$ for $x_2 > P_{\{3\}}$.

Now consider $x_2 \leqslant \frac{P_N - P_{\{3\}}}{2}$. Restricting attention to the lower horizontal boundary of B_N^3, since $\underline{x}_3(x_1, x_2)$ is continuous over $B_N^3 \cup B_{\{1,3\}}^3$, for $x_1 \geqslant P_{\{1,3\}} - P_{\{3\}}$,

$$\begin{aligned} &T_N^3(x_3(x_1, P_N - P_{\{1,3\}}), x_1, P_N - P_{\{1,3\}}) \\ &\quad = T_{\{1,3\}}^3(x_3(x_1, P_N - P_{\{1,3\}}), x_1, P_N - P_{\{1,3\}}). \end{aligned}$$

Furthermore, (22) implies $\underline{x}_3(x_1, x_2)$ is either constant or decreasing in x_2 in the region $B_N^3 \cap \{(x_1, x_2) \mid x_1 \geqslant x_2\}$. Since $(x_1, x_2') \in B_N^3$ implies $x_2' \geqslant P_N - P_{\{1,3\}}$,

$$\begin{aligned} T_N^3(x_3(x_1, x_2'), x_1, x_2') &\leqslant T_{\{1,3\}}^3(x_3(x_1, P_N - P_{\{1,3\}}), x_1, P_N - P_{\{1,3\}}), \\ &\qquad (x_1, x_2') \in B_N^3,\, x_1 \geqslant \max\{x_2, P_{\{1,3\}} - P_{\{3\}}\}. \end{aligned} \tag{23}$$

Thus, for $x_2 \in [P_N - P_{\{1,3\}}, \frac{P_N - P_{\{3\}}}{2}]$,

$$\begin{aligned} G^2(x_2) &= \int_{P_N - P_{\{3\}} - x_2}^{P_{\{1,3\}} - P_{\{3\}}} T_N^3(x_3(x_1, x_2), x_1, x_2)\, dx_1 \\ &\quad + \int_{P_{\{1,3\}} - P_{\{3\}}}^{1} T_N^3(x_3(x_1, x_2), x_1, x_2)\, dx_1 \\ &\leqslant \int_{P_{\{1,3\}} - P_{\{3\}}}^{1} T_N^3(x_3(x_1, x_2), x_1, x_2)\, dx_1 \\ &\leqslant \int_{P_{\{1,3\}} - P_{\{3\}}}^{1} T_{\{1,3\}}^3(x_3(x_1, P_N - P_{\{1,3\}}), x_1, P_N - P_{\{1,3\}})\, dx_1 \\ &= 0. \end{aligned}$$

The first inequality comes because $T_N^3 \leqslant 0$ for $x_1 \leqslant P_{\{1,3\}} - P_{\{3\}} \leqslant P_{\{3\}}$ (Part i). The second comes from (23), the equality comes from (17).

For $x_2 \in [\frac{P_N - P_{\{3\}}}{2}, P_N/3]$,

$$G^2(x_2) = \int_{x_2}^{\frac{P_N - x_2}{2}} \{4(P_N - x_1 - x_2) - 3\}\, dx_1 + \int_{\frac{P_N - x_2}{2}}^{1} \{4x_1 - 3\}\, dx_1.$$

Differentiating with respect to x_2 twice (using the continuity of the integrand in x_1) gives

$$\frac{d^2 G^2(x_2)}{dx_2^2} = 12 > 0,$$

so $G^2(x_2)$ is convex in this range.

For $x_2 \in [P_N/3, P_{\{3\}}]$,

$$G^2(x_2) = \int_{x_2}^{1} \{4x_1 - 3\}\, dx_1.$$

Differentiating gives

$$\frac{dG^2(x_2)}{dx_2} = -4x_2 + 3 > 0,$$

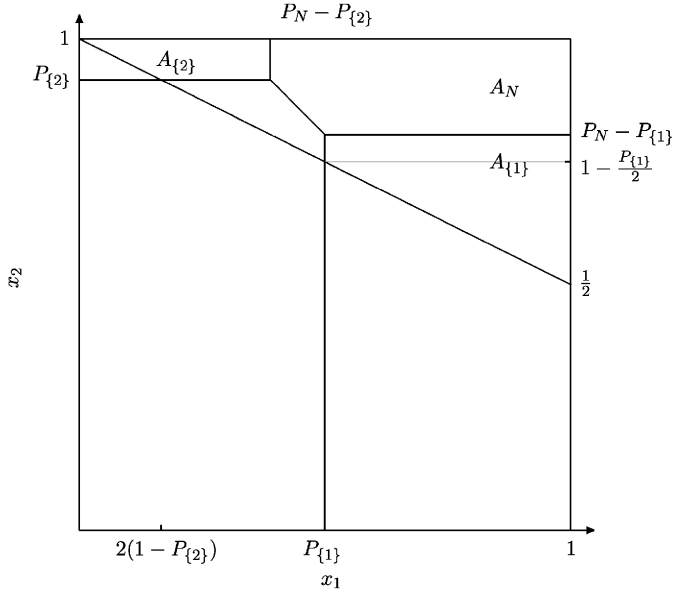

Fig. 4. A three price mechanism.

using the fact that $x_2 \leqslant P_{\{3\}} = 3/4$, so $G^2(x_2)$ is increasing in this range. Thus, $G^2(x_2)$ is quasi-convex and Part (ii) follows.

(Part (iii)) Lemma 4(ii) and symmetry implies

$$0 = \int_{P_N - P_{\{1,3\}}}^{1} G^2(x_2)\, dx_2 + \int_{P_N - P_{\{2,3\}}}^{1} G^1(x_1)\, dx_1.$$

$G^1 = G^2$ and $P_{\{1,3\}} = P_{\{2,3\}}$ then implies Part (iii). □

A.2. Proof of suboptimality of three price mechanisms for counterexample

A mechanism with prices such that $1 > P_{\{1\}} \geqslant 2(P_N - 1)$ (not shown but is a mechanism with the point $(P_{\{1\}}, P_N - P_{\{1\}})$ below the line $x_1/2 + x_2 = 1$) is never optimal. This mechanism is dominated by instead offering $P_{\{1\}} > 1$, holding $P_N, P_{\{2\}}$ fixed. A seller can induce buyers who purchased only one good at $1 > P_{\{1\}}$ to buy two goods at $P_N > 1$ because, in this case, $x \in A_{\{1\}}$ and $P_{\{1\}} \geqslant 2(P_N - 1)$ implies

$$\begin{aligned} x_1 + x_2 &= \frac{x_1}{2} + \frac{x_1}{2} + x_2 \\ &\geqslant \frac{P_{\{1\}}}{2} + 1 \\ &\geqslant P_N. \end{aligned}$$

Therefore, Fig. 4 illustrates a typical three price mechanism.

The uniform density and some geometry imply

$$\begin{aligned}\int_{A_{\{1\}}} dx &= \frac{1}{2}(1 - P_{\{1\}})\left((P_N - P_{\{1\}}) - \frac{1}{2} + (P_N - P_{\{1\}}) - \left(1 - \frac{P_{\{1\}}}{2}\right)\right)\\ &= \frac{1}{2}(1 - P_{\{1\}})\left(2P_N - \frac{3}{2}P_{\{1\}} - \frac{3}{2}\right),\\ \int_{A_{\{2\}}} dx &= \frac{1}{2}(1 - P_{\{2\}})(P_N - P_{\{2\}} + P_N - P_{\{2\}} - 2(1 - P_{\{2\}}))\\ &= (1 - P_{\{2\}})(P_N - 1),\\ \int_{A_N} dx &= \frac{1}{2}(2 + P_{\{2\}} - P_{\{1\}} - P_N)(P_{\{2\}} + P_{\{1\}} - P_N) + (1 - P_{\{2\}})(1 + P_{\{2\}} - P_N).\end{aligned}$$

For any three-price profile,

$$R(P) = P_{\{1\}} \int_{A_{\{1\}}} dx + P_{\{2\}} \int_{A_{\{2\}}} dx + P_N \int_{A_N} dx.$$

Partially differentiating this with respect to $P_{\{2\}}$ yields

$$\hat{P}_{\{2\}}(P_N) = \frac{2P_N - 1}{3P_N - 2}$$

as before. Partially differentiating $R(P)$ with respect to $P_{\{1\}}$ yields

$$\frac{\partial R(P)}{\partial P_{\{1\}}} = \frac{9}{4}P_{\{1\}}^2 - 3P_{\{1\}}P_N + \left(2P_N - \frac{3}{4}\right).$$

Differentiating again gives

$$\tfrac{9}{2}P_{\{1\}} - 3P_N$$

which is negative only if $P_{\{1\}} \leqslant \frac{2}{3}P_N$. The roots of $\frac{\partial R(P)}{\partial P_{\{1\}}}$ are

$$\hat{P}^+_{\{1\}}(P_N) = \frac{2}{3}\left(P_N + \sqrt{P_N^2 - 2P_N + \frac{3}{4}}\right),$$

and

$$\hat{P}^-_{\{1\}}(P_N) = \frac{2}{3}\left(P_N - \sqrt{P_N^2 - 2P_N + \frac{3}{4}}\right).$$

The first equation exceeds $\frac{2}{3}P_N$ so only the second root is a potential solution. Since the term $P_N^2 - 2P_N + \frac{3}{4} = (P_N - \frac{1}{2})(P_N - \frac{3}{2})$, the root has a real solution only if $P_N \leqslant \frac{1}{2}$ or $P_N \geqslant \frac{3}{2}$. (If this condition is violated, then the objective function is always increasing in $P_{\{1\}}$.) The case $P_N \leqslant \frac{1}{2}$ is clearly suboptimal ($P_N = 1$ dominates this.) Thus, we restrict attention to the case, $P_N \geqslant \frac{3}{2}$.

We now examine $R(\hat{P}_{\{1\}}(P_N), \hat{P}_{\{2\}}(P_N), P_N)$ computationally for $P_N \geqslant \frac{3}{2}$. This function has its maximum at $P_N = \frac{3}{2}$ which implies $P_{\{1\}} = 1$ or $A_{\{1\}} = \emptyset$.

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